Assignment 12

Oral question

1. All hyperbolic rotations fixing the point i in the Poincaré upper half plane model are fractional linear transformations $z \mapsto \frac{az+b}{cz+d}$ sending i into i. Using this fact, and assuming that we have scaled our coefficients to satisfy $ad-bc=1$, show that
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$
for some angle θ.

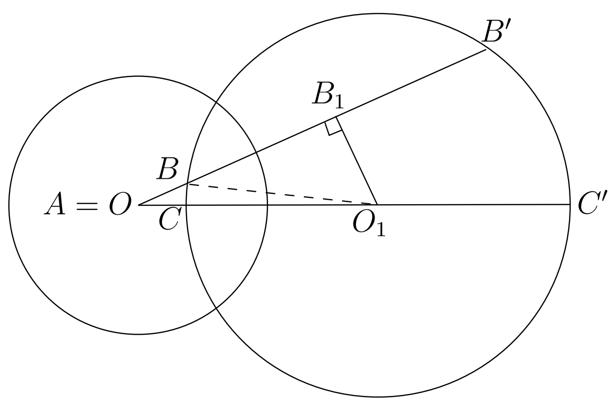
2. Consider the fractional linear transformation $z \mapsto \frac{az+b}{cz+d}$ where $a, b, c, d \in \mathbb{R}$ and $ad-bc \neq 0$. Introduce $z = z_1 + z_2 i$ and calculate explicitly the imaginary part of $\frac{az+b}{cz+d}$. Prove that the imaginary part of the image is positive for all $z_2 > 0$ if and only if $ad - bc > 0$.

   Now show that a conjugate fractional linear map $z \mapsto \frac{a\bar{z}+b}{c\bar{z}+d}$ takes the upper half plane into itself if and only if $ad - bc < 0$.

Question to be answered in writing

1. Complete the following proof of the *hyperbolic Pythagorean theorem* which states the following:

   Any right triangle $\triangle ABC$ with $\angle C$ being the right angle satisfies $\cos(A) = \tanh(b)/\tanh(c)$.



   Use the Poincaré disc model and assume that the vertex A is at the center of the disk. (The right angle of $ABC_\triangle$ is at C.) The lines AB and AC are represented by straight lines, the line BC is represented by an arc of a circle centered at O_1. Let B' resp. C' be the second intersection of OB resp OC with this circle and B_1 be the orthogonal projection of O to the line OB.

   Using that the Euclidean distance OB equals $\tanh(c/2)$ and that $OB \cdot OB' = 1$ (justify why), prove that the Euclidean distance $BB' = 2/\sinh(c)$. Observe that the Euclidean distance CC' is similarly equal to $2/\sinh(b)$. Due to the Star Trek Lemma, the angle $\angle BO_1B_1$ is equal to $\angle B$. (Why?) Hence
$$\sin(B) = \frac{BB_1}{O_1B} = \frac{BB'}{2O_1C} = \frac{BB'}{CC'} = \frac{\sinh(b)}{\sinh(c)}.$$

Finally, using that $\cos(A) = AB_1/AO_1$, where $AB_1 = OB + BB'/2$ and $AO_1 = AC + CC'/2$, prove that

$$\cos(A) = \frac{\tanh(b)}{\tanh(c)}.$$