Sample Test II.

he actual test will have less questions and perhaps one bonus question. You will have 75 minutes to answer them, without using your notes or communicating with other students. You will have to give the simplest possible answer and show all your work.

1. Given an integer $b > 1$, prove that every number has a base b representation. Illustrate your proof by finding the base 3 representation of 251.
2. State and prove a criterion for divisibility of N by 8 in terms of the hexadecimal (=base 16) representation of n.
3. State and prove a criterion for divisibility of N by 17 in terms of the hexadecimal (=base 16) representation of n.
4. Using the Chinese Remainder Theorem, solve the system of congruences

$$\begin{aligned} x &\equiv 3 \pmod{4} \\ x &\equiv 5 \pmod{6} \\ 2x &\equiv 7 \pmod{5} \end{aligned}$$

5. State and outline the proof of Fermat's little theorem.
6. Use Fermat's little theorem to find a number between 0 and 16 that is congruent to $3^{3000000}$ modulo 17.
7. State and outline the proof of Wilson's theorem.
8. Let p be an odd prime. Using Wilson's theorem, prove that

$$(p-2)! \equiv 1 \pmod{p}.$$

9. If f is a multiplicative function and F is defined by $F(n) = \sum_{d|n} f(d)$, then F is multiplicative. Prove this claim.
10. State and prove the Möbius inversion formula.
11. Let $\tau(n)$ be the number of divisors of n, and $\mu(n)$ the Möbius function. Prove that

$$\sum_{d|n} \mu(d)\tau\left(\frac{n}{d}\right) = 1.$$

12. Let $\sigma(n)$ the sum of divisors of n and $\mu(n)$ the Möbius function. Prove that

$$\sum_{d|n} \mu(d)\sigma\left(\frac{n}{d}\right) = n.$$

13. Explain why σ and τ are multiplicative functions.
14. Determine the exponent of the highest power of 7 appearing in the prime factorization of 140!.
15. Prove, by means of number theory that the binomial coefficient $\binom{n}{k}$ is an integer. (You are *not* allowed to use a combinatorial argument.)
16. Is the Euler function ϕ multiplicative? Justify your answer!
17. Calculate $\phi(1234)$.
18. Prove that the product of two multiplicative functions is multiplicative (We define the product $f \cdot g$ by $(f \cdot g)(n) := f(n) \cdot g(n)$.)
19. Which powers of $\rho = \cos\left(\frac{2\pi}{30}\right) + i \sin\left(\frac{2\pi}{30}\right)$ are primitive thirtieth (complex) roots of unity? Give a formula for the number of primitive n-th complex roots of unity. Justify the formula.
20. State and outline the proof of Euler's generalization of Fermat's little theorem.

Good luck. Gábor Hetyei