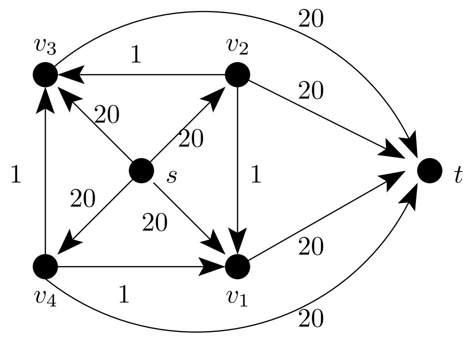
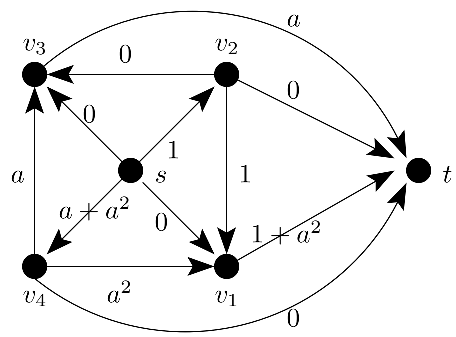
“Complete failure” of the Ford-Fulkerson algorithm

Let a be the only positive root of the equation $x^3 + x - 1 = 0$. (There is only one positive root, according to Descartes' rule of signs.) Since $\left(\frac{1}{2}\right)^3 + \frac{1}{2} - 1 < 0$ and $1^3 + 1 - 1 > 0$, by the intermediate value theorem, a must be strictly between $\frac{1}{2}$ and 1.

Consider the graph with the capacities and initial flow below:

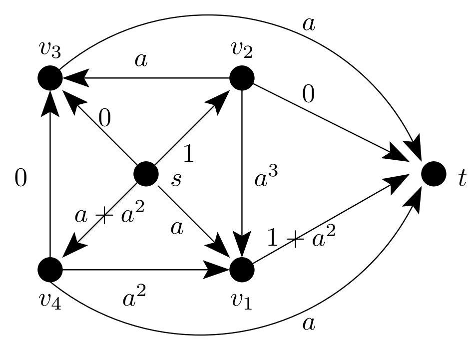


Capacities

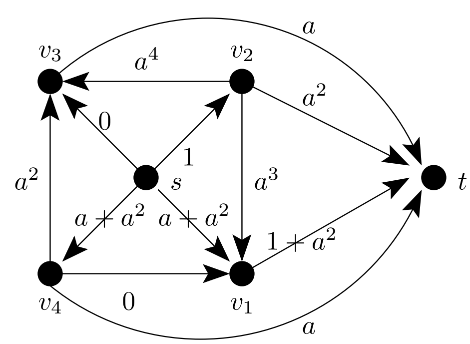


First flow

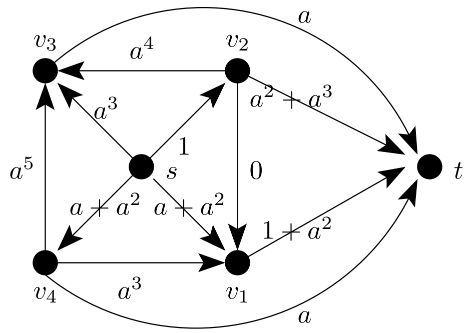
Using the correcting path $s \to v_1 \to v_2 \to v_3 \to v_4 \to t$ we can increase this flow by at most a, and get the following second flow:



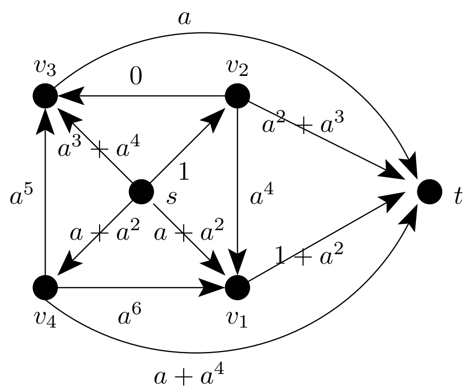
We used $1 - a = a^3$ to simplify the value on the edge $v_2 \to v_1$. Using the correcting path $s \to v_1 \to v_4 \to v_3 \to v_2 \to t$ we can increase this flow by at most a^2 and get the following third flow:



Here we used $a - a^2 = a(1 - a) = a \cdot a^3 = a^4$ to simplify the value on the edge $v_2 \to v_3$. Let us use now the correcting path $s \to v_3 \to v_4 \to v_1 \to v_2 \to t$, which allows to increase the flow by at most a^3, and get the following fourth flow:



Here we used $a^2 - a^3 = a^2(1-a) = a^2 \cdot a^3 = a^5$ to simplify the value on the edge $v_4 \to v_3$. Let us use finally the correcting path $s \to v_3 \to v_2 \to v_1 \to v_4 \to t$, which allows to increase the flow by at most a^4, and get the fifth flow:



The value on the edge $v_4 \to v_1$ was simplified using $a^3 - a^4 = a^3(1-a) = a^3 \cdot a^3 = a^6$.

Compare now the first and the fifth flow. Note that for those edges of the graph that go between two vertices from $\{v_1, v_2, v_3, v_4\}$ the values in the fifth flow may be obtained by multiplying the corresponding value in the first flow by a^4. Observe furthermore, that for each correcting path above the increase was limited by a power of a written on an edge with both ends in $\{v_1, v_2, v_3, v_4\}$. If we repeat the same sequence of correcting paths one more times, we can increase the flow by $a^4(a + a^2 + a^3 + a^4)$ and end up with a flow that is "similar" to the first flow again, except for the edges connecting two vertices from $\{v_1, v_2, v_3, v_4\}$ the numbers from the first flow have to be multiplied by a^4. Repeating the same sequence of correcting paths infinitely, the flow value converges to

$$(1 + a + a^2) + (a + a^2 + a^3 + a^4) + (a^5 + a^6 + a^7 + a^8) + \cdots = a + a^2 + \frac{1}{1-a}$$

Since $1 - a = a^3$, the flow value converges to $a + a^2 + a^{-3} < 1 + 1 + 2^3 = 10$ (by $\frac{1}{2} < a < 1$). This is definitely less then the value of the maximum flow which is obviously $4 \cdot 20 = 80$.