Name: **Student ID:**

Sample Test II.

The actual test will have less questions and perhaps one bonus question. You will have 75 minutes to answer them, without using your notes or communicating with other students. You will have to give the simplest possible answer and show all your work. Besides trying to answer the sample questions listed below, make sure you also review all homework questions that we already discussed in class as any of them may appear on the test.

1. Using the Chinese Remainder Theorem, solve the system of congruences

$$\begin{aligned} x &\equiv 3 \pmod{4} \\ x &\equiv 5 \pmod{6} \\ 2x &\equiv 7 \pmod{5} \end{aligned}$$

2. State and prove the Chinese Remainder Theorem.

3. State and outline the proof of Fermat's little theorem.

4. Use Fermat's little theorem to find a number between 0 and 16 that is congruent to $3^{3000000}$ modulo 17.

5. State and outline the proof of Wilson's theorem.

6. Let p be an odd prime. Using Wilson's theorem, prove that

$$(p-2)! \equiv 1 \pmod{p}.$$

7. If f is a multiplicative function and F is defined by $F(n) = \sum_{d|n} f(d)$, then F is multiplicative. Prove this claim.

8. State and prove the Möbius inversion formula.

9. Let $\tau(n)$ be the number of divisors of n, and $\mu(n)$ the Möbius function. Prove that

$$\sum_{d|n} \mu(d)\tau\left(\frac{n}{d}\right) = 1.$$

10. Prove the converse of the statement in Question 7.

11. Let $\sigma(n)$ the sum of divisors of n and $\mu(n)$ the Möbius function. Prove that

$$\sum_{d|n} \mu(d)\sigma\left(\frac{n}{d}\right) = n.$$

12. Explain why σ and τ are multiplicative functions.
13. Use the greatest integer function to express the highest exponent k for which the k^{th} power p^k of a prime p divides $n!$. Prove your formula.
14. Determine the exponent of the highest power of 7 appearing in the prime factorization of 140!.
15. Prove, by means of number theory that the binomial coefficient $\binom{n}{k}$ is an integer. (You are *not* allowed to use a combinatorial argument.)
16. Is the Euler function ϕ multiplicative? Justify your answer!
17. Calculate $\phi(1234)$.
18. Prove that the product of two multiplicative functions is multiplicative (We define the product $f \cdot g$ by $(f \cdot g)(n) := f(n) \cdot g(n)$.)
19. State and outline the proof of Euler's generalization of Fermat's little theorem.
20. Prove Gauss' formula for $\sum_{d|n} \phi(d)$.

Good luck. Gábor Hetyei