Assignment 5

Oral questions

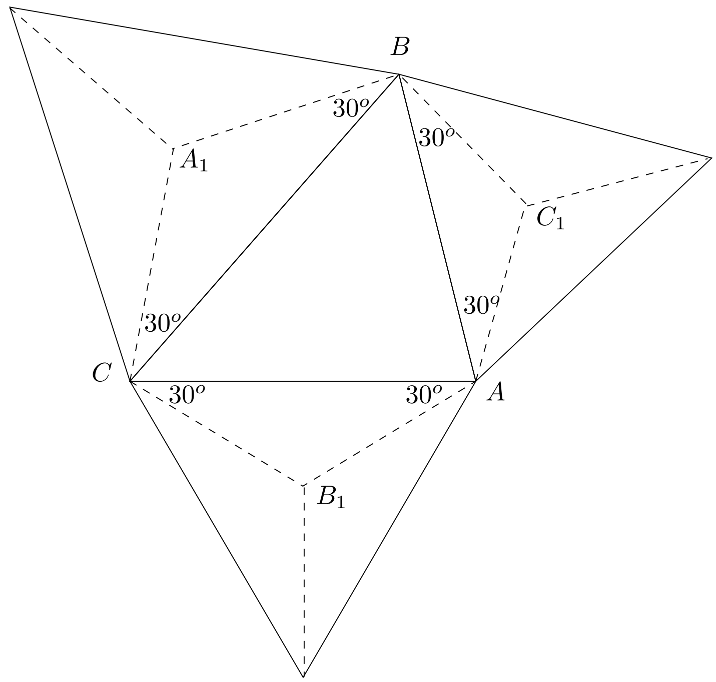
1. Use Ceva's theorem to prove that the Nagel point exists. (See your notes for the definition.)

2. Let $ABC_\triangle$ be a right triangle with a right angle at C and let C_1 be the orthogonal projection of C on AB. Prove that $|CC_1|$ is the *geometric mean* of $|AC_1|$ and $|C_1B|$, that is $|CC_1| = \sqrt{|AC_1| \cdot |C_1B|}$. Deduce the inequality between the arithmetic and geometric mean: $\sqrt{ab} \leq \frac{a+b}{2}$ for all $a, b \geq 0$.

3. Let a and b the side lengths of a parallelogram, and c and d the lengths of its diagonals. Prove that $2(a^2 + b^2) = c^2 + d^2$. (In other words, the sum of the lengths of the squares of the diagonals equals the sum of the squares of the side lengths.)

Question to be answered in writing

1. Prove Napoleon's theorem: Given an arbitrary triangle $ABC_\triangle$, the centers of the equilateral triangles exterior to $ABC_\triangle$ form an equilateral triangle. (Illustration and hints on next page.)



Hints: Represent the points A, B, C, A_1, B_1, C_1 with complex numbers a, b, c, a_1, b_1, c_1. Observe that multiplying with

$$\rho := \frac{1}{\sqrt{3}} \left(\cos(30^o) + i \cdot \sin(30^o)\right)$$

rotates the vector $\overrightarrow{BA} = a - b$ into $\overrightarrow{BC_1} = c_1 - b$. Use this observation to express c_1 in terms of a, b and ρ. Express then a_1 and c_1 similarly in terms of a, b, c and ρ. Show that $c_1 - a_1$ is obtained by multiplying $b_1 - a_1$ with

$$\frac{\rho}{1-\rho} = \frac{2\rho - 1}{\rho} = \frac{\rho - 1}{2\rho - 1}.$$

It is probably easier to do so if you find the quadratic equation whose roots are ρ and its conjugate. Finally show that

$$\frac{\rho}{1-\rho} = \cos(60^o) + i \cdot \sin(60^o)$$

meaning that $\overrightarrow{A_1C_1}$ is obtained from $\overrightarrow{A_1B_1}$ by a 60^o rotation.