The dual to the transportation problem

The transportation problem may be phrased as follows. Suppose we have m warehouses with a supply vector $(s_1, \ldots, s_m)$ and n stores with a demand vector $(d_1, \ldots, d_n)$. We may assume that

$$s_1 + s_2 + \cdots + s_m = d_1 + d_2 + \cdots + d_n.$$

(If not, we add a dummy warehouse or a dummy store.) We also assume that we are given a unit transportation cost $c_{i,j}$ between warehouse i and store j. The *primal* transportation problem is the following: minimize $\sum_{i,j} c_{i,j} x_{i,j}$, subject to

$$\sum_{j=1}^{n} x_{i,j} = s_i \quad \text{for } i = 1, 2, \ldots, m \text{ and}$$

$$\sum_{i=1}^{n} x_{i,j} = d_j \quad \text{for } j = 1, 2, \ldots, n.$$

In other word, we want to find a flow from the warehouses to the stores, minimizing transportation costs. The *dual* transportation problem is the following. Maximize $\sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{m} s_i u_i$ subject to

$$c_{i,j} \geq v_j - u_i \quad \text{for all } i, j.$$

In other words, we want to maximize profit for a transportation company that offers buying price u_i at warehouse i and selling price v_j at store j, such that the price difference $v_j - u_i$ does not exceed the actual transportation cost along the edge.

Lemma 1 *For any solution $(x_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n)$ of the primal problem, and any solution of $(u_1, \ldots, u_m; v_1, \ldots, v_n)$ of the dual problem we have*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} \geq \sum_{j=1}^{m} v_j d_j - \sum_{i=1}^{n} u_i s_i.$$

In other words, any solution to the primal problem is greater than equal to any solution of the dual problem.

Proof: We have

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} \geq \sum_{i=1}^{m} \sum_{j=1}^{n} (v_j - u_i) x_{i,j} = \sum_{j=1}^{m} v_j \sum_{i=1}^{n} x_{i,j} - \sum_{i=1}^{n} u_i \sum_{j=1}^{m} x_{i,j} = \sum_{j=1}^{m} v_j d_j - \sum_{i=1}^{n} u_i s_i.$$

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In Tucker's "Applied Combinatorics", 6th Ed we first find a spanning tree solution to the primal problem, using the NW corner rule. The corresponding system of prices $(u_1, \ldots, u_m; v_1, \ldots, v_n)$ satisfies $c_{i,j} = (v_j - u_i)$ for the edges in the spanning tree but we may have $c_{i,j} < (v_j - u_i)$ for some edge not in the spanning tree. We change the tree we change the flow, repeat until $c_{i,j} \geq (v_j - u_i)$ holds for all edges. At this point we have a valid solution for the dual problem and

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{i,j} x_{i,j} = \sum_{j=1}^{m} v_j d_j - \sum_{i=1}^{n} u_i s_i,$$

since $c_{i,j} = v_j - u_i$ for edges in the spanning tree and these are the edges where $x_{i,j}$ may be positive.