Complex numbers

Complex numbers are of the form $a + bi$, where a and b are real numbers and $i^2 = -1$. They are added and multiplied in the obvious way, e.g,

$$(1+2i)+(3-4i) = 4-2i \quad \text{and} \quad (1+2i)\cdot(3-4i) = 3+6i-4i+8 = 11+2i.$$

The rules are

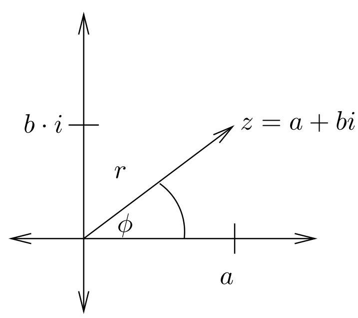
$$(a_1+b_1i) \pm (a_2+b_2i) = (a_1 \pm a_2) + (b_1 \pm b_2)i \quad \text{and}$$
$$(a_1+b_1i)\cdot(a_2+b_2i) = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i.$$

To divide complex numbers we use *complex conjugates*. The conjugate of $z = a+bi$ is $\overline{z} = a - bi$. The product $z\overline{z} = a^2 + b^2$ is a real number. Thus

$$\frac{a_1+b_1i}{a_2+b_2i} = \frac{(a_1+b_1i)(a_2-b_2i)}{(a_2+b_2i)(a_2-b_2i)} = \frac{(a_1+b_1i)(a_2-b_2i)}{a_2^2+b_2^2}.$$

For example

$$\frac{1+2i}{3-4i} = \frac{(1+2i)(3+4i)}{3^2+4^2} = \frac{-5+10i}{25} = -\frac{1}{5} + \frac{2}{5}i.$$



They may be identified with the point in the plane whose rectangular coordinates are (a, b). Under this identification the length or *modulus* of $a+bi$ is $r = \sqrt{a^2+b^2}$. By taking out the modulus we may rewrite the complex number in *polar form*:

$$z = a+bi = r(\cos\phi + i\sin\phi) \quad \text{where} \quad r = \sqrt{a^2+b^2}, \quad \cos(\phi) = a/r \quad \text{and} \quad \sin(\phi) = b/r.$$

For example

$$2 + 2\sqrt{3}i = 4\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = 4(\cos(\pi/3) + \sin(\pi/3)i)$$

since

$$r = \sqrt{4+12} = 4 \quad \text{and} \quad \frac{2}{4} = \cos(\pi/3), \quad \frac{2\sqrt{3}}{4} = \sin(\pi/3).$$

The angle ϕ is called the *argument* of the complex number. Multiplying complex numbers results in multiplying the moduli and adding the arguments:

$$\begin{aligned} z_1z_2 &= r_1(\cos\phi_1 + i\sin\phi_1)\cdot r_2(\cos\phi_2 + i\sin\phi_2) \\ &= r_1r_2(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2 + (\cos\phi_1\sin\phi_2 + \sin\phi_1\cos\phi_2)i) \\ &= r_1r_2(\cos(\phi_1+\phi_2) + \sin(\phi_1+\phi_2)i). \end{aligned}$$

In particular, multiplying by a complex number dilates by its modulus and rotates by its argument.

Dividing complex numbers results in taking the quotient of their moduli and the difference of their arguments:

$$\begin{aligned}
z_1/z_2 &= \frac{r_1(\cos\phi_1 + i\sin\phi_1)}{r_2(\cos\phi_2 + i\sin\phi_2)} = \frac{r_1}{r_2}\frac{(\cos\phi_1 + i\sin\phi_1)(\cos\phi_2 - i\sin\phi_2)}{\cos^2\phi_2 + \sin^2\phi_2} \\
&= \frac{r_1}{r_2}(\cos\phi_1\cos\phi_2 + \sin\phi_1\sin\phi_2 + (\cos\phi_1\sin\phi_2 - \sin\phi_1\cos\phi_2)i) \\
&= \frac{r_1}{r_2}(\cos(\phi_1 - \phi_2) + \sin(\phi_1 - \phi_2)i).
\end{aligned}$$

In particular, the argument of the quotient of two complex numbers is the angle between them.

Geometrically, taking the complex conjugate corresponds to reflecting with respect to the horizontal axis.

Using the facts that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \quad \text{and}$$

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!} == \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

it is easy to derive *Euler's formula*

$$e^{i\cdot\phi} = \cos(\phi) + i \cdot \sin(\phi).$$