Network flows

A *network* is a directed graph $G = (V, E)$ with a pair (s, t) of distinguished vertices and a positive real number $k(e)$ associated to each edge e. The vertex s is the *source*, the vertex t is the *sink* and the number $k(e)$ is the *capacity* of the directed edge e. A *flow* is a function $f : E \to \mathbb{R}$ satisfying the following conditions:

1. $0 \leq f(e) \leq k(e)$ holds for all $e \in E$. (The flow is subject to the capacity constraints.)
2. Introducing $\mathrm{In}(v)$ and $\mathrm{Out}(v)$ for the set of edges ending in, respectively starting at v,
$$\sum_{e \in \mathrm{In}(v)} f(e) = \sum_{e \in \mathrm{Out}(v)} f(e)$$
holds for all $v \in V \setminus \{s, t\}$. (For any vertex that is not the source or the sink, what flows into the vertex is what flows out of the vertex.)

An *$s - t$ cut* (S, T) is an ordered set partition of the vertex set V into two parts, such that $s \in S$ and $t \in T$. The *net flow from S to T* is
$$f(S, T) := \sum_{u \in S, v \in T} \sum_{e \in \mathrm{Out}(u) \cap \mathrm{In}(v)} f(e) - \sum_{v \in S, u \in T} \sum_{e \in \mathrm{Out}(u) \cap \mathrm{In}(v)} f(e).$$
To simplify writing sums we introduce the notation
$$\overrightarrow{f}(X, Y) := \sum_{u \in X, v \in Y} \sum_{e \in \mathrm{Out}(u) \cap \mathrm{In}(v)} f(e)$$
for any pair (X, Y) of disjoint sets of vertices. We may rewrite $f(S, T)$ as $\overrightarrow{f}(S, T) - \overrightarrow{f}(T, S)$.

Proposition 1 *For a fixed flow $f : E \to \mathbb{R}$ the value of $f(S, T)$ is the same for all $s - t$ cuts (S, T). In particular, it is the same as $f(\{s\}, V \setminus \{s\})$ (the net flow from the source) and the same as $f(V \setminus \{t\}, \{t\})$ (the flow into the sink).*

Proof: Let us fix an $s - t$ cut (S, T). Summing the second flow condition for all $v \in S - \{s\}$ we obtain
$$\sum_{v \in S - \{s\}} \sum_{e \in \mathrm{Out}(v)} f(e) = \sum_{v \in S - \{s\}} \sum_{e \in \mathrm{In}(v)} f(e).$$
Observe that $f(e)$ is counted on both sides if both ends of e belong to $S \setminus \{s\}$. Subtracting all such values, we obtain
$$\overrightarrow{f}(S \setminus \{s\}, T) + \overrightarrow{f}(S \setminus \{s\}, \{s\}) = \overrightarrow{f}(T, S \setminus \{s\}) + \overrightarrow{f}(\{s\}, S \setminus \{s\}).$$
Adding $\overrightarrow{f}(\{s\}, T)$ to both sides and simplifying yields
$$\overrightarrow{f}(S, T) + \overrightarrow{f}(S \setminus \{s\}, \{s\}) = \overrightarrow{f}(T, S \setminus \{s\}) + \overrightarrow{f}(\{s\}, V \setminus \{s\}).$$
Adding $\overrightarrow{f}(T, \{s\})$ to both sides yields
$$\overrightarrow{f}(S, T) + \overrightarrow{f}(V \setminus \{s\}, \{s\}) = \overrightarrow{f}(T, S) + \overrightarrow{f}(\{s\}, V \setminus \{s\}).$$
Rearranging yields
$$\overrightarrow{f}(S, T) - \overrightarrow{f}(T, S) = \overrightarrow{f}(\{s\}, V \setminus \{s\}) - \overrightarrow{f}(V \setminus \{s\}, \{s\}).$$

The left hand side is $f(S,T)$, the right hand side is $f(\{s\}, V \setminus \{s\})$. $\diamond$

We call the common value of $f(S,T)$ for all s-t cuts the *(net) total flow.*

The *capacity* $k(S,T)$ of an s-t cut is defined by

$$k(S,T) = \sum_{u \in S, v \in T} k(u,v).$$

Lemma 1 *For any s-t cut (S,T) we have $f(S,T) \le k(S,T)$*

This is obvious, the value of $\overrightarrow{f}(S,T)$ is at most $k(S,T)$, and the value of $-\overrightarrow{f}(T,S)$ is at most 0.

Theorem 1 (Ford-Fulkerson) *For any network, the maximum flow value is equal to the minimum s-t cut capacity.*

Proof: We can think of a flow as an $|E|$-dimensional vector. The flow value is a linear function of the input coordinates, hence it is continuous. The flow conditions defined a closed bounded domain, hence the flow value does have a maximum on this domain, there is a maximum flow.

For any edge $e \in \mathrm{Out}(\{u\}) \cap \mathrm{In}(\{v\})$, let us introduce a new edge $e^* \in \mathrm{Out}(\{v\}) \cap \mathrm{In}(\{u\})$. We denote the set of new (reversed) edges by E^*.

Define the *slack* $s(e)$ as $k(e) - f(e)$ for each $e \in E$ and the slack $s(e^*)$ as $f(e)$ for each $e^* \in E^*$. An *augmenting path* is a directed path $s = u_0 \to u_1 \to \cdots \to u_{m-1} \to u_m$ such that for each (u_i, u_{i+1}) there is an edge in $E \cup E^*$ from u_i to u_{i+1} whose slack is positive. If there is an augmenting path from s to t we may increase the flow value by a small $\varepsilon > 0$ as follows: for each i if there is an $e \in E$ from u_i to u_{i+1}, we increase $f(e)$ by ε, and if there is an $e^* \in E^*$ from u_i to u_{i+1} then we decrease $f(e)$ by ε. The value of the resulting flow increases by ε.

Hence, for a maximum flow, there is no augmenting path from s to t. Define the s-t cut (S,T) by setting S as the set of vertices that are reachable from s with an augmenting path. (The set T is its complement and it contains the sink). By definition the slack of all edges (e or e^*) starting in S and ending in T is zero: we have $k(S,T) - \overrightarrow{f}(S,T) = 0$ (for the edges in E) and $\overrightarrow{f}(T,S) = 0$ (for the edges in E^*). Therefore $f(S,T) = k(S,T)$. $\diamond$