Connecting the two Poincaré models

The domain of the fractional linear transformation $\phi : z \mapsto \dfrac{z-i}{1-iz}$ is $\mathbb{C} \setminus \{-i\}$. It may be rewritten as

$$\frac{z-i}{1-iz} = \frac{i(1-iz)}{1-iz} + \frac{-2i}{1-iz} = i + \frac{-2i}{1-iz}.$$

This equivalent form shows that ϕ may be written as the composition $\phi = \phi_7\phi_6\phi_5\phi_4\phi_3\phi_2\phi_1$, where

1. $\phi_1 : z \mapsto -iz$ is the rotation around the origin by $-\pi/2$ (thus $\phi_1(z) = -iz$);
2. $\phi_2 : z \mapsto z + 1$ is the horizontal translation to the right by 1 (thus $\phi_2\phi_1(z) = 1 - iz$);
3. $\phi_3 : z \mapsto 1/\overline{z}$ the inversion about the unit circle, centered at the origin, (thus $\phi_3\phi_2\phi_1(z) = 1/(\overline{1-iz})$);
4. $\phi_4 : z \mapsto \overline{z}$ is the reflection about the real axis (thus $\phi_4 \cdots \phi_1(z) = 1/(1 - iz)$);
5. $\phi_5 : z \mapsto -iz$ is the rotation around the origin by $-\pi/2$ (thus $\phi_5 \cdots \phi_1(z) = -i/(1 - iz)$);
6. $\phi_6 : z \mapsto 2z$ is the dilation centered at the origin by a factor of 2 (thus $\phi_6 \cdots \phi_1(z) = -2i/(1-iz)$);
7. $\phi_7 : z \mapsto z + i$ is the vertical translation up by 1 (thus $\phi_7 \cdots \phi_1(z) = \phi(z)$);

Theorem 1 *The map $\phi : z \mapsto \dfrac{z-i}{1-iz}$, restricted to the upper half plane formed by all complex numbers whose imaginary part is non-negative, establishes a bijection between this half plane and the closed unit disk, centered at the origin.*

Proof: Let us analyze the sequence of maps ϕ_1,...,ϕ_7 described above. These maps are all injective, therefore it suffices to describe the surjective image of the upper half plane after applying $\phi_j \cdots \phi_1$, for each $j \in \{1, \ldots, 7\}$.

1. ϕ_1 takes the set of complex numbers whose imaginary part is non-negative into the set of complex numbers whose real part is non-negative.
2. ϕ_2 takes this half plane into the set of complex numbers whose real part is at least 1.
3. ϕ_3 takes the half plane on the previous line into the disk of radius $1/2$, centered at $1/2$.
4. ϕ_4 takes the disk of radius $1/2$, centered at $1/2$, into itself.
5. ϕ_5 takes of radius $1/2$, centered at $1/2$, into the disk of radius $1/2$, centered at $-i/2$.
6. ϕ_6 takes the disk of radius $1/2$, centered at $-i/2$, into the unit disk, centered at $-i$.
7. ϕ_7 takes the unit disk, centered at $-i$, into the unit disk, centered at the origin.

$\Diamond$

Since ϕ is a composition of transformations that take lines and circles into lines and circles, preserving angles and the cross ratio, we may use ϕ^{-1} to "export" the geometry of the Poincaré disk model to the upper half plane, thus obtaining the Poincaré upper half plane model:

1. The set of ideal points on the unit circle in the Poincaré disk model corresponds to the real line in the upper half plane model.

2. Since lines in the Poincaré disk model were defined as parts of lines and circles inside the unit disk that are perpendicular to the unit circle, lines in the upper half plane model are parts of lines and circles in the upper half plane that are perpendicular to the real line. These are exactly the vertical, upwards infinite half-lines and concave-down semicircles whose endpoint(s) is (are) on the real line.

3. Angles are the angles of the "lines" (defined as the angles of their tangents at the intersection) in either models.

4. The distance if A and B in the Poincaré disk model was defined as $\ln(A, B, P, Q)$ where P and Q are the ideal points of the line connecting A and B. Since ϕ^{-1} preserves cross-ratio, we may define distance in the upper half plane model in the same way. In the case when A and B are not on the same vertical line, P and Q are the endpoints of the semicircle connecting them, located on the real line. In the case when A and B are on the same vertical line, P is the intersection of this vertical line with the real line, and Q is the point at infinity. The sensed ratio (A, B, Q) is -1 so the cross ratio (A, B, P, Q) is the negative of the sensed ratio (A, B, P), i.e. AP/BP. The distance of A and B is $\ln(AP/PB)$.

References

[1] D. Royster, "Non-Euclidean Geometry and a Little on How We Got There," Lecture notes, December 11, 2011.