Sample Final Exam questions

The final exam will be cumulative and will have about 10 questions. This study guide gives sample questions on the material covered since Test 3. For questions on the material covered before Test 3, please refer to the study guides provided before the tests.

1. What is the amount of time needed for an investment to double, if it is invested at the rate of 5%, compounded monthly?
2. Find the accumulated amount if 200 dollars are invested for 5 years at the rate of 3%, compounded continuously.
3. Find the derivative of $f(x) = 3^x$. (Such a question may be part of a more complicated question.)
4. Find the antiderivative of $f(x) = 3x^5 - \frac{1}{x}$.
5. Solve the differential equation $f'(x) = 3\sqrt{x} - (x+1)^2$, subject to the initial condition $f(1) = \frac{5}{3}$.
6. Using integration by substitution, find the antiderivative of $\dfrac{1}{\sqrt{2x+3}}$.
7. Using integration by substitution, find the antiderivative of $3 \cdot x \cdot e^{x^2-1}$.
8. Find the definite integral $\displaystyle\int_1^2 (x^2 - 1)\, dx$.
9. Find the average value of the function $f(x) = e^x - \dfrac{1}{x}$ on the interval $[1, 3]$.
10. Annual sales (in millions of units) of a certain product are expected to grow according to the function $f(t) = 100 - (t - 10)^2$ per year, where t is measured in years. How many units will be sold in the next 20 years?
11. Find the area bounded by the curves $f(x) = \frac{x+1}{2}$ and $g(x) = x^2$.
12. An investor is investing a $5e^{0.05t}$ thousand dollars/year at time t for the next 15 years into an account returning 6% interest, compounded continuously. What is the future value of the investment in 15 years? Use the formula $e^{rT} \int_0^T R(t)e^{-rt}dt$
13. The demand function of a product is given by $D(x) = 6,000,000 - 5,000 \cdot x$, the supply function is $S(x) = x^2$ where x is the number of items sold. Find the market equilibrium quantity $\overline{x}$, the market equilibrium price $\overline{p}$, the consumers' surplus and the producers' surplus.

Solutions:

1. We use the formula
$$A = P \cdot \left(1 + \frac{r}{m}\right)^{m \cdot t}$$
with $A = 2P$, $r = 0.05$ and $m = 12$. After simplifying by P we get
$$2 = \left(1 + \frac{0.05}{12}\right)^{12 \cdot t}$$
Taking the natural logarithm on both sides, we get
$$\ln(2) = 12 \cdot t \cdot \ln\left(1 + \frac{0.05}{12}\right).$$
Thus
$$t = \frac{\ln(2)}{12 \cdot \ln\left(1 + \frac{0.05}{12}\right)} \approx 13.89 \quad \text{(years).}$$

2. We use the formula
$$A = P \cdot e^{r \cdot t}$$
with $P = 200$, $r = 0.03$ and $t = 5$ and get
$$A = 200 \cdot e^{0.03 \cdot 5} = 200 \cdot e^{0.15} \approx 232.37.$$

3. Either we remember that the derivative of a^x is $\ln(a) \cdot a^x$, or we use the chain rule to find the derivative of $3^x = e^{\ln(3) \cdot x}$ with $f(u) = e^u$ and $u = \ln(3) \cdot x$, so $\frac{du}{dx} = \ln(3)$. Either way, the answer is $\ln(3) \cdot 3^x$.

4. Rules to remember: the antiderivative of x^m is $\frac{x^{m+1}}{m+1}$ when $m \neq 1$, and the antiderivative of $\frac{1}{x}$ is $\ln(x)$. Hence
$$\int \left(3x^5 - \frac{1}{x}\right) \, dx = 3 \cdot \frac{x^6}{6} - \ln(x) + C = \frac{x^6}{2} - \ln(x) + C$$
(Don't forget the $+C$ at the end!)

5. We can rewrite $3\sqrt{x} - (x+1)^2$ as $3\sqrt{x} - x^2 - 2x - 1$, and integrate term by term:
$$\int (3\sqrt{x} - x^2 - 2x - 1) \, dx = 3 \cdot \frac{2x^{3/2}}{3} - \frac{x^3}{3} - x^2 - x + C = 2x^{3/2} - \frac{x^3}{3} - x^2 - x + C$$

To find C we substitute $x = 1$:

$$2 - \frac{1}{3} - 1 - 1 + C = \frac{5}{3},$$

which gives $C = 2$. Thus the final answer is

$$f(x) = 2x^{3/2} - \frac{x^3}{3} - x^2 - x + 2.$$

6. We use the substitution $u = 2x + 3$, then $\frac{du}{dx} = 2$. Hence

$$\int \frac{1}{\sqrt{2x+3}}\, dx = \int \frac{1}{\sqrt{u}} \frac{1}{2} \frac{du}{dx} \cdot du = \int \frac{1}{\sqrt{u}} \frac{1}{2}\, du = \frac{1}{2} \cdot \frac{\sqrt{u}}{2} + C = \sqrt{2x+3} + C.$$

7. We use the substitution $u = x^2 - 1$, then $\frac{du}{dx} = 2x$. Hence

$$\int 3 \cdot x \cdot e^{x^2-1}\, dx = \int \frac{3}{2} \cdot e^u \cdot \frac{du}{dx} du = \int \frac{3}{2} \cdot e^u\, du = \frac{3}{2} e^u + C = \frac{3e^{x^2-1}}{2} + C.$$

8. Using the fundamental theorem of calculus

$$\int_1^2 (x^2 - 1)\, dx = \left(\frac{x^3}{3} - x \right) \Big|_1^2 = \frac{8}{3} - 2 - \left(\frac{1}{3} - 1 \right) = \frac{4}{3}.$$

9. The average value of a function $f(x)$ on the interval $[a, b]$ is $\dfrac{1}{b-a} \displaystyle\int_a^b f(x)\, dx$. Hence the average value is

$$\frac{1}{3-1} \int_1^3 \left(e^x - \frac{1}{x} \right) dx = \frac{1}{2} \cdot (e^x - \ln(x)) \Big|_1^3 = \frac{(e^3 - \ln(3)) - (e^1 - \ln(1))}{2} = \frac{e^3 - e^1 - \ln(3)}{2}.$$

10. We need to evaluate $\int_0^{20} (100 - (t-10)^2)\, dt$. If you are not afraid of substitution, use $u = t - 10$ (so $du/dt = 1$), and get that the answer is

$$\int_{-10}^{10} 100 - u^2\, du = \left(100 \cdot u - \frac{u^3}{3} \right) \Big|_{-10}^{10} = 2 \cdot \left(1{,}000 - \frac{1{,}000}{3} \right) \approx 1{,}333.$$

(In the second step we used that $100 \cdot u - \frac{u^3}{3}$ is an odd function.) If you want to play it safe, expand $(t-10)^2$, get $(t-10)^2 = 100 - 20 \cdot t + t^2$. Thus the integral is

$$\int_0^{20} (100 - (t-10)^2)\, dt = \int_0^{20} (20 \cdot t - t^2)\, dt = \left(20 \cdot \frac{t^2}{2} - \frac{t^3}{3} \right) \Big|_0^{20} \approx 1{,}333.$$

11. The x-coordinates of the intersection are the solutions of the equation $x^2 = \frac{x+1}{2}$, that is, $x^2 - \frac{x}{2} - \frac{1}{2} = 0$. The quadratic formula gives

$$x = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2}}{2} = \frac{\frac{1}{2} \pm \frac{3}{2}}{2} = -\frac{1}{2}, 1.$$

On the interval $\left[-\frac{1}{2}, 1\right]$ we have $g(x) \geq f(x)$. Hence the area is

$$\int_{-\frac{1}{2}}^{1} \left(\frac{x+1}{2} - x^2\right)\, dx = \left(\frac{x^2}{4} + \frac{x}{2} - \frac{x^3}{3}\right)\Big|_{-\frac{1}{2}}^{1} = \left(\frac{1}{4} + \frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{16} - \frac{1}{4} + \frac{1}{24}\right)$$
$$= \frac{5}{12} + \frac{7}{48} = \frac{27}{48} = \frac{9}{16}.$$

12. We will use the formula with $R(t) = 5e^{0.05t}$, $r = 0.06$ and $T = 15$. We get

$$e^{0.06\cdot 15}\int_0^{15} 5 \cdot e^{0.05t} \cdot e^{-0.06t}\, dt = e^{0.9}\int_0^{15} 5 \cdot e^{-0.01t}\, dt = 5 \cdot e^{0.9} \cdot \frac{e^{-0.01t}}{-0.01}\Big|_0^{15} = 500 \cdot e^{0.9}(1 - e^{-0.15})$$
$$\approx 171.30$$

13. To find the market equilibrium amount, we solve the equation $x^2 = 6,000,000 - 5000x$, which has the negative solution $x = -6,000$ and the positive solution $x = 1,000$. The positive solution is the market equilibrium quantity. The market equilibrium price is $\overline{p} = \overline{x}^2 = 1,000,000$. The consumer's surplus is

$$CS = \int_0^{\overline{x}} D(x)\, dx - \overline{p} \cdot \overline{x} = \int_0^{1,000} 6,000,000 - 5,000 \cdot x\, dx - 1,000,000 \cdot 1000$$
$$= \left(6,000,000 \cdot x - 5,000 \cdot \frac{x^2}{2}\right)\Big|_0^{1,000} - 10^9 = 2,500,000,000,$$

and the supplier's surplus is

$$DS = \overline{p} \cdot \overline{x}\int_0^{\overline{x}} S(x)\, dx - = 1,000,000 \cdot 1000 \int_0^{1,000} x^2\, dx$$
$$= 10^9 - \frac{x^3}{3}\Big|_0^{1,000} = 666,666,666.66.$$