Sample Test II.

The real test will have less questions and you will have about 75 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. The questions below are sample questions related to stating and proving theorems. Besides trying to answer these questions, make sure you also review all homework exercises. The test may also have questions similar to those exercises. During the test, the usage of books or notes, or communicating with other students will not be allowed.

1. Is the Cartesian product of two integral domains an integral domain? Justify your answer!
2. Let R be a ring and let a be any ring element. Prove that the solution x of the equation $a + x = 0_R$ is unique. Explain how this may be used to prove that $a + b = a + c$ implies $b = c$.
3. What is $a \cdot 0_R$ equal to in a ring? Prove your claim!
4. Prove that $-(-a) = a$ in a ring.
5. Prove that $-(a + b) = (-a) + (-b)$ in a ring.
6. Prove that a subset S is a subring if it is not empty, and it is closed under subtraction and multiplication.
7. Describe the unique solution of the equation $a + x = b$ in a ring.
8. If $ac = bc$ in a ring, does it always follow that $a = b$? When does it follow? Justify your claim with example and/or proof, as appropriate.
9. Prove that every field is an integral domain. Is the converse true?
10. Give an example of a zero divisor and an idempotent element.
11. Let $f : R \to S$ be a ring homomorphism. Prove that $f(0_R) = 0_S$ and that $f(-a) = -f(a)$ for all $a \in R$.
12. Let R be a ring. When is it true that $\deg(f \cdot g) = \deg(f) + \deg(g)$ holds for all nonzero polynomials $f, g \in R[x]$?
13. Let F be a field. Describe the units of $F[x]$. Justify your description.
14. State the division algorithm theorem in $F[x]$ and prove the uniqueness part.
15. State and prove the remainder theorem for polynomials in $F[x]$.
16. State and prove the factor theorem for polynomials with $F[x]$.
17. Explain why reducible polynomials of degree at most 3 in $F[x]$ must have a root.

18. State and prove the rational zeros (rational root test) theorem.

19. Find all rational zeros of the polynomial $10x^4 + 7x^3 + 6x^2 - 4x - 1$.

20. Define the greatest common divisor of two polynomials in $F[x]$ and explain how the Euclidean algorithm may be used to find it. (You do not have to prove your claim.)

21. Let F be a field. Prove that every irreducible polynomial $p(x) \in F[x]$ has the following property: If $p(x)$ divides $f(x)g(x)$ then it either divides $f(x)$ or it divides $g(x)$. Explain how this statement may be used to prove unique factorization in $F[x]$. (Are we showing uniqueness or existence using this claim? How do we prove the rest?)

22. Let F be a field. Prove that every nonconstant polynomial in $F[x]$ is the product of finitely many irreducible polynomials.

23. When can we say that two polynomials $f, g \in F[x]$ induce the same function from F to F if and only if they are equal? Prove your claim, and give an example for the situation when the claim is false.

Good luck. Gábor Hetyei