Sample Test II.

The real test will have less questions and you will have about 75 minutes to answer them. The usage of books or notes, or communicating with other students will not be allowed. You will have to give the simplest possible answer and show all your work. Below I am only listing questions related to the definitions, theorems, and proofs I expect you to know. There will be also application exercises, similar to the already discussed homework questions.

1. Write a recurrence formula for the Stirling numbers $S(n,k)$ of the second kind using only two smaller Stirling numbers. Prove the formula and use it to provide a table for $S(n,k)$ for $n \leq 6$.
2. Define the Bell numbers and express them in terms of the Stirling numbers of the second kind.
3. What is the number of onto functions from $[n]$ to $[m]$? Express your answer in terms of the Stirling numbers of the second kind. Justify your answer.
4. Explain why the Stirling numbers of the second kind satisfy the identity
$$S(n,k) = \sum_{j=0}^{n} \binom{n-1}{j} S(n-j-1, k-1) \quad \text{for } n \geq 1 \text{ and } k \geq 1.$$
5. Explain how the previous identity (or the idea in its proof) may be used to prove
$$B(n) = \sum_{j=0}^{n} \binom{n-1}{j} B(j).$$
6. Write a recurrence formula for the integer partition numbers $P(n,k)$ prove it, and provide a table for them for $n \leq 6$.
7. Draw the Ferrers diagram of the set partition whose type vector is $1^3 2^0 3^2$.
8. What kind of functions are counted by the partition number $P(n,k)$? (What is the size of the domain, the target, are the elements distinct or identical, are the functions one-to-one or onto?)
9. State and prove the inclusion-exclusion formula.
10. n persons attend a party. A fire breaks out in the building, while outside there is a heavy rain. Everybody rushes to the wardrobe, picks up an umbrella, and leaves. What is the probability that no one picked their own umbrella? Give an exact formula as a function of n and an approximate number for large n. Justify your answer.
11. Use the inclusion-exclusion formula to prove that the Stirling numbers of the second kind are given by the formula
$$S(n,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

12. Prove by induction that
$$1^2 + 2^2 + \cdots + k^2 = \frac{n(n+1)(2n+1)}{6}$$
holds for every nonnegative integer n.

13. Find $\binom{1/3}{5}$.

14. What is the coefficient of x^m in $\dfrac{1}{(1-x)^n}$? Justify your answer!

15. Write $1 + x + x^2 + \cdots + x^n$ and $\sum_{n=0}^{\infty} x^n$ in closed form.

16. State the convolution formula expressing the coefficient of x^k in $f(x) \cdot g(x)$ where $f(x) = \sum_{n\geq 0} a_n x^n$ and $g(x) = \sum_{n\geq 0} b_n x^n$.

17. Write the convolution formula for the exponential generating function for two sequences of numbers. Explain how this follows from the ordinary convolution formula.

18. There are 3 questions on a quiz, the first worth 3 points, the second 2 points, the third 4 points. Write the ordinary generating function for number a_n of ways to make n points on the quiz.

19. Give a closed formula for the sequence a_n given by $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 6a_n - 9a_{n-1}$.

20. Give a closed formula for the sequence a_n given by $a_0 = 5$, $a_1 = 16$, and $a_{n+1} = 4a_n - 4a_{n-1}$.

Good luck. Gábor Hetyei