Sample Final Exam Questions (Mandatory Part)

This document is subject to changes until Tuesday April 30

The actual final exam will have a mandatory and an optional section. The optional questions will be similar to the ones on the previous (sample) tests, and need to be answered only if you do not want me to re-use your average (adjusted) test score. The list of questions below is supposed to help you prepare for the mandatory part of the final. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below I am only listing questions related to the definitions, theorems, and proofs I expect you to know. There will be also application exercises, similar to the already discussed homework questions.

1. Express $(-1)^n\binom{1/2}{n}$ as a multiple of the Catalan number C_n.
2. The Catalan number C_n is defined as the number of sequences $a_1, \ldots, a_{2n}$ such that exactly n of the a_is is 1, the remaining a_is are -1, and we have $a_1 + a_2 + \cdots + a_m \geq 0$ for all $m \leq n$. Express C_n using binomial coefficients. Prove your formula, using the reflection principle.
3. Prove that the Catalan numbers satisfy the recurrence formula $C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i}$.
4. Using the recurrence formula $C_{n+1} = \sum_{i=0}^{n} C_i \cdot C_{n-i}$, write a quadratic equation for the generating function of the Catalan numbers, and solve it. Explain, which sign would you use in the quadratic formula and why.
5. Prove that the number of ways to climb a stairway of n steps by taking 1 or 2 steps at a time is the Fibonacci number F_n. Use this observation to express F_n as a sum of binomial coefficients of the form $\binom{n-i}{i}$. Prove your formula.
6. Give a closed-form formula for the Fibonacci number F_n and prove it.
7. Use the closed-form formula for F_n to show that, for large n, the quotient F_{n+1}/F_n approximately equals the golden ratio $\frac{1+\sqrt{5}}{2}$.
8. Prove by strong induction that the Lucas number L_n is given by $L_n = F_{n-2} + F_n$. Explain why this formula shows that L_n counts the tilings of the circular n-board with 1- and 2-tiles.
9. Find the ordinary generating function $f_k(x) = \sum_{n=0}^{\infty} S(n,k)x^n$ of the Stirling numbers of the second kind $S(n,k)$. Prove your formula.
10. Write $x^4 - 2x$ as a linear combination of the polynomials $(x)_4$, $(x)_3$, $(x)_2$, $(x)_1$ and $(x)_0$.
11. Write $(x)_4 - 2(x)_2$ as a linear combination of the powers of x.
12. Prove the formula
$$\Delta^m f(n) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} f(n+m-k).$$

13. Prove the formula
$$f(n) = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(0) \quad \text{for } n \geq 0,$$
and explain how this formula may be used to find a closed form formula for a higher order arithmetic sequence.

14. Using difference tables, find a closed-form formula for $f(n) = 1^3 + 2^3 + \cdots + n^3$.

15. Prove that the Stirling number of the first kind $s(n,k)$ is given by $s(n,k) = (-1)^{n-k}c(n,k)$ where $c(n,k)$ is the number of permutations of $\{1,2,\ldots,n\}$ with k cycles.

16. Use generating functions or a direct bijection to prove that the number of partitions of n into odd parts is the same as the number of partitions into distinct parts.

17. Prove that the number of self-conjugate integer partitions of n is the same as the number of its integer partitions into distinct odd parts.

18. Use generating functions to find $P(n,2)$.

19. Prove that the integer partition number $P(n,k)$ satisfies
$$\frac{\left(\!\binom{k}{n-k}\!\right)}{k!} \leq P(n,k).$$

20. Prove that the integer partition number $P(n,k)$ satisfies
$$P(n,k) \leq \frac{\left(\!\binom{k}{n+\binom{k}{2}-k}\!\right)}{k!}.$$

21. Use the inequalities stated in the previous two exercises to show that, for a fixed k,
$$P(n,k) \sim \frac{n^{k-1}}{k!(k-1)!}$$

Good luck. Gábor Hetyei