Sines and cosines in the Poincaré disk model of the hyperbolic plane

**Theorem 1** Assume that $ABC_\triangle$ is a right triangle, with its right angle at $C$, in the hyperbolic plane represented by the Poincaré disk model. Then

$$\sin(B) = \frac{\sinh(b)}{\sinh(c)} \quad \text{and} \quad \cos(A) = \frac{\tanh(b)}{\tanh(c)}.$$ 

**Proof:** Without loss of generality we may assume that $A$ is at the center of the Poincaré disk.

![Diagram](https://via.placeholder.com/150)

The lines $AB$ and $AC$ are represented by straight lines, the line $BC$ is represented by an arc of a circle $C_1$ centered at $O_1$. Let $B'$ resp. $C'$ be the second intersection of $OB$ resp $OC$ with this circle and $B_1$ be the orthogonal projection of $O$ to the line $OB$.

Since the Poincaré disk and the circle $C_1$ are orthogonal to each other, the power of $A = O$ with respect to $C_1$ is 1 (=the radius of the Poincaré disk). Hence the Euclidean distance $OB$ satisfies $OB \cdot OB' = 1$. We also know that the Euclidean distance $OB$ equals $\tanh(c/2)$. Thus

$$BB' = OB' - OB = 1/\tanh(c/2) - \tanh(c/2) = \frac{\cosh(c/2) - \sinh(c/2)}{\sinh(c/2) \cdot \cosh(c/2)} = \frac{2}{2 \cdot \sinh(c/2) \cdot \cosh(c/2)} = \frac{2}{\sinh(c)}$$

Similarly, since the Euclidean distance $OC$ equals $\tanh(b/2)$, we get $CC' = 2/\sinh(b)$. The angle of $ABC_\triangle$ at $B$ is the angle between the tangent of $C_1$ at $B$ and the line $OB$. Due to the Star Trek Lemma, this is the half of the central angle $\angle BO_1B'$, which is equal to $\angle BO_1B_1$. Hence $\sin(B)$ may be calculated from the right triangle $O_1B_1B_\triangle$, and we get

$$\sin(B) = \frac{BB_1}{O_1B} = \frac{BB'}{2O_1C} = \frac{BB'}{CC'} = \frac{\sinh(b)}{\sinh(c)}.$$ 

We may calculate $\cos(A)$ using $\cos(A) = AB_1/|AO_1|$. Here

$$AB_1 = OB + BB'/2 = \tanh(c/2) + 1/\sinh(c) = \frac{\sinh(c/2)}{\cosh(c/2)} + \frac{1}{2 \sinh(c/2) \cosh(c/2)}$$

$$= \frac{2 \sinh^2(c/2) + 1}{2 \sinh(c/2) \cosh(c/2)} = \frac{2 \sinh^2(c/2) + \cosh^2(c/2) - \sinh^2(c/2)}{\sinh(c)} = \frac{\cosh^2(c/2) + \sinh^2(c/2)}{\sinh(c)}$$

$$= \frac{1}{\sinh(c)}.$$
Similarly, $AO_1 = AC + CC'/2$ yields $AO_1 = 1/ \tanh(c)$ and so we obtain

$$\cos(A) = AB_1/AO_1 = \frac{\tanh(b)}{\tanh(c)}.$$ 

In analogy to the formulas for $\sin(B)$ and $\cos(A)$ we also have

$$\sin(A) = \frac{\sinh(a)}{\sinh(c)} \quad \text{and} \quad \cos(B) = \frac{\tanh(A)}{\tanh(c)}.$$ 

Since $1 = \sin^2(A) + \cos^2(A)$, we get

$$1 = \frac{\sinh^2(a)}{\sinh^2(c)} + \frac{\tanh^2(b)}{\tanh^2(c)} = \frac{\sinh^2(a) + \tanh^2(b) \cdot \cosh^2(c)}{\sinh^2(c)} = \frac{\sinh^2(a) \cosh^2(b) + \sinh^2(b) \cdot \cosh^2(c)}{\cosh^2(b) \sinh^2(c)}.$$ 

Multiplying both sides with $\cosh^2(b) \sinh^2(c)$ we get

$$\cosh^2(b) \sinh^2(c) = \sinh^2(a) \cosh^2(b) + \sinh^2(b) \cdot \cosh^2(c).$$ 

Using the identity $\sinh^2(x) = \cosh^2(x) - 1$ we may get rid of the hyperbolic sines and write

$$\cosh^2(b)(\cosh^2(c) - 1) = (\cosh^2(a) - 1) \cosh^2(b) + (\cosh^2(b) - 1) \cdot \cosh^2(c), \quad \text{i.e.,}$$

$$\cosh^2(b) \cosh^2(c) - \cosh^2(b) = \cosh^2(a) \cosh^2(b) - \cosh^2(b) + \cosh^2(b) \cosh^2(c) - \cosh^2(c).$$ 

Adding $\cosh^2(b) + \cosh^2(c) - \cosh^2(b) \cosh^2(c)$ yields

$$\cosh^2(c) = \cosh^2(a) \cosh^2(b).$$ 

Since the range of the hyperbolic cosine function is a subset of the positive real numbers, we may take the square root on both sides and get the hyperbolic Pythagorean theorem:

**Theorem 2** If $a, b, c$ are the sides of a hyperbolic right triangle, $c$ is the hypotenuse and the hyperbolic plane is the Poincaré disk model then

$$\cosh(c) = \cosh(a) \cosh(b).$$