The hyperbolic laws of sines and cosines for general triangles

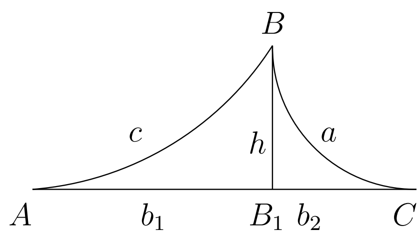
We consider the Poincaré disk model. Recall that, in this model, any right triangle $\triangle ABC$ with the right angle at C satisfies

$$\cosh(c) = \cosh(a)\cosh(b), \tag{1}$$

$$\cos(A) = \frac{\tanh(b)}{\tanh(c)} \quad \text{and} \tag{2}$$

$$\sin(B) = \frac{\sinh(b)}{\sinh(c)}. \tag{3}$$

To prove the hyperbolic laws of sines and cosines, we will use the following figure:



Theorem 1 (Hyperbolic law of sines) *Any triangle in the Poincaré disk model satisfies*

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)} = \frac{\sin(C)}{\sinh(c)}.$$

Proof: Applying (3) to the right triangle ABB_1 yields

$$\sin(A) = \frac{\sinh(h)}{\sinh(c)}.$$

This equation allows us to express $\sinh(h)$ as follows:

$$\sinh(h) = \sin(A)\sinh(c).$$

Similarly, applying (3) to the right triangle CBB_1 allows us to write

$$\sinh(h) = \sin(C)\sinh(a).$$

Therefore we have

$$\sin(A)\sinh(c) = \sin(C)\sinh(a),$$

since both sides equal $\sinh(h)$. Dividing both sides by $\sinh(a)\sinh(c)$ yields

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(C)}{\sinh(c)}.$$

The equality

$$\frac{\sin(A)}{\sinh(a)} = \frac{\sin(B)}{\sinh(b)}$$

may be shown in a completely similar fashion. $\diamond$

Theorem 2 (Hyperbolic law of cosines) *Any triangle in the in the Poincaré disk model satisfies*

$$\cosh(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\cos(A).$$

Proof: Applying (1) to the right triangle $\triangle BB_1C$ yields

$$\cosh(a) = \cosh(b_2)\cosh(h)$$

Let us replace b_2 with $b - b_1$ in the above equation. After applying the formula $\cosh(x-y) = \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$ we obtain

$$\cosh(a) = \cosh(b)\cosh(b_1)\cosh(h) - \sinh(b)\sinh(b_1)\cosh(h).$$

Applying (1) to the right triangle $\triangle BB_1A$ we may replace both occurrences of $\cosh(h)$ above with $\cosh(c)/\cosh(b_1)$ and obtain

$$\cosh(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(b_1)\frac{\cosh(c)}{\cosh(b_1)}, \quad \text{that is,}$$

$$\cosh(a) = \cosh(b)\cosh(c) - \sinh(b)\sinh(c)\frac{\tanh(b_1)}{\tanh(c)}.$$

Finally, (2) applied to the right triangle $\triangle BB_1A$ allows replacing $\tanh(b_1)/\tanh(c)$ with $\cos(A)$. $\diamond$

Theorem 3 (Second hyperbolic law of cosines) *Any triangle in the in the Poincaré disk model satisfies*

$$\cos(B) = -\cos(A)\cos(C) + \sin(A)\sin(C)\cosh(b).$$

Proof: Let us write (1) for the triangles $\triangle BB_1A$ and $\triangle CB_1A$, and multiply them together. We get

$$\cosh(a)\cosh(c) = \cosh^2(h)\cosh(b_1)\cosh(b_2).$$

After multiplying both sides by $\cosh(b_1 + b_2) = \cosh(b)$ we obtain

$$\cosh(a)\cosh(c)(\cosh(b_1)\cosh(b_2) + \sinh(b_1)\sinh(b_2)) = \cosh(b)\cosh^2(h)\cosh(b_1)\cosh(b_2).$$

Replacing $\cosh^2(h)$ with $1 + \sinh^2(h)$ and dividing both sides by $\cosh(b_1)\cosh(b_2)$ yields

$$\cosh(a)\cosh(c)(1 + \tanh(b_1)\tanh(b_2)) = \cosh(b)(1 + \sinh^2(h)).$$

After rearranging we get

$$\cosh(a)\cosh(c) - \cosh(b) = -\tanh(b_1)\tanh(b_2)\cosh(c)\cosh(a) + \sinh^2(h)\cosh(b).$$

By Theorem 2 we may replace $\cosh(a)\cosh(c) - \cosh(b)$ on the left hand side by $\cos(B)\sinh(a)\sinh(c)$ and get

$$\cos(B)\sinh(a)\sinh(c) = -\tanh(b_1)\tanh(b_2)\cosh(c)\cosh(a) + \sinh^2(h)\cosh(b).$$

Dividing both sides by $\sinh(a)\sinh(c)$ yields

$$\cos(B) = -\frac{\tanh(b_1)}{\tanh(c)}\frac{\tanh(b_2)}{\tanh(a)} + \frac{\sinh(h)}{\sinh(c)}\frac{\sinh(h)}{\sinh(a)}\cosh(b).$$

Finally, applying (2) to $\triangle BB_1A$ and $\triangle BB_1C$ allows replacing $\tanh(b_1)/\tanh(c)$ with $\cos(A)$ and $\tanh(b_2)/\tanh(a)$ with $\cos(C)$, and applying (3) to $\triangle BB_1A$ and $\triangle BB_1C$ allows replacing $\sinh(h)/\sinh(c)$ with $\sin(A)$ and $\sinh(h)/\sinh(a)$ with $\sin(C)$. $\diamond$