A bijective proof of Euler's theorem on integer partitions

Euler's theorem says that the number of partitions of n into odd parts is the same as the number of integer partitions of n into distinct parts

Proof: Given a partition of n into distinct parts $a_1, a_2, \ldots, a_k$, write each a_i in the form $a_i = 2^{b_i} \cdot c_i$ where c_i is odd and b_i is a nonnegative integer. (There is exactly one way to do so: 2^{b_i} is the highest power of 2 that divides a_i. Replace each a_i with 2^{b_i} copies of c_i.

For example $1 + 2 + 5 + 20 = 28$ is transformed first into $1 \times 1 + 2 \times 1 + 1 \times 5 + 4 \times 5 = 28$ and then into $1 + 1 + 1 + 5 + 5 + 5 + 5 + 5 = 28$.

The fact that the a_is are pairwise distinct is equivalent to stating that the ordered pairs (b_i, c_i) are pairwise distinct. This is true because of the uniqueness of the form $2^{\text{integer}} \times \text{odd}$.

The inverse of the above operation is easy to find. Collect first the copies of the same odd integer into products. For example, $1 + 1 + 1 + 5 + 5 + 5 + 5 + 5 = 28$ may be rewritten as $3 \times 1 + 5 \times 5 = 28$. In general, you get a formula $n_1 \times d_1 + n_2 \times c_2 + \cdots n_m \times c_m = n$, where $d_1, \ldots, d_m$ are distinct odd integers. Now rewrite each n_i in base 2: $n_i = 2^{b_{i,1}} + \cdots 2^{b_{i,k_i}}$. The distinct parts are then $2^{b_{i,j}} c_i$ where $1 \leq j \leq k_i$. For example, for $1 + 1 + 1 + 5 + 5 + 5 + 5 + 5 = 28$, we have $3 = 2^0 + 2^1$ and $5 = 2^0 + 2^2$. The distinct parts are $2^0 \times 1 = 1$, $2^1 \times 1 = 2$, $2^0 \times 5 = 5$ and $2^2 \times 5 = 20$.