Study Guide for the mandatory part of the final

This document may undergo substantial changes until our class on April 29. After that, only minor corrections may be possible

The actual final exam will have a mandatory and an optional section. The optional questions will be similar to the ones on the previous (sample) tests, and need to be answered only if you do not want me to re-use your average (unadjusted) test score. The list of questions below is supposed to help you prepare for the mandatory part of the final. Review also all homework questions assigned after our second test as questions similar to them might appear on the mandatory part of the final.

1. List the defining properties of an inner product. Indicate which of these properties are related to the bilinearity, and which are related to the notion of a distance.
2. Prove the Cauchy-Schwartz inequality for a real inner product space.
3. Prove the triangle inequality for a real inner product space.
4. Provide the definition of an orthogonal and an orthonormal basis.
5. How would you expand a vector v in an orthogonal basis $\{v_1, \ldots, v_n\}$? Prove your formula.
6. Prove that any finite dimensional subspace of an inner product space has an orthogonal complement.
7. Explain how Gram-Schmidt orthogonalization works.
8. Define the adjoint of a linear operator with respect to an inner product. Prove that your definition uniquely defines a linear operator.
9. Prove the following properties of adjoint operators: $T^{**} = T$ and $T^* + U^* = (T + U)^*$.
10. State and prove Schur's theorem on linear operators whose characteristic polynomial splits.
11. Define normal and self-adjoint operators. Give an example of a normal operator that is not self-adjoint.
12. Let T be a normal operator on a finite dimensional complex vector space V. Prove that V has an orthonormal basis consisting of eigenvectors of T.
13. Prove that a self-adjoint matrix has only real eigenvalues.
14. Let T be a self-adjoint operator on a real inner product space V. Prove that V has an orthonormal basis consisting only of the eigenvectors of T.
15. Prove that a complex $n \times n$ matrix is normal if and only if it is unitary equivalent to a diagonal matrix.

16. Define a T-cyclic subspace.

17. State and prove the Cayley-Hamilton Theorem.

18. Prove that every Euclidean domain is a principal ideal domain.

19. Which statement holds in all domains: "every irreducible is prime" or "every prime is irreducible"? Justify your answer.

20. State the structure theorem of finitely generated modules over principal ideal domains. Use this theorem to prove that every linear operator on a finite dimensional vector space has a Jordan normal form.

21. Let $T : V \to V$ be a linear operator on a complex vector space. Express the trace of T in terms of its eigenvalues. (Hint: use the Jordan normal form.)

Good luck. Gábor Hetyei