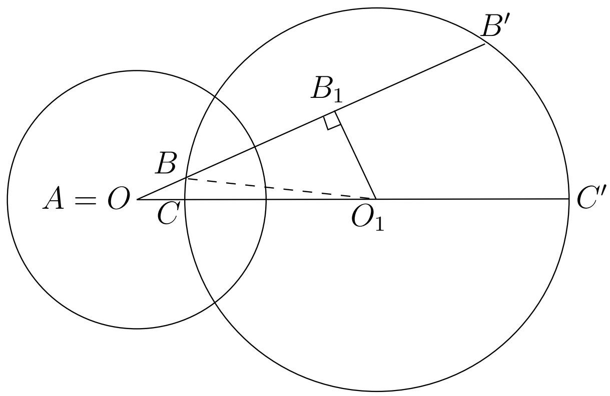
Sines and cosines in the Poincaré disk model of the hyperbolic plane

Theorem 1 *Assume that $ABC_\triangle$ is a right triangle, with its right angle at C, in the hyperbolic plane represented by the Poincaré disk model. Then*

$$\sin(B) = \frac{\sinh(b)}{\sinh(c)} \quad \textit{and} \quad \cos(A) = \frac{\tanh(b)}{\tanh(c)}.$$

Proof: Without loss of generality we may assume that A is at the center of the Poincaré disk.



The lines AB and AC are represented by straight lines, the line BC is represented by an arc of a circle C_1 centered at O_1. Let B' resp. C' be the second intersection of OB resp OC with this circle and B_1 be the orthogonal projection of O to the line OB.

Since the Poincaré disk and the circle C_1 are orthogonal to each other, the power of $A = O$ with respect to C_1 is 1 (=the radius of the Poincaré disk). Hence the Euclidean distance OB satisfies $OB \cdot OB' = 1$. We also know that the Euclidean distance OB equals $\tanh(c/2)$. Thus

$$\begin{aligned} BB' &= OB' - OB = 1/OB - OB = 1/\tanh(c/2) - \tanh(c/2) = \frac{\cosh(c/2)}{\sinh(c/2)} - \frac{\sinh(c/2)}{\cosh(c/2)} \\ &= \frac{\cosh^2(c/2) - \sinh^2(c/2)}{\sinh(c/2)\cdot\cosh(c/2)} = \frac{2}{2\cdot\sinh(c/2)\cdot\cosh(c/2)} = \frac{2}{\sinh(c)} \end{aligned}$$

Similarly, since the Euclidean distance OC equals $\tanh(b/2)$, we get $CC' = 2/\sinh(b)$. The angle of $ABC_\triangle$ at B is the angle between the tangent of C_1 at B and the line OB. Due to the Star Trek Lemma, this is the half of the central angle $\angle BO_1B'$, which is equal to $\angle BO_1B_1$. Hence $\sin(B)$ may be calculated from the right triangle $O_1B_1B_\triangle$, and we get

$$\sin(B) = \frac{BB_1}{O_1B} = \frac{BB'}{2O_1C} = \frac{BB'}{CC'} = \frac{\sinh(b)}{\sinh(c)}.$$

We may calculate $\cos(A)$ using $\cos(A) = AB_1/AO_1$. Here

$$\begin{aligned} AB_1 &= OB + BB'/2 = \tanh(c/2) + 1/\sinh(c) = \frac{\sinh(c/2)}{\cosh(c/2)} + \frac{1}{2\sinh(c/2)\cosh(c/2)} \\ &= \frac{2\sinh^2(c/2)+1}{2\sinh(c/2)\cosh(c/2)} = \frac{2\sinh^2(c/2) + \cosh^2(c/2) - \sinh^2(c/2)}{\sinh(c)} = \frac{\cosh^2(c/2)+\sinh^2(c/2)}{\sinh(c)} \\ &= \frac{\cosh(c)}{\sinh(c)} = \frac{1}{\tanh(c)}. \end{aligned}$$

Similarly, $AO_1 = AC + CC'/2$ yields $AO_1 = 1/\tanh(c)$ and so we obtain

$$\cos(A) = AB_1/AO_1 = \frac{\tanh(b)}{\tanh(c)}.$$

$\diamond$

In analogy to the formulas for $\sin(B)$ and $\cos(A)$ we also have

$$\sin(A) = \frac{\sinh(a)}{\sinh(c)} \quad \text{and} \quad \cos(B) = \frac{\tanh(a)}{\tanh(c)}.$$

Since $1 = \sin^2(A) + \cos^2(A)$, we get

$$1 = \frac{\sinh^2(a)}{\sinh^2(c)} + \frac{\tanh^2(b)}{\tanh^2(c)} = \frac{\sinh^2(a) + \tanh^2(b) \cdot \cosh^2(c)}{\sinh^2(c)} = \frac{\sinh^2(a)\cosh^2(b) + \sinh^2(b) \cdot \cosh^2(c)}{\cosh^2(b)\sinh^2(c)}.$$

Multiplying both sides with $\cosh^2(b)\sinh^2(c)$ we get

$$\cosh^2(b)\sinh^2(c) = \sinh^2(a)\cosh^2(b) + \sinh^2(b) \cdot \cosh^2(c).$$

Using the identity $\sinh^2(x) = \cosh^2(x) - 1$ we may get rid of the hyperbolic sines and write

$$\cosh^2(b)(\cosh^2(c) - 1) = (\cosh^2(a) - 1)\cosh^2(b) + (\cosh^2(b) - 1) \cdot \cosh^2(c), \quad \text{i.e.,}$$

$$\cosh^2(b)\cosh^2(c) - \cosh^2(b) = \cosh^2(a)\cosh^2(b) - \cosh^2(b) + \cosh^2(b)\cosh^2(c) - \cosh^2(c).$$

Adding $\cosh^2(b) + \cosh^2(c) - \cosh^2(b)\cosh^2(c)$ yields

$$\cosh^2(c) = \cosh^2(a)\cosh^2(b).$$

Since the range of the hyperbolic cosine function is a subset of the positive real numbers, we may take the square root on both sides and get the hyperbolic Pythagorean theorem:

Theorem 2 *If a, b, c are the sides of a hyperbolic right triangle, c is the hypotenuse and the hyperbolic plane is the Poincaré disk model then*

$$\cosh(c) = \cosh(a)\cosh(b).$$