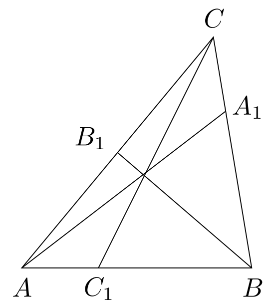
Ceva's theorem

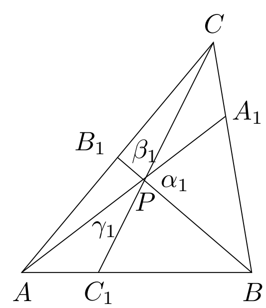
Let ABC be any triangle and choose a point A_1, B_1, C_1 on the line segments BC, AC, AB, respectively.



Theorem 1 (Ceva) *The lines AA_1, BB_1, and CC_1 are concurrent if and only if*

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

Proof: Assume first the three lines meet in the point P and use the notation shown in the picture below:



The triangles $AC_1P_\triangle$ and $C_1BP_\triangle$ have a common altitude at P, so the proportion of their areas is the proportion of the corresponding bases. Thus we may write

$$\frac{AC_1}{C_1B} = \frac{PA \cdot PC_1 \cdot \sin(\gamma_1)/2}{PB \cdot PC_1 \cdot \sin(\beta_1)/2} = \frac{PA \cdot \sin(\gamma_1)}{PB \cdot \sin(\beta_1)}.$$

Similarly we have

$$\frac{BA_1}{A_1C} = \frac{PB \cdot \sin(\alpha_1)}{PC \cdot \sin(\gamma_1)} \quad \text{and} \quad \frac{CB_1}{B_1A} = \frac{PC \cdot \sin(\beta_1)}{PA \cdot \sin(\alpha_1)}.$$

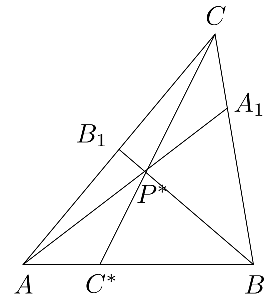
Multiplying the three fractions we get 1. For the converse, assume that A_1, B_1 and C_1 satisfy

$$\frac{AC_1}{C_1B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A} = 1.$$

We may rewrite this as

$$\frac{AC_1}{C_1B} = \frac{A_1C}{BA_1} \cdot \frac{B_1A}{CB_1}.$$

Define P^* as the intersection of AA_1 and BB_1 and let $C*$ be the intersection of CP^* with AB:



By the already shown implication of Ceva's theorem we have

$$\frac{AC^*}{C^*B} \cdot \frac{BA_1}{A_1C} \cdot \frac{CB_1}{B_1A},$$

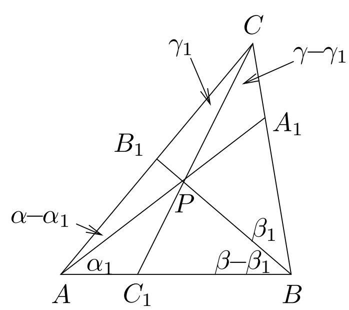
and so

$$\frac{AC^*}{C^*B} = \frac{A_1C}{BA_1} \cdot \frac{B_1A}{CB_1}.$$

We obtained that $C^* = C_1$ since they both subdivide AB into two segments of the same proportions. Therefore CC_1 also passes through P^*. ◇

The following equivalent form of Ceva's theorem is often useful.

Theorem 2 (Ceva) *Using the notation of the picture below*



the lines AA_1, BB_1, and CC_1 are concurrent if and only if

$$\sin(\alpha_1)\sin(\beta_1)\sin(\gamma_1) = \sin(\alpha - \alpha_1)\sin(\beta - \beta_1)\sin(\gamma - \gamma_1).$$

Proof: The triangles $AC_1C_\triangle$ and $C_1BC_\triangle$ have a common altitude at C, so the proportion of their areas is the proportion of the corresponding bases. Thus we may write

$$\frac{AC_1}{C_1B} = \frac{AC \cdot CC_1 \cdot \sin(\gamma_1)/2}{BC \cdot CC_1 \cdot \sin(\gamma - \gamma_1)/2} = \frac{AC \cdot \sin(\gamma_1)}{BC \cdot \sin(\gamma - \gamma_1)}.$$

Similarly we have

$$\frac{BA_1}{A_1C} = \frac{AB \cdot \sin(\alpha_1)}{AC \cdot \sin(\alpha - \alpha_1)} \quad \text{and} \quad \frac{CB_1}{B_1A} = \frac{BC \cdot \sin(\beta_1)}{AB \cdot \sin(\beta - \beta_1)}.$$

Multiplying the three equations we get that Ceva's condition is equivalent to

$$\frac{\cancel{AC} \cdot \sin(\gamma_1)}{\cancel{BC} \cdot \sin(\gamma - \gamma_1)} \cdot \frac{\cancel{AB} \cdot \sin(\alpha_1)}{\cancel{AC} \cdot \sin(\alpha - \alpha_1)} \cdot \frac{\cancel{BC} \cdot \sin(\beta_1)}{\cancel{AB} \cdot \sin(\beta - \beta_1)} = 1.$$

Multiplying both sides with $\sin(\alpha - \alpha_1)\sin(\beta - \beta_1)\sin(\gamma - \gamma_1)$ yields the statement. ◇