The hyperbolic Pythagorean theorem

The hyperbolic Pythagorean theorem is the following statement.

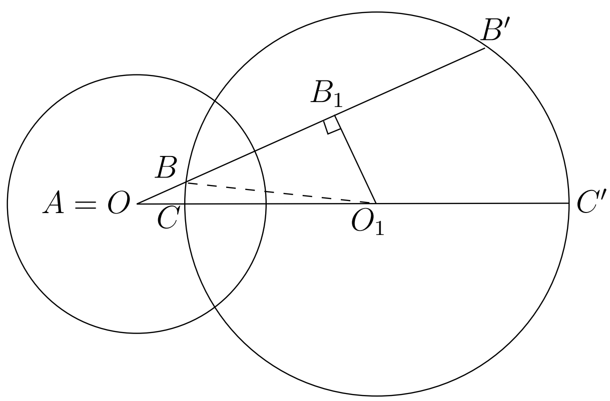
Proposition 1 *Any right triangle* $\triangle ABC$ *with* $\angle C$ *being the right angle satisfies* $\cosh(c) = \cosh(a)\cosh(b)$.

Proof: See [1, page 178]. $\Diamond$

To prove the rest of the formulas of hyperbolic trigonometry, we need to show the following.

Proposition 2 *Any right triangle* $\triangle ABC$ *with* $\angle C$ *being the right angle satisfies* $\cos(A) = \tanh(b)/\tanh(c)$.

Proof: It is your homework to fill in the details in the following proof.



Use the Poincaré disc model and assume that the vertex A is at the center of the disk. (The right angle of $ABC_\triangle$ is at C.) The lines AB and AC are represented by straight lines, the line BC is represented by an arc of a circle centered at O_1. Let B' resp. C' be the second intersection of OB resp OC with this circle and B_1 be the orthogonal projection of O to the line OB.

Using that the Euclidean distance OB equals $\tanh(c/2)$ and that $OB \cdot OB' = 1$ (justify why), prove that the Euclidean distance $BB' = 2/\sinh(c)$. Observe that the Euclidean distance CC' is similarly equal to $2/\sinh(b)$. Due to the Star Trek Lemma, the angle $\angle BO_1B_1$ is equal to $\angle B$. (Why?) Hence

$$\sin(B) = \frac{BB_1}{O_1B} = \frac{BB'}{2O_1C} = \frac{BB'}{CC'} = \frac{\sinh(b)}{\sinh(c)}.$$

Finally, using that $\cos(A) = AB_1/AO_1$, where $AB_1 = OB + BB'/2$ and $AO_1 = AC + CC'/2$, prove that

$$\cos(A) = \frac{\tanh(b)}{\tanh(c)}.$$

$\Diamond$

Proposition 3 *The previous two statements also imply the following equalities:*

$$\sin(A) = \frac{\sinh(a)}{\sinh(c)}, \tag{1}$$

$$\frac{\cos(A)}{\sin(B)} = \cosh(a) \quad \textit{and} \tag{2}$$

$$\cot(A)\cot(B) = \cosh(a)\cosh(b) \tag{3}$$

Proof: Before proving equation (1), note that this equation was actually shown during the proof of Proposition 2 (for B whose role is exchangeable with the role of A). That said, here we show that it follows algebraically from the previous two propositions. By Proposition 2 we have

$$\sin^2(A) = 1 - \cos^2(A) = \frac{\tanh^2(c) - \tanh^2(b)}{\tanh^2(c)}.$$

Using the fact that $\tanh(x) = \sinh(x)/\cosh(x)$, the above equation may be rewritten as

$$\sin^2(A) = \frac{\sinh^2(c)\cosh^2(b) - \cosh^2(c)\sinh^2(b)}{\sinh^2(c)\cosh^2(b)}.$$

Replacing each $\sinh^2(x)$ with $\cosh^2(x) - 1$ in the numerator we get

$$\sin^2(A) = \frac{(\cosh^2(c) - 1)\cosh^2(b) - \cosh^2(c)(\cosh^2(b) - 1)}{\sinh^2(c)\cosh^2(b)} = \frac{\cosh^2(c) - \cosh^2(b)}{\sinh^2(c)\cosh^2(b)}.$$

By Proposition 1 we may replace $\cosh^2(c)$ with $\cosh^2(a)\cosh^2(b)$ and get

$$\sin^2(A) = \frac{\cosh^2(a)\cosh^2(b) - \cosh^2(b)}{\sinh^2(c)\cosh^2(b)} = \frac{\cosh^2(a) - 1}{\sinh^2(c)} = \frac{\sinh^2(a)}{\sinh^2(c)}.$$

Since A is an acute angle, $\sin(A)$ is positive and we may take the square root on both sides to obtain equation (1). Combining equation (1) with Proposition 2 yields

$$\frac{\cos(A)}{\sin(B)} = \frac{\tanh(b)}{\tanh(c)} \cdot \frac{\sinh(c)}{\sinh(b)} = \frac{\cosh(c)}{\cosh(b)}.$$

By Proposition 1 we may replace $\cosh(c)$ with $\cosh(a)\cosh(b)$ and get

$$\frac{\cos(A)}{\sin(B)} = \frac{\cosh(a)\cosh(b)}{\cosh(b)}.$$

Equation (2) follows after simplifying by $\cosh(b)$. Finally, using equation (2) for $\cos(A)/\sin(B)$ and for $\cos(B)/\sin(A)$ yields

$$\cot(A)\cot(B) = \frac{\cos(A)}{\sin(B)} \cdot \frac{\cos(B)}{\sin(A)} = \cosh(a)\cosh(b).$$

$\diamondsuit$

References

[1] D. Royster, "Non-Euclidean Geometry and a Little on How We Got There," Lecture notes, May 7, 2012.