Assignment 4

Oral questions

1. Prove the following converse of the Star Trek Lemma: given a circle centered at O, and three points A, B, and C such that

   (i) B and C are on the circle,
   (ii) The angle $\angle BAC$ is the half of $\angle BOC$,

   the angle A is also on the circle. (I ask you to work out only the case when $\angle BAC$ is acute and O lies in its interior, keeping in mind that there are also other cases, see the first written question. It might help if you consider, how $\angle BAC$ changes when you move the point A on a line containing O, towards O or away from it.)

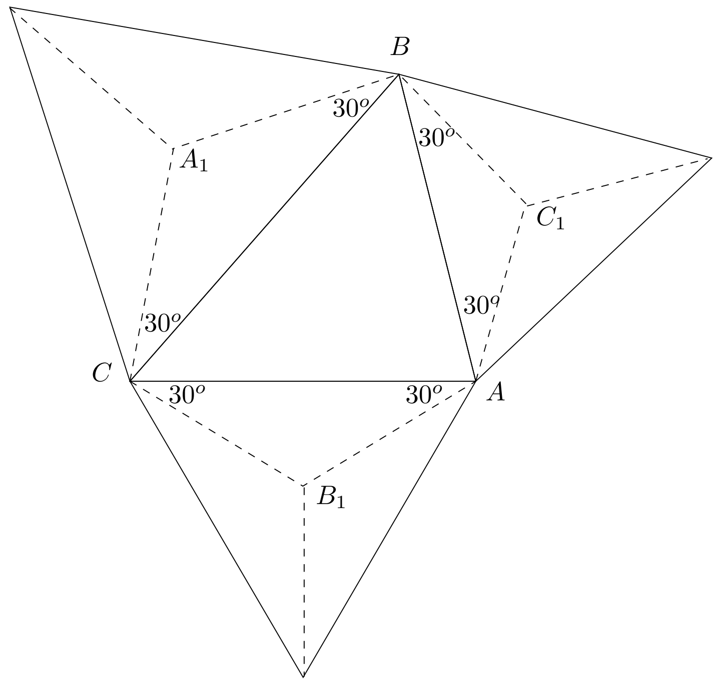
2. Prove that a quadrilateral is cyclic if and only if the sum of two of its opposite angles is 180^o. Explain which implication is related to the Star Trek Lemma, and which to its converse.

3. Let a, b, and c be the sides of a triangle, and A its area. Prove that the excircle at side a has radius $2A/(-a+b+c)$.

Questions to be answered in writing

1. Use Ceva's theorem to prove that the orthocenter exists.

2. Prove Napoleon's theorem: Given an arbitrary triangle $ABC_\triangle$, the centers of the equilateral triangles exterior to $ABC_\triangle$ form an equilateral triangle. (Illustration and hints on next page.)



Hints: Represent the points A, B, C, A_1, B_1, C_1 with complex numbers a, b, c, a_1, b_1, c_1. Observe that multiplying with

$$\rho := \frac{1}{\sqrt{3}}\left(\cos(30^o) + i \cdot \sin(30^o)\right)$$

rotates the vector $\overrightarrow{BA} = a - b$ into $\overrightarrow{BC_1} = c_1 - b$. Use this observation to express c_1 in terms of a, b and ρ. Express then a_1 and c_1 similarly in terms of a, b, c and ρ. Show that $c_1 - a_1$ is obtained by multiplying $b_1 - a_1$ with

$$\frac{\rho}{1-\rho} = \frac{2\rho - 1}{\rho} = \frac{\rho - 1}{2\rho - 1}.$$

It is probably easier to do so if you find the quadratic equation whose roots are ρ and its conjugate. Finally show that

$$\frac{\rho}{1-\rho} = \cos(60^o) + i \cdot \sin(60^o)$$

meaning that $\overrightarrow{A_1C_1}$ is obtained from $\overrightarrow{A_1B_1}$ by a 60^o rotation.