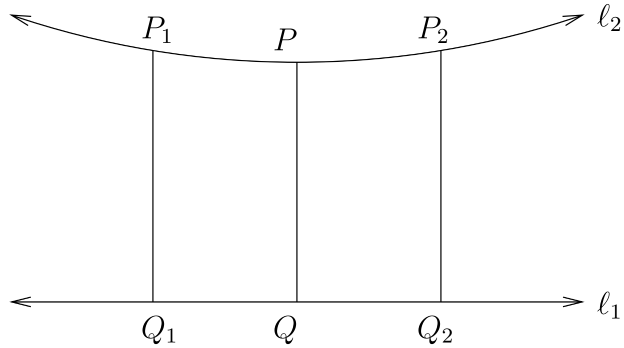
Hypercycles and horocycles

Definition 1 *Two parallel lines are* hyperparallel *if they have a common perpendicular.*

Proposition 1 *Two parallel lines ℓ_1 and ℓ_2 are hyperparallel, if and only if there exists two distinct points P_1 and P_2 on ℓ_2 such that the distance of P_1 from ℓ_1 is the same as the distance from P_2 from ℓ_1.*

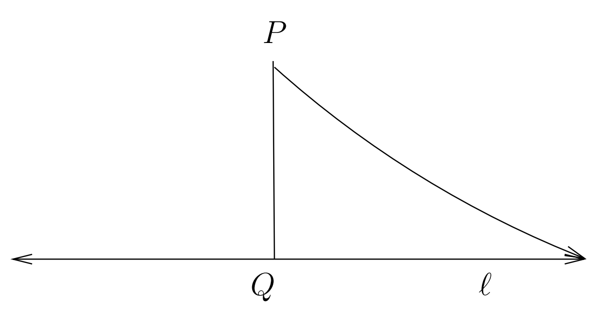
Proof:



Suppose there are two such points P_1 and P_2. Let the perpendicular projections of P_1 and P_2 onto ℓ_1 be Q_1, respectively Q_2. Then $P_1P_2Q_2Q_1$ is a Saccheri quadrilateral whose median PQ is the common perpendicular PQ.

Conversely assume PQ is the common perpendicular. Let P_1 and P_2 be points on ℓ_2 such that P_1P is congruent to P_2P. Reflection about the line PQ takes the line ℓ_1 into itself and the points P_1 and P_2 into each other. Thus the distance of P_1 from ℓ_1 is the same as the distance from P_2 from ℓ_1. ◇

Let ℓ be a line and P be a point not on ℓ. Let Q be the perpendicular projection of P onto ℓ A *limiting parallel ray starting at* P is a ray starting at P that does not intersect ℓ but has the property that decreasing the angle between the ray and the line PQ by any amount results in a ray intersecting ℓ.

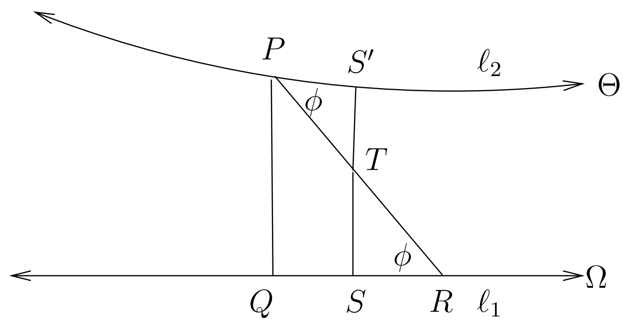


The angle between the limiting parallel ray and the line PQ is the *angle of parallelism* $\Pi(PQ)$. It can be shown that $\Pi(PQ)$ only depends on the length of PQ.

Definition 2 *Two parallel lines are* horoparallel *if one line contains a limiting parallel ray to the other line.*

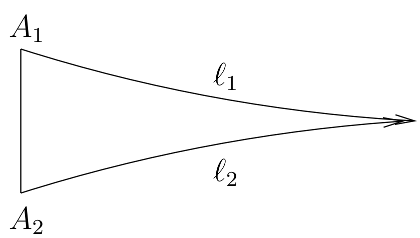
Proposition 2 *If two parallel lines ℓ_1 and ℓ_2 are not horoparallel then they are hyperparallel.*

Proof:



Let P be a point on ℓ_2 and let Q be its perpendicular projection onto ℓ_1. Let Θ and Ω be limit points on ℓ_2 and ℓ_1 respectively, on the same side of PQ such that $\angle QP\Theta$ is acute. (If it is a right angle, we are done.) The two lines being not horoparallel, the angle $\angle QP\Theta$ is larger than the angle of parallelism $\Pi(PQ)$. Let R be a point on the line ℓ_1, on the same side of PQ as Ω. If R is very close to Q then $\angle QRP$ is almost $\pi/2$, more than $\angle RP\Theta$ which is less than the acute angle $\angle QP\Theta$. If R is very close to Ω, then $\angle QRP$ is almost 0 which is less than $\angle RP\Theta$ that is more than $\angle QP\Theta - \Pi(PQ)$. Between these two extremes there is a point R such that $\angle QRP = \angle RP\Theta$. Let T be the midpoint of PR and let S be the perpendicular porjection of S onto ℓ_1. Let S' be the point on ℓ_2 that is on the same side of PQ as Θ and satisfies $|PS'| = |RS|$ The triangle $\triangle S'PT$ is congruent to the right triangle $\triangle SRT$ by SAS. Therefore the $\angle S'TP$ is congruent to $\angle STP$ and the points S, T and S' are collinear. The line SS' is the common perpendicular of ℓ_1 and ℓ_2. $\diamond$

Given a line ℓ with limit point Ω consider the set L of all lines that are horoparallel to ℓ and have limit point Ω. A point A_1 on line $\ell_1 \in L$ *corresponds* to a point A_2 on line ℓ_2 if the angle between the line A_1A_2 and ℓ_1 is the same as the angle between the line A_1A_2 and ℓ_2.



Proposition 3 *Correspondence is an equivalence relation.*

Definition 3 *A* horocycle *is an equivalence class of points under correspondence.*

Note that a horocycle is uniquely defined by one point and the limit point Ω used to define correspondence.

Finally we define hypercycles.

Definition 4 *A* hypercycle *is the set of all points that are at the same fixed distance d from a given line ℓ, on the same side of the line ℓ.*

Note that a hypercycle is uniquely defined by the line ℓ, the distance d and the side of the line ℓ containing the hypercycle. We may also define hypercycles as equivalence classes, where the points P and Q are equivalent when they are on the same side of ℓ and at the same distance from ℓ.

References

[1] D. Royster, "Non-Euclidean Geometry and a Little on How We Got There," Lecture notes, May 7, 2012.