

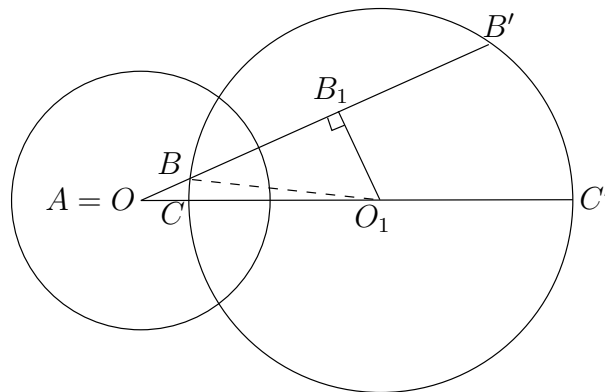
# Assignment 11

## Oral question

1. Prove that the distance function  $d(A, B) = |\log(AB, PQ)|$  of the Poincaré disk model is additive: if  $A * C * B$  on a Poincaré line then  $d(AC) + d(CB) = d(AB)$ . Fix a Poincaré line with ideal points  $P$  and  $Q$  and a point  $A$  on it. Move another point  $B$  along the Poincaré line from  $P$  to  $Q$ . Show that  $d(A, B)$  changes from  $\infty$  to 0 and then back to  $\infty$ .

## Question to be answered in writing

1. A hyperbolic circle centered at  $C$  of radius  $r$  is the set of all points  $A$  satisfying  $d(A, C) = r$ . Prove that a hyperbolic circle corresponds to a circle in the Poincaré disk model, although its center may not be equal to its Euclidean center. (Hint: prove the statement in the case when  $C = P$  first, where  $P$  is the center of the Poincaré disk. Use then the fact that reflection about the perpendicular bisector of  $PC$  takes a hyperbolic circle centered at  $P$  into a hyperbolic circle centered at  $C$ , and that this reflection corresponds to an inversion about a circle.)
2. Complete the following proof of the *hyperbolic Pythagorean theorem* (Theorem 16.1) which states the following:  
*Any right triangle  $\triangle ABC$  with  $\angle C$  being the right angle satisfies  $\cos(A) = \tanh(b) / \tanh(c)$ .*



Use the Poincaré disc model and assume that the vertex  $A$  is at the center of the disk. (The right angle of  $ABC_{\Delta}$  is at  $C$ .) The lines  $AB$  and  $AC$  are represented by straight lines, the line  $BC$  is represented by an arc of a circle centered at  $O_1$ . Let  $B'$  resp.  $C'$  be the second intersection of  $OB$  resp  $OC$  with this circle and  $B_1$  be the orthogonal projection of  $O$  to the line  $OB$ .

Using that the Euclidean distance  $OB$  equals  $\tanh(c/2)$  and that  $OB \cdot OB' = 1$  (justify why), prove that the Euclidean distance  $BB' = 2 / \sinh(c)$ . Observe that the Euclidean distance  $CC'$  is similarly equal to  $2 / \sinh(b)$ . Due to the Star Trek Lemma, the angle  $\angle BO_1B_1$  is equal to  $\angle B$ . (Why?) Hence

$$\sin(B) = \frac{BB_1}{O_1B} = \frac{BB'}{2O_1C} = \frac{BB'}{CC'} = \frac{\sinh(b)}{\sinh(c)}.$$

Finally, using that  $\cos(A) = AB_1/AO_1$ , where  $AB_1 = OB + BB'/2$  and  $AO_1 = AC + CC'/2$ , prove that

$$\cos(A) = \frac{\tanh(b)}{\tanh(c)}.$$