Assignment 4

Mandatory questions to be answered orally

1. Consider again the following relation on the set of ordered pairs of positive integers: $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 + y_2 = x_2 + y_1$. Recall, that the sum of two equivalence classes was defined by

$$(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 + y_2),$$

and multiplication was defined by

$$(x_1, x_2) \cdot (y_1, y_2) := (x_1y_1 + x_2y_2, x_1y_2 + x_2y_1).$$

Prove that multiplication is distributive with respect to addition.

2. For the same ordered pairs, set $(x_1, x_2) < (y_1, y_2)$ if $x_1 + y_2 < y_1 + x_2$. Prove that the relation $<$ is compatible with the equivalence relation defined in the previous exercise. That is, show that $(x_1, x_2) \sim (x_1', x_2')$, $(y_1, y_2) \sim (y_1', y_2')$, and $(x_1, x_2) < (y_1, y_2)$ imply $(x_1', x_2') < (y_1', y_2')$.

3. Given a positive integer x, we denote the equivalence class $\{(x + t, t) \mid t \in \mathbb{P}\}$ with A_x, the equivalence class $\{(t, x+t) \mid t \in \mathbb{P}\}$ with B_x. We also denote the equivalence class $\{(t, t) \mid t \in \mathbb{P}\}$ with 0. We know that every equivalence class satisfies exactly one of the following: it is either 0, or of the form A_x or of the form B_x. Identify each positive integer x with the equivalence class A_x with the set of positive integers, and introduce $-x$ as the equivalence class B_x. Given positive integers x and y, define $y - x$ as the equivalence class $A_y + B_x$. Prove that $y > x$ if and only if $A_y + B_x$ is the equivalence class of a positive integer.

4. Prove that the equivalence class of (x_1, x_2) is a class of the form A_x if and only if $(x_1, x_2) > (t, t)$ for all t. (In other words, "x is positive if and only if $x > 0$".)

5. Call the equivalence classes of the form B_x "negative integers", and identify each positive integer x with the equivalence class A_x. Prove that multiplying two negative integers yields a positive integer.

6. Prove that if $(x_1, x_2) < (y_1, y_2)$ and (z_1, z_2) is in the equivalence class of a positive integer then we have $(x_1, x_2) \cdot (z_1, z_2) < (y_1, y_2) \cdot (z_1, z_2)$.

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Mandatory question to be answered in writing

1. Define subtraction by the formula

$$(x_1, x_2) - (y_1, y_2) = (x_1 + y_2, x_2 + y_1).$$

Prove that $(x_1, x_2) - (x_1, x_2)$ belongs to the equivalence class of zero, and that $((x_1, x_2) - (y_1, y_2)) \cdot (z_1, z_2)$ is equivalent to $(x_1, x_2) \cdot (z_1, z_2) - (y_1, y_2) \cdot (z_1, z_2)$. (You may assume that subtraction is compatible with the equivalence relation.)

Bonus question

1. Consider the "integers modulo 4", that is, the set $\{1, 2, 3, 4\}$ with the successor relation $1' = 2$, $2' = 3$, $3' = 4$ and $4' = 1$. Convince yourself that addition and multiplication may be defined similarly to addition and multiplication of positive integers, as introduced in our textbook. (I do not need to see this in writing.) Consider ordered pairs (x_1, x_2) from this set and set again $(x_1, x_2) \sim (y_1, y_2)$ if $x_1 + y_2 = x_2 + y_1$. Look at again the 6 statements in the oral exercises of Assignment 3. Which would be *not* true any more in this model? Explain only why the false statement(s) do not hold any more, and what would be true instead of them.