

A remarkable example of the transportation problem
from Section 4.5 in Tucker's "Applied Combinatorics", 5th Ed.

Assume your current spanning tree solution is

		1	2	Dummy
1	4	2	0	
2	6	7	0	
3	5	6	0	
	60	0	20	
	10			

Let us work out the corresponding set of prices, starting with an arbitrary price $u_1 = \$10$.

$$\begin{array}{ll}
 \text{warehouse 1: } & u_1 = \$10 \quad \text{store 1: } \quad v_1 = \$11 \\
 \text{warehouse 2: } & u_2 = \$5 \quad \text{store 2: } \quad v_2 = \$12 \\
 \text{warehouse 3: } & u_3 = \$6 \quad \text{dummy: } \quad v_3 = \$6
 \end{array}$$

$$\begin{array}{ll}
 \text{edge (1, 1): } & c_{11} = 4 > v_1 - u_1 = \$11 - \$10 = \$1 \text{ increase of } \$3 \\
 \text{edge (1, 3): } & c_{13} = 0 > v_3 - u_1 = \$6 - \$10 = -\$4 \text{ increase of } \$4 \\
 \text{edge (2, 3): } & c_{23} = 0 < v_3 - u_2 = \$6 - \$5 = \$1 \text{ decrease of } \$1 \\
 \text{edge (3, 1): } & c_{31} = 5 = v_1 - u_3 = \$11 - \$6 = \$5 \text{ no change}
 \end{array}$$

Thus the solution may be improved by adding the edge (2,3) to the current spanning tree, whose edge set is $\{(1, 2), (2, 1), (2, 2), (3, 2), (3, 3)\}$. Adding the edge (2,3) to the tree creates the circuit $\{(2, 2), (2, 3), (3, 2), (3, 3)\}$. Since nothing is transported along the edge (2,2), we can not change the solution of the transportation problem. But we can change the associated spanning tree to $\{(1, 2), (2, 1), (2, 3), (3, 2), (3, 3)\}$ and this will improve our set of selling prices. Note that both $\{(1, 2), (2, 1), (2, 2), (3, 2), (3, 3)\}$ and $\{(1, 2), (2, 1), (2, 3), (3, 2), (3, 3)\}$ have an edge with 0 flow in it, present only to make sure our solution is associated to a spanning tree. We changed this "superfluous" edge. Our solution is now represented by the following table.

		1	2	Dummy
1	4	2	0	
2	6	7	0	
3	5	6	0	
	60	0	20	
	10			

Now we work out a new set of prices, starting with an arbitrary price $u_1 = \$10$.

$$\begin{array}{ll}
 \text{warehouse 1: } & u_1 = \$10 \quad \text{store 1: } \quad v_1 = \$12 \\
 \text{warehouse 2: } & u_2 = \$6 \quad \text{store 2: } \quad v_2 = \$12 \\
 \text{warehouse 3: } & u_3 = \$6 \quad \text{dummy: } \quad v_3 = \$6
 \end{array}$$

Again we look for a better solution:

- edge (1, 1): $c_{11} = 4 > v_1 - u_1 = \$12 - \$10 = \2 increase of \$2
- edge (1, 3): $c_{13} = 0 > v_3 - u_1 = \$6 - \$10 = -\4 increase of \$4
- edge (2, 2): $c_{22} = 7 > v_2 - u_2 = \$12 - \$6 = \6 increase of \$1
- edge (3, 1): $c_{31} = 5 < v_1 - u_3 = \$12 - \$6 = \6 decrease of \$1

The solution may be improved by adding the edge (3, 1) to the spanning tree $\{(1, 2), (2, 1), (2, 3), (3, 2), (3, 3)\}$. This move creates the circuit $\{(2, 1), (3, 1), (3, 3), (2, 3)\}$. The lowest even edge has 20 on it, this is the amount by which we can change the flow in the edges. The new solution is shown in the following table.

	1	2	Dummy
1	4	2	0
2	6	7	0
3	5	6	0

Note that our flow decreased to zero in the edge (3, 3) and we remove this edge to have a spanning tree again. Our new spanning tree is $\{(1, 2), (2, 1), (2, 3), (3, 1), (3, 2)\}$. We use this tree to work out a new set of prices, starting with an arbitrary price $u_1 = \$10$.

- warehouse 1: $u_1 = \$10$ store 1: $v_1 = \$11$
- warehouse 2: $u_2 = \$5$ store 2: $v_2 = \$12$
- warehouse 3: $u_3 = \$6$ dummy: $v_3 = \$5$

Again we look for a better solution:

- edge (1, 1): $c_{11} = 4 > v_1 - u_1 = \$11 - \$10 = \1 increase of \$3
- edge (1, 3): $c_{13} = 0 > v_3 - u_1 = \$5 - \$10 = -\5 increase of \$5
- edge (2, 2): $c_{22} = 7 = v_2 - u_2 = \$12 - \$5 = \7 no change
- edge (3, 3): $c_{33} = 0 > v_3 - u_3 = \$5 - \$6 = -\1 increase of \$1

Now we really have an optimal solution. The cost of this solution is

$$40 \times \$2 + 40 \times \$6 + 20 \times \$0 + 20 \times \$5 + 10 \times \$6 = \$480,$$

\$20 less expensive than the one we started with.