

Stirling numbers

1 Stirling numbers of the second kind

The Stirling numbers $S(m, n)$ of the *second kind* count the number of ways to partition an m -element set into n equivalence classes. As a consequence, the number of functions from an m -element set onto an n -element set (with distinct elements on both sides) is

$$S(m, n) \cdot n!$$

The numbers $S(m, n)$ satisfy the recursion formula

$$S(m, n) = S(m - 1, n - 1) + nS(m - 1, n) \quad (1)$$

Using this formula we may calculate a table for $S(m, n)$:

$m \backslash n$	0	1	2	3	4
0	1				
1	0	1			
2	0	1	1		
3	0	1	3	1	
4	0	1	7	6	1

Table 1: Stirling numbers of the second kind

Another way to compute the Stirling numbers of the second kind is to use the formula

$$S(m, n) = \frac{1}{n!} \sum_{i=0}^n (-1)^i \binom{n}{i} (n - i)^m \quad (2)$$

which is a consequence of the Principle of Inclusion and Exclusion applied to the number of onto functions from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$.

Counting all functions from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, x\}$ in two different ways yields

$$x^m = \sum_{j=0}^m S(m, j)(x)_j \quad \text{where } (x)_j = x(x - 1) \cdots (x - j + 1). \quad (3)$$

We may generalize (3) to arbitrary (non-integer) values of x since two polynomials are equal for every value of x if they agree for infinitely many values.

2 Stirling numbers of the first kind

The Stirling number $s(n, m)$ of the *first kind* is the coefficient of x^m in the polynomial $(x)_n = x(x-1)\cdots(x-n+1)$. In other words,

$$(x)_n = \sum_{m=0}^n s(n, m)x^m \quad (4)$$

It is easy to verify that the Stirling numbers of the second kind satisfy the recursion formula

$$s(m, n) = s(m-1, n-1) - (m-1)s(m-1, n). \quad (5)$$

Using this formula we may calculate a table for $s(m, n)$:

$m \backslash n$	0	1	2	3	4
0	1				
1	0	1			
2	0	-1	1		
3	0	2	-3	1	
4	0	-6	11	-6	1

Table 2: Stirling numbers of the first kind

The absolute value of $s(m, n)$ is often denoted by $c(m, n)$, since it is the number of permutations on m letters with exactly n cycles. This may be shown by induction using the recursion formula (5) and

$$c(m, n) = c(m-1, n-1) + (m-1)c(m-1, n)$$

3 Relation between the Stirling numbers of both kinds

Equations (3) and (4) indicate that the matrix of Stirling numbers of the first (second) kind allows to express the falling factorials of x in the basis of powers of x (and vice versa). Hence if we take the first k columns and first k rows for any fixed k of Tables 1 and 2, the resulting matrices are inverses of each other. In other words

$$\sum_{m=0}^{\min(m,n)} s(m, k)S(k, n) = \delta_{m,n} \quad (6)$$

and

$$\sum_{m=0}^{\min(m,n)} S(m, k)s(k, n) = \delta_{m,n} \quad (7)$$

Here $\delta_{m,n}$ is the *Kronecker-delta* whose value is 1 if $m = n$ and 0 otherwise.