ON THE DISTRIBUTION OF ZEROS OF DERIVATIVES OF THE RIEMANN ξ-FUNCTION

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Abstract. For the completed Riemann zeta function $\xi(s)$, it is known that the Riemann hypothesis for $\xi(s)$ implies the Riemann hypothesis for $\xi^{(m)}(s)$, where m is any positive integer. In this paper, we investigate the distribution of the fractional parts of the sequence $(\alpha\gamma)$, where α is any fixed non-zero real number and γ runs over the imaginary parts of the zeros of $\xi^{(m)}(s)$. We also obtain a zero density estimate and an explicit formula for the zeros of $\xi^{(m)}(s)$. In particular, all our results hold uniformly for $0 \le m \le g(T)$, where the function $g(T)$ tends to infinity with T and $g(T) = o(\log\log T)$.

1. Introduction

The Riemann ξ-function is defined by

$$\xi(s) = H(s)\zeta(s), \tag{1.1}$$

where

$$H(s) := \frac{s}{2}(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \tag{1.2}$$

and $\zeta(s)$ denotes the Riemann zeta function. The non-trivial zeros of $\zeta(s)$ are identical to the zeros of $\xi(s)$. It is well known that the real parts of the zeros of $\xi(s)$ lie in the critical strip $0 < \operatorname{Re} s < 1$. The Riemann hypothesis for $\xi(s)$ states that these zeros lie on the critical line $\operatorname{Re} s = 1/2$. Moreover, the Riemann hypothesis for $\xi(s)$ implies that the zeros of $\xi^{(m)}(s)$ also lie on the critical line $\operatorname{Re} s = 1/2$. In 1983, Conrey [5] showed that for $m \ge 0$, the real parts of the zeros of $\xi^{(m)}(s)$ also lie in the critical strip $0 < \operatorname{Re} s < 1$.

There has also been a great interest in studying the vertical distribution of the zeros of $\xi(s)$. Under the assumption of the Riemann hypothesis, Rademacher [29] first proved that the sequence $(\alpha\gamma_1)$, where γ_1 denotes the imaginary part of a non-trivial zero of $\zeta(s)$ and α is any fixed non-zero real number, is uniformly distributed modulo one. Hlawka [18] proved this result unconditionally.

Let $\{x\}$ denote the fractional part of a real number x. Let $\rho_1 = \beta_1 + i\gamma_1$ denote a non-trivial zero of $\zeta(s)$. The discrepancy of the set $\{\{\alpha\gamma_1\} : 0 < \gamma_1 \le T\}$ is defined by

$$D^*_\alpha(T) := \sup_{0\le y\le 1}\left|\frac{\#\{0 \le \gamma_1 \le T;\ 0 \le \{\alpha\gamma_1\} < y\}}{N(T)} - y\right|,$$

where $N(T)$ denotes the number of zeros of $\zeta(s)$ such that $0 \le \beta_1 \le 1$ and $0 < \gamma_1 \le T$.

For any integer x, let $\alpha = \frac{\log x}{2\pi}$. In 1975, Hlawka [18] showed that

$$D^*_\alpha(T) \ll \frac{\log x}{\log T} \tag{1.3}$$

under the Riemann hypothesis, while

$$D^*_\alpha(T) \ll \frac{\log x}{\log\log T} \tag{1.4}$$

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unconditionally. In 1993, Fujii [16] improved this bound and showed that

$$D_\alpha^*(T) \ll_\alpha \frac{\log\log T}{\log T}.$$

Recently, Ford and Zaharescu [14] investigated this result on discrepancy in more general settings. In particular, they showed that the discrepancy of the set $\{h(\alpha\gamma_1) : 0 < \gamma_1 \le T\}$ is of the order $O(1/\log T)$ for a large class of functions $h : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$. Also, Akbary and Murty [1] obtained similar results on the uniform distribution and the discrepancy for a large class of Dirichlet series on the assumption of average density hypothesis.

Another very important result in this direction is due to Montgomery. In [25], he studied the pair correlation of zeros of $\zeta(s)$ and showed that the distribution of consecutive spacing of imaginary parts of zeros follows GUE distribution. These results were also extended for general L functions by Murty and Perelli [27], and Murty and Zaharescu [28]. In his work [25], Montgomery mentioned the connection between Landau-Siegel zeros and the gap between consecutive zeros of the Riemann zeta function. Conrey and Iwaniec [4] showed that the existence of Landau-Siegel zeros implies that spacing of consecutive zeros of $\zeta(s)$ are close to multiples of half the average spacing.

The vertical distributions of zeros of $\xi'(s)$ have also been studied recently. In [10], Farmer, Gonek and Lee initiated the study of consecutive spacing of zeros of $\xi'(s)$. They investigated the pair correlation of the zeros of $\xi'(s)$ under the Riemann hypothesis. They obtained various estimates on the consecutive spacing and multiplicity of the zeros of $\xi'(s)$. Bui [2] improved some of their results on consecutive spacing of zeros of $\xi'(s)$.

One motivation of studying such distributions of $\xi^{(m)}(s)$ is to understand the distribution of zeros of an entire function under differentiation. From the functional equation

$$\xi(s) = \xi(1-s) \tag{1.5}$$

one can see that the entire function $\xi(1/2+it)$ is real on the real axis and has order one. Also, from the work of Craven, Csordas and Smith [7], Ki and Kim [21], and Kim [22], one may observe that for sufficiently large m, the Riemann hypothesis is true for $\xi^{(m)}(s)$ in a bounded region. Also, the zeros of $\xi^{(m)}(s)$ approach equal spacing as m tends to infinity. For details, readers are directed to the work of Farmer and Rhoades [11], Coffey [3], and Ki [20].

Since the small gaps between zeros become larger under differentiation, by the work of Conrey and Iwaniec [4], one may disprove the existence of Landau-Siegel zeros by showing the gap between consecutive zeros of $\xi^{(m)}(s)$ to be less than half of the average spacing for sufficiently many zeros; for details, also see [10].

In 2009, Ford, Soundararajan and Zaharescu [13] established some connections between Montgomery's pair correlation function and the distribution of the fractional parts of $\alpha\gamma_1$. So one might expect that the pair correlation result of Gonek, Farmer and Lee [10] would have connections with the distribution of fractional parts of $\alpha\gamma$, where $\rho = \beta + i\gamma$ denotes a complex zero of $\xi^{(m)}(s)$.

Although much information on the distribution of fractional parts of $\alpha\gamma_1$ is known, the authors cannot recall any results of the distribution of fractional parts of $\alpha\gamma$. The main goal of this paper to obtain some classical results on the distribution of the fractional parts of $\alpha\gamma$ analogous to the results of Rademacher [29] and Hlawka [18]. Our first result in this direction is stated below.

Theorem 1.1. *For $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and a positive integer m, the sequence $(\alpha\gamma)$ is uniformly distributed modulo one, where γ runs over the imaginary parts of zeros of $\xi^{(m)}(s)$.*

Next, we are interested in the discrepancy of the sequence $(\alpha\gamma)$. Let $N_m(T)$ denote the number of zeros of $\xi^{(m)}(s)$ such that $0 \le \beta \le 1$ and $0 < \gamma \le T$. Conrey [5] proved that

$$N_m(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O_m(\log T). \tag{1.6}$$

Let $D^*(\alpha;T)$ denote the discrepancy

$$D^*(\alpha;T) := \sup_{0\le y\le 1}\left|\frac{\#\{0\le\gamma\le T;\ 0\le\{\alpha\gamma\}<y\}}{N_m(T)} - y\right|,$$

of the set $\{\{\alpha\gamma\}:\ 0<\gamma\le T\}$.

We have the following bound for $D^*(\alpha;T)$, which generalizes the results of Hlawka [18] for $\zeta(s)$.

Theorem 1.2. *Let $\alpha \ge \frac{\log 2}{2\pi}$ and m be a non-negative integer. Then*

$$D^*(\alpha;T) \le \frac{a_1\alpha}{\log\log T} + \frac{e^{a_2 m}\log\log T}{\sqrt{\log T}}$$

as $T\to\infty$, where a_1 and a_2 are absolute constants. Under the assumption of the Riemann hypothesis,

$$D^*(\alpha;T) \le \frac{c_1\alpha}{\log T} + \exp\left(\frac{c_2 m\log T}{\log\log T}\right)\frac{(\log\log T)^2}{T^{1/3}(\log T)^2}$$

as $T\to\infty$, where c_1 and c_2 are absolute constants.

Remark: Let $g(T)$ tend to infinity with T and $g(T)=o(\log\log T)$. Then for all $0\le m\le g(T)$, the bound

$$D^*(\alpha;T) \ll \frac{\alpha}{\log\log T}$$

holds unconditionally and

$$D^*(\alpha;T) \ll \frac{\alpha}{\log T}$$

holds under the assumption of the Riemann hypothesis. Theorem 1.2 shows that the distribution of the sequence $\{\alpha\gamma\}$ depends on m. If we take $m\le g(T)$, where $g(T)$ tends to infinity with T and $g(T)=o(\log\log T)$, then the discrepancy vanishes as m tends to infinity. In other words, the sequence $\{\alpha\gamma\}$ becomes more and more well spaced as m approaches infinity. This result can be compared with that of Ki [20] who showed that

There exist sequences A_n and C_n, with $C_n\to 0$ slowly, such that

$$\lim_{n\to\infty} A_n\xi^{(2n)}(C_n s) = \cos s$$

uniformly on compact subset of $\mathbb{C}$,

which was conjectured by Farmer and Rhoades [11]. In other words, one can say, the zeros of derivatives become more well spaced as m increases.

Hlawka's discrepancy bounds (1.3) and (1.4) rely on the explicit formula of Landau [23]

$$\sum_{0<\gamma_1\le T} x^{\rho_1} = -\Lambda(x)\frac{T}{2\pi} + O(\log T), \tag{1.7}$$

where $\Lambda(n)$ is the von-Mangoldt function. Gonek [17] gave an explicit formula, similar to (1.7), which is uniform in both x and T. Fujii [15] also obtained a similar result independently. Gonek's explicit formula can be stated as follows:

$$\sum_{0<\gamma_1\le T} x^{\rho_1} = -\Lambda(x)\frac{T}{2\pi} + O\left(x\log^2(2xT) + \frac{\log 2T}{\log x}\right) + O\left(\log x\min\left(T,\frac{x}{\langle x\rangle}\right)\right), \tag{1.8}$$

where $\langle x\rangle$ is the distance to the nearest integer prime power other than x itself.

In order to prove Theorems 1.1 and 1.3, we also need an explicit formula for the zeros of $\xi^{(m)}(s)$. An essential ingredient in obtaining the explicit formulas (1.7) and (1.8) in the case of $\zeta(s)$ is the Dirichlet series representation of $\frac{\zeta'}{\zeta}(s)$ for $\operatorname{Re}s>1$. There are no such Dirichlet series for $\frac{\xi^{(m+1)}}{\xi^{(m)}}(s)$.

We give an explicit formula for $\xi^{(m)}(s)$ with an extra parameter, which can be absorbed in the error terms for small values of x.

Theorem 1.3. *Let $x > 1$ and n_x be the nearest prime power to x. Let $g(T)$ tends to infinity with T and $g(T) = o(\log\log T)$. Then, for $T > 1$ and any integer $K \geq 1$,*

$$\sum_{0\leq\gamma\leq T} x^{\rho} = -\frac{\Lambda(n_x)}{2\pi}\delta_{x,T} + O\left(T\sum_{k=1}^{K}\frac{k2^{mk}\log^{k+1}x}{\log^k T}\right) + O\left(x\log^2 4x\sum_{k=1}^{K}(A_m\log x)^k\right)$$
$$+ O\left(x\log(2xT)\log\log 2x + \log 2T\min\left(T, \frac{1}{\log x}\right)\right) + O\left(xT\left(\frac{B_m\log\log T}{\log T}\right)^{\frac{K+1}{3}}\right), \tag{1.9}$$

holds uniformly for $0 \leq m \leq g(T)$. Here A_m and B_m are function of m with $\log A_m \ll m$ and $\log B_m \ll m$,

$$\delta_{x,T} = T$$

if $x = n_x$, and

$$\delta_{x,T} \ll \min\left(T, \frac{1}{\left|\log\frac{x}{n_x}\right|}\right),$$

if x is not an integer.

The first error term in (1.9) can be written as a main term with some more efforts. This error term may also disappear if x is not an integer. Also the second error term in (1.9) can be improved by a result of Erdös [8] for small values of K.

Differentiating (1.5) gives the functional equation

$$\xi^{(m)}(s) = (-1)^m\xi^{(m)}(1-s). \tag{1.10}$$

Since $\xi^{(m)}(s)$ is real-valued for real values of s, it is clear from (1.10) that the zeros of $\xi^{(m)}(s)$ are symmetric with respect to the line $\sigma = 1/2$. Therefore, for $0 < x < 1$, we have

$$\sum_{0\leq\gamma\leq T} x^{\rho} = \sum_{0\leq\gamma\leq T} x^{1-\bar{\rho}} = x\overline{\sum_{0\leq\gamma\leq T}\left(\frac{1}{x}\right)^{\rho}}. \tag{1.11}$$

If we choose

$$K = \left\lfloor\frac{3\log T}{\log\log T}\right\rfloor,$$

then for a fixed x and for T sufficiently large, one can show that (1.9) can be written as

$$\sum_{0\leq\gamma\leq T} x^{\rho} \ll Tx^{\epsilon} + xT^{\epsilon},$$

for $\epsilon > 0$, which may depend on T. Therefore, by the Riemann hypothesis we find that

$$\sum_{0\leq\gamma\leq T} x^{i\gamma} \ll Tx^{-\frac{1}{2}+\epsilon} + x^{\frac{1}{2}}T^{\epsilon}, \tag{1.12}$$

which is non trivial for $2 \leq x \leq T^{2-\epsilon}$ by (1.6). Now, if one assumes that $\{x^{i\gamma}\}_{\gamma}$ behave like independent random variables, then we may expect that

$$\sum_{0\leq\gamma\leq T} x^{i\gamma} \ll T^{\frac{1}{2}+\epsilon} \tag{1.13}$$

for all $x > 0$. Clearly, this is not true for every x.

By observing the bounds in (1.12) and (1.13), we have the following conjecture.

Conjecture 1.4. *For all real numbers $x, T \geq 2$ and any $\epsilon > 0$,*

$$\sum_{0 \leq \gamma \leq T} x^{i\gamma} \ll T x^{-\frac{1}{2}+\epsilon} + T^{\frac{1}{2}} x^{\epsilon}$$

holds uniformly for $0 \leq m \leq g(T)$.

To obtain the bounds in (1.3) and (1.4), another important result needed is to obtain a non-trivial upper bound for

$$\sum_{0<\gamma_1 \leq T} \left| \beta_1 - \frac{1}{2} \right|.$$

In 1924, Littlewood [24] proved that

$$\sum_{0<\gamma_1 \leq T} \left| \beta_1 - \frac{1}{2} \right| \ll T \log\log T,$$

which was later improved by Selberg [30] in 1942. In particular, he obtained

$$\int_{\frac{1}{2}}^{1} N(\sigma, T)\ d\sigma \ll T,$$

where $N(\sigma, T)$ denotes the number of zeros ρ_1 of $\zeta(s)$ such that $\beta_1 > \sigma$ and $0 < \gamma_1 < T$.

For a fixed σ, let $N_m(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\xi^{(m)}(s)$ such that $\beta > \sigma$ and $0 < \gamma < T$. Our next result provides a zero density estimate for $\xi^{(m)}(s)$.

Theorem 1.5. *Let $g(T)$ tends to infinity with T and $g(T) = o(\log\log T)$. Then*

$$\int_{\frac{1}{2}}^{1} N_m(\sigma, T)\ d\sigma \leq C(m) T$$

holds uniformly for $0 \leq m \leq g(T)$, where $\log C(m) \ll m$.

Since the prior works suggest that the zeros of $\xi^{(m)}(s)$ migrate to the line $\sigma = \frac{1}{2}$, we have the following conjecture.

Conjecture 1.6. *The function $C(m)$ is a decreasing function of m.*

Remark: Note that for $\sigma > \frac{1}{2}$,

$$\int_{\frac{1}{2}}^{1} N_m\left(\sigma', T\right)\ d\sigma' \geq \left(\sigma - \frac{1}{2}\right) N_m\left(\sigma, T\right).$$

Therefore,

$$N_m(\sigma, T) = O_m\left(\frac{T}{\sigma - \frac{1}{2}}\right) \tag{1.14}$$

holds for $\frac{1}{2} < \sigma \leq 1$. Combining (1.6) and (1.14) we find that the zeros of $\xi^{(m)}(s)$ are clustered near the line $\sigma = \frac{1}{2}$.

2. Auxiliary lemmas

For a positive real number θ, and $X = T^\theta$, define

$$M_X(s) = \sum_{n \le X} \frac{\mu(n)}{n^{s+R/\log T}} P\left(1 - \frac{\log n}{\log X}\right), \tag{2.1}$$

where P is a polynomial with $P(0) = 0$ and $P(1) = 1$. We have the following result from [6].

Lemma 2.1. *Let $V(s) = Q\left(-\frac{1}{\log T}\ \frac{d}{ds}\right)\zeta(s)$ for some polynomial Q, and let $M_X(s)$ be defined as in* (2.1)*. For $\theta < 4/7$*

$$\int_2^T \left| V M_X\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2\ dt \sim cT,$$

where $0 < R \ll 1$ and the constant c depends on P, Q, and R only.

For fixed P and R one has $c \ll |Q(1)|^2$ (see [6, p. 10]). We also need the following result of Conrey [5].

Lemma 2.2. *Let $L(s) = \frac{H'(s)}{H(s)}$, where $H(s)$ is defined in* (1.2)*, and $s = \sigma + it$. Then, for any fixed integer $k \ge 1$, the following holds:*

(1) *for $|t| \ge 1$,*

$$L(s) = \frac{1}{2}\log\frac{s}{2\pi} + O\left(\frac{1}{|t|}\right)$$

and

$$L^{(k)}(s) \ll \frac{1}{|t|^k}.$$

(2) *For $t > 10$ and $0 < \sigma < A\log\log T$, where A is a constant,*

$$\frac{H^{(k)}(s)}{H(s)} = (L(s))^k + O\left(\frac{\log^{k-1} t}{t}\right).$$

We also need the following lemma from [17].

Lemma 2.3. *For $x, T \ge 1$ and $c = 1 + \frac{1}{\log 2x}$,*

$$\sum_{\substack{n=2 \\ n \ne x}}^{\infty} \frac{\Lambda(n)}{n^c} \min\left(T, \frac{1}{|\log x/n|}\right) \ll \log 2x \log\log 2x + \log x \min\left(\frac{T}{x}, \frac{1}{\langle x\rangle}\right),$$

where $\langle x\rangle$ is the distance to the nearest integer prime power other than x itself.

Weyl's criterion [32] for uniformly distributed sequences is given by the following lemma.

Lemma 2.4. *A sequence $(x_n)_{n\ge1}$, is uniformly distributed mod 1 if and only if*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} = 0,$$

for all integers $k \ne 0$.

The following inequality is due to Erdös and Turán [9].

Lemma 2.5. *Let D_N denote the discrepancy of a sequence $(x_n)_{n\geq 1}$ of real numbers. Then, for any positive integer M,*

$$D_N \leq \frac{C_1}{M+1} + C_2 \sum_{k=1}^{M} \frac{1}{k} \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k x_n} \right|,$$

where C_1 and C_2 are absolute positive constants.

The following lemma is due to Montgomery and Vaughan [26].

Lemma 2.6. *If $\sum_{n=1}^{\infty} n|a_n|^2$ converges, then*

$$\int_0^T \left| \sum_{n=1}^{\infty} a_n n^{-it} \right|^2 dt = \sum_{n=1}^{\infty} |a_n|^2 (T + O(n)).$$

The following lemma from [31, p. 213] will be used to bound the argument of an analytic function.

Lemma 2.7. *Let $f(s)$ be an analytic function except for a pole at $s=1$ and be real for real s. Let $0 \leq a < b < 2$. Suppose that T is not an ordinate of any zero of $f(s)$. Let $|f(\sigma+it)| \leq M$ for $\sigma \geq a$, $1 \leq t \leq T+2$ and $\mathrm{Re}(f(2+it)) \geq c > 0$ for some $c \in \mathbb{R}$. Then, for $\sigma \geq b$,*

$$|\arg f(\sigma + iT)| \leq \frac{c}{\log \frac{2-a}{2-b}} \left(\log M + \log \frac{1}{c} \right) + \frac{3\pi}{2}.$$

Let Λ_k denote the generalized von-Mangoldt defined by

$$\Lambda_k(n) := \sum_{d|n} \mu(d) \log^k \frac{n}{d}.$$

Therefore, for $\mathrm{Re}(s) > 1$,

$$\sum_{n=1}^{\infty} \frac{\Lambda_k(n)}{n^s} = (-1)^k \frac{\zeta^{(k)}(s)}{\zeta(s)}. \tag{2.2}$$

Let Λ_k^{*l} denote the l-fold convolutions of Λ_k, i.e.,

$$\Lambda_k^{*l} = \underbrace{\Lambda_k * \cdots * \Lambda_k}_{l \text{ times}}. \tag{2.3}$$

Then, we have the following inequality.

Lemma 2.8. *With the notation from (2.2) and (2.3)*

$$(\Lambda_k \log * \Lambda_{k_1}^{*l_1} * \cdots * \Lambda_{k_m}^{*l_m})(n) \leq (\log n)^{1+k+k_1+l_1+\cdots+k_n+l_n}. \tag{2.4}$$

Proof. From [19, p. 35], we have

$$\Lambda_k(n) \leq \log^k n.$$

Using the above inequality and (2.2), we find that

$$(\Lambda_k \log * \Lambda_{k_1})(n) = \sum_{ab=n} \Lambda_k(a) \log(a) \Lambda_{k_1}(b) \leq \Lambda_k(n) \log(n) (1 * \Lambda_{k_1})(n) \leq \log^{k+k_1+1} n.$$

By repeating this argument, we complete the proof of the lemma. □

As an application of the Faà di Bruno formula [12, p. 188], we obtain the following result.

Lemma 2.9. *For any non-zero analytic function f, we have*

$$\frac{f^{(n)}}{f}(s) = \sum_{\substack{\mu_1+2\mu_2+\cdots+k\mu_k=n \\ \mu_1+\mu_2+\cdots+\mu_k=k}} \prod_{i=1}^{k} \frac{n!}{\mu_i!(i!)^{\mu_i}} \left(\left(\frac{f'}{f} \right)^{(i-1)} (s) \right)^{\mu_i}. \tag{2.5}$$

3. Proof of the explicit formula

Applying Leibnitz's rule in (1.1), we find that

$$\xi^{(m)}(s) = H^{(m)}(s)F_m(s) \tag{3.1}$$

where

$$F_m(s) := \zeta(s) + \sum_{j=1}^{m} c_j \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \zeta^{(j)}(s) \tag{3.2}$$

with $c_j = \binom{m}{j}$. For $t > 10$, $\frac{H^{(m)}(s)}{H(s)}$ is non-zero by Lemma 2.2 and $H(s)$ never vanishes, therefore $H^{(m)}(s)$ does not have any complex zero for $t > 10$. Therefore, the complex zeros of $F_m(s)$ are the only zeros of $\xi^{(m)}(s)$. The logarithmic derivative of (3.2) yields

$$\frac{F_m'(s)}{F_m(s)} = \frac{\zeta'(s)}{\zeta(s)} + \frac{E_m'(s)}{E_m(s)}, \tag{3.3}$$

where

$$E_m(s) = 1 + \sum_{j=1}^{m} c_j \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \frac{\zeta^{(j)}(s)}{\zeta(s)}. \tag{3.4}$$

Also,

$$\begin{aligned} E_m'(s) &= \sum_{j=1}^{m} c_j \left(\frac{H^{(m-j)}(s)}{H^{(m)}(s)} \frac{\zeta^{(j)}(s)}{\zeta(s)} \right)' \\ &= \sum_{j=1}^{m} c_j \left(\frac{\zeta^{(j)}(s)}{\zeta(s)} \right)' \frac{H^{(m-j)}(s)}{H^{(m)}(s)} + \sum_{j=1}^{m} c_j \left(\frac{H^{(m-j)}(s)}{H^{(m)}(s)} \right)' \frac{\zeta^{(j)}(s)}{\zeta(s)} \\ &=: E_{m1}'(s) + E_{m2}'(s). \end{aligned} \tag{3.5}$$

Let $c = 1 + \frac{1}{\log 2x}$ and consider the rectangle $\mathcal{R}$ defined by the vertices $1 - c + iT_0,\ c + iT_0,\ c + iT$ and $1 - c + iT$, where T_0 is chosen later. Then, by the residue theorem,

$$\frac{1}{2\pi i} \int_{\mathcal{R}} \frac{F_m'(s)}{F_m(s)} x^s ds = \sum_{T_0 \le t \le T} x^{\rho}. \tag{3.6}$$

Since $\xi^{(m)}(s)$ is an entire function of order 1, by the Hadamard's factorization theorem, one can rewrite it as

$$\xi^{(m)}(s) = e^{A+Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{-s/\rho},$$

where the product runs over all the zeros of $\xi^{(m)}(s)$, and A, B are certain constants. Also note that all complex zeros of $F_m(s)$ lie in the strip $0 < \sigma < 1$. This implies that $\frac{F_m'}{F_m}(s)$ is bounded at $2 + it$ for any $t \in \mathbb{R}$. Therefore, by logarithmic differentiation, (3.1), and Lemma 2.2, we obtain

$$\begin{aligned} \frac{F_m'}{F_m}(\sigma + it) &= \sum_{\rho} \left(\frac{1}{s - \rho} - \frac{1}{\rho} \right) + O(\log t) \\ &= \sum_{\rho} \left(\frac{1}{s - \rho} - \frac{1}{2 + it - \rho} \right) + O(\log t). \end{aligned} \tag{3.7}$$

Now, we consider the terms in the sum on the right side of (3.7) for which $|\gamma - t| \geq 1$. From (1.6), we have

$$N_m(t+1) - N_m(t) \ll f(m) \log t, \tag{3.8}$$

where $\log f(m) \ll m$. From now on $f(m)$ denotes a function of m, not necessarily same at each occurence, and $\log f(m) \ll m$. Using (3.8) we find that

$$\begin{aligned}
\sum_{n=1}^{\infty} \sum_{n \leq |\gamma - t| < n+1} \frac{2-\sigma}{(s-\rho)(2+it-\rho)} &\ll \sum_{n=1}^{\infty} \sum_{n \leq |\gamma - t| < n+1} \frac{1}{(\gamma - t)^2} \\
&\ll \sum_{n=1}^{\infty} \sum_{n \leq |\gamma - t| < n+1} \frac{1}{n^2} \\
&\ll \sum_{n=1}^{\infty} \frac{\log(t+n)}{n^2} \\
&\ll \log t.
\end{aligned} \tag{3.9}$$

Since $0 \leq \beta \leq 1$, by (3.8) we have

$$\sum_{|\gamma - t| < 1} \frac{1}{2+it-\rho} \ll \log t. \tag{3.10}$$

Invoking (3.9) and (3.10) in (3.7) we obtain

$$\frac{F_m'}{F_m}(s) = \sum_{|\gamma - t| < 1} \frac{1}{s-\rho} + O(\log t). \tag{3.11}$$

From (3.11), in (3.6) the integral along the top horizontal side of the rectangle $\mathcal{R}$ can be written as

$$\sum_{|\gamma - T| < 1} \int_{c+iT}^{1-c+iT} \frac{x^s}{s-\rho}\, ds + O\left(\log 2T \int_{1-c}^{c} x^\sigma \, d\sigma\right) =: \sum_{|\gamma - T| < 1} I_\gamma + O\left(x \frac{\log 2T}{\log 2x}\right). \tag{3.12}$$

In order to compute I_γ, we shift the line of integration from $\operatorname{Im} s = T$ to $\operatorname{Im} s = T+1$. For $|\gamma - T| < 1$, by the residue theorem, we see that

$$\begin{aligned}
I_\gamma &= \left(\int_{c+i(T+1)}^{1-c+i(T+1)} + \int_{c+iT}^{c+i(T+1)} - \int_{1-c+iT}^{1-c+i(T+1)}\right) \frac{x^s}{s-\rho}\, ds + O(1) \\
&\ll 1 + \int_{1-c}^{c} \frac{x^\sigma}{\sqrt{(\sigma-\beta)^2 + (T+1-\gamma)^2}} d\sigma + x \int_{T}^{T+1} \frac{dt}{\sqrt{(c-\beta)^2 + (t-\gamma)^2}} + \frac{x^{1-c}}{\beta - 1 + c} \\
&\ll x \log\log 2x.
\end{aligned}$$

Note that the sum on the right side of (3.12) has $\log(2T)$ terms. Therefore, the contribution from the top horizontal integral is

$$\frac{1}{2\pi i} \int_{c+iT}^{1-c+iT} \frac{F_m'(s)}{F_m(s)} x^s ds \ll f(m) x \log(2T) \log\log(2x). \tag{3.13}$$

Since $\frac{F_m'(s)}{F_m(s)}$ is bounded in the interval $[1-c+iT_0, c+iT_0]$, the contribution from the integral along the lower horizontal of the rectangle $\mathcal{R}$ in (3.6) is given by

$$\frac{1}{2\pi i} \int_{c+iT_0}^{1-c+iT_0} \frac{F_m'(s)}{F_m(s)} x^s \, ds \ll f(m) \frac{x}{\log 2x}. \tag{3.14}$$

Next, we compute the integral on the right vertical line of the rectangle $\mathcal{R}$ in (3.6). From (3.3), one has

$$\int_{c+iT_0}^{c+iT} \frac{F_m'}{F_m}(s)x^s ds = \int_{c+iT_0}^{c+iT} \frac{\zeta'}{\zeta}(s)x^s\ ds + \int_{c+iT_0}^{c+iT} \frac{E_{m1}'}{E_m}(s)x^s\ ds + \int_{c+iT_0}^{c+iT} \frac{E_{m2}'}{E_m}(s)x^s\ ds$$
$$=: I_1 + I_2 + I_3. \tag{3.15}$$

From [31, sect. 6.19], we have the following bound for the Riemann zeta function

$$\frac{\zeta'}{\zeta}(\sigma+it) \ll \log^{\frac{2}{3}} t \log^{\frac{1}{3}} \log t,$$

which holds uniformly on $\sigma > 1 - A\log^{-\frac{2}{3}} t \log^{-\frac{1}{3}} \log t$, where A is an absolute constant. Using the Cauchy integral formula, for any positive integer n, we obtain

$$\left(\frac{\zeta'}{\zeta}(\sigma+it)\right)^{(n)} \ll \log^{\frac{2}{3}} t \log^{\frac{1}{3}} \log t,$$

which holds uniformly on $\sigma > 1 - A\log^{-\frac{2}{3}} t \log^{-\frac{1}{3}} \log t$. Hence, by Lemma 2.5,

$$\frac{\zeta^{(n)}}{\zeta}(\sigma+it) \ll f(n) \log^{\frac{2n}{3}} t \log^{\frac{n}{3}} \log t \tag{3.16}$$

and

$$\left(\frac{\zeta^{(n)}}{\zeta}(\sigma+it)\right)^{(l)} \ll f(n) \log^{\frac{2n}{3}} t \log^{\frac{n}{3}} \log t, \tag{3.17}$$

for $\sigma > 1 - A\log^{-\frac{2}{3}} t \log^{-\frac{1}{3}} \log t$. As an application of Lemma 2.2, we deduce that

$$\frac{H^{(m)}}{H^{(m-j)}}(\sigma+it) = \frac{1}{2^j} \log^j \frac{s}{2\pi}\left(1 + O\left(\frac{1}{t\log t}\right)\right) \tag{3.18}$$

and

$$\left(\frac{H^{(m)}}{H^{(m-j)}}(\sigma+it)\right)' \ll f(j)\frac{\log^{j-1} t}{t} \tag{3.19}$$

for t large. Combining (3.16) and (3.18) with (3.4), we find that

$$|E_m(\sigma+it) - 1| \ll f(m)\sum_{j=1}^{m} c_j \frac{\log^{\frac{2j}{3}} t \log^{\frac{j}{3}} \log t}{\log^j t} \ll f(m)\frac{\log^{\frac{1}{3}} \log t}{\log^{\frac{1}{3}} t} < \frac{1}{2} \tag{3.20}$$

for large t and uniformly for $\sigma > 1$ and $m \le g(T)$. Now, we choose T_0 so that (3.18), (3.19), and (3.20) hold for all $t \ge T_0$. Using (3.16), (3.18), (3.19), and (3.20) in (3.5), we have

$$\frac{E_{m2}'}{E_m}(\sigma+it) \ll f(m)\sum_{j=1}^{m} c_j \frac{\log^{\frac{2j}{3}} t \log^{\frac{j}{3}} \log t}{t\log^{j+1} t} \tag{3.21}$$

for $t \ge T_0$ and uniformly for $\sigma > 1$. Therefore, integrating by parts, and using (3.20) and (3.21), one deduces that

$$I_3 \ll f(m)x. \tag{3.22}$$

To compute I_2, we first rewrite it as

$$I_2 = \sum_{k=0}^{K-1}(-1)^k \int_{c+iT_0}^{c+iT} E_{m1}'(s)(E_m(s)-1)^k x^s\ ds + \int_{c+iT_0}^{c+iT} \frac{E_{m1}'(s)(E_m(s)-1)^K}{E_m(s)} x^s\ ds$$
$$=: I_{21} + I_{22}. \tag{3.23}$$

From (3.5), (3.17), and (3.18), we find that

$$E'_{m1}(\sigma+it) \ll f(m)\sum_{j=1}^{m} c_j \frac{\log^{\frac{2j}{3}} t \log^{\frac{j}{3}} \log t}{\log^j t} \ll f(m)\frac{\log^{\frac{1}{3}} \log t}{\log^{\frac{1}{3}} t}, \tag{3.24}$$

for $t \geq T_0$ and uniformly for $\sigma > 1$. Hence, from (3.20), (3.24), and the definition of I_{22} in (3.23), we have

$$I_{22} \ll xT\left(\frac{f(m)\log^{\frac{1}{3}} \log T}{\log^{\frac{1}{3}} T}\right)^{K+1}, \tag{3.25}$$

where the implied constant in the bound is absolute, and the constant C_m depends only on m. From (3.4) and (3.5), we have

$$\begin{aligned}\sum_{k=0}^{K-1} E'_{m1}(s)(E_m(s)-1)^k = &\sum_{j=1}^{m} c_j \left(\frac{\zeta^{(j)}(s)}{\zeta(s)}\right)' \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \\ &\times \sum_{k=0}^{K-1} \sum_{l_1+l_2+\cdots+l_m=k} k! \prod_{i=1}^{m} \frac{1}{l_i!}\left(c_i \frac{\zeta^{(i)}(s)}{\zeta(s)}\frac{H^{(m-i)}(s)}{H^{(m)}(s)}\right)^{l_i}.\end{aligned} \tag{3.26}$$

If $\tilde{k} = l_1 + 2l_2 + \cdots + ml_m$, then by (3.17) and (3.18), we obtain the following bound

$$\left(\frac{\zeta^{(j)}(s)}{\zeta(s)}\right)' \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \prod_{i=1}^{m}\left(\frac{\zeta^{(i)}(s)}{\zeta(s)}\frac{H^{(m-i)}(s)}{H^{(m)}(s)}\right)^{l_i} \ll \left(\frac{f(m)\log^{\frac{1}{3}} \log t}{\log^{\frac{1}{3}} t}\right)^{\tilde{k}+j}. \tag{3.27}$$

The sum of coefficients of terms in (3.26) bounded by (3.27) is at most

$$\sum_{j=1}^{m} c_j \sum_{k=0}^{K-1} (-1)^k \sum_{l_1+l_2+\cdots+l_m=k} k! \prod_{i=1}^{m} \frac{1}{l_i!}(c_i)^{l_i} \leq K2^{mK}.$$

Therefore, the contribution from all terms on the right side of (3.26) those bounded by (3.27) with $\tilde{k}+j > K$ is at most

$$K2^{mK} \sum_{k=K+1}^{\infty}\left(\frac{C_m \log^{\frac{1}{3}} \log t}{\log^{\frac{1}{3}} t}\right)^k \ll K\left(\frac{f(m)\log\log t}{\log t}\right)^{\frac{K+1}{3}} \tag{3.28}$$

for $t \geq T_0$. Let $\tilde{k}+j = L \leq K$. Then, from (3.16), (3.17), (3.18), and using trivial bounds, one has

$$\begin{aligned}\left(\frac{\zeta^{(j)}(s)}{\zeta(s)}\right)' \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \prod_{i=1}^{m}\left(\frac{\zeta^{(i)}(s)}{\zeta(s)}\frac{H^{(m-i)}(s)}{H^{(m)}(s)}\right)^{l_i} &= \frac{2^L}{\log^L(s/2\pi)} \prod_{i=1}^{m}\left(\frac{\zeta^{(i)}(s)}{\zeta(s)}\right)^{l_i}\left(\frac{\zeta^{(j)}(s)}{\zeta(s)}\right)' \\ &\quad + O\left(\frac{2^L \log^{\frac{L}{3}} \log t}{t\log^{\frac{L}{3}+1} t}\right) \\ &\ll \left(\frac{f(m)\log^{\frac{1}{3}} \log t}{\log^{\frac{1}{3}} t}\right)^L.\end{aligned} \tag{3.29}$$

Thus, the sum of the coefficients of terms in (3.26) bounded by (3.29) is at most

$$\sum_{j=1}^{m} c_j \sum_{k=0}^{L-1} (-1)^k \sum_{l_1+l_2+\cdots+l_m=k} k! \prod_{i=1}^{m} \frac{1}{l_i!}(c_i)^{l_i} \leq L2^{mL}. \tag{3.30}$$

Also, from the definition of Λ_k (see (2.4)), the following Dirichlet series can be written as

$$\frac{\zeta^{(k)}(s)}{\zeta(s)} = (-1)^k \sum_{n=1}^{\infty} \frac{\Lambda_k(n)}{n^s} \quad \text{and} \quad \left(\frac{\zeta^{(k)}(s)}{\zeta(s)}\right)' = (-1)^{k+1} \sum_{n=1}^{\infty} \frac{\Lambda_k(n)\log n}{n^s}$$

for $\sigma > 1$. Hence,

$$\prod_{i=1}^{m} \left(\frac{\zeta^{(i)}(s)}{\zeta(s)}\right)^{l_i} \left(\frac{\zeta^{(j)}(s)}{\zeta(s)}\right)' = (-1)^L \sum_{n=1}^{\infty} \frac{b_L(n)}{n^s} \tag{3.31}$$

for $\sigma > 1$. Moreover, from Lemma 2.8, we find that $b_L(n) \le \log^{L+1} n$. Combining (3.29), (3.30) and (3.31), we have

$$\sum_{\substack{\tilde{k}+j=L \\ l_1+\cdots+l_m=k}} \left(\frac{\zeta^{(j)}(s)}{\zeta(s)}\right)' \frac{H^{(m-j)}(s)}{H^{(m)}(s)} k! \prod_{i=1}^{m} \frac{c_j^{l_i}}{l_i!} \left(\frac{\zeta^{(i)}(s)}{\zeta(s)} \frac{H^{(m-i)}(s)}{H^{(m)}(s)}\right)^{l_i}$$
$$= \frac{(-2)^L}{\log^L(s/2\pi)} \sum_{n=1}^{\infty} \frac{a_L(n)}{n^s} + O\left(\frac{C_m^L \log^{\frac{L}{3}} \log t}{t \log^{\frac{L}{3}+1} t}\right), \tag{3.32}$$

where $a_L(n) \le L2^{mL} \log^{L+1} n$. Therefore, from (3.28), (3.32) and an integration by parts, we deduce that

$$I_{21} = \sum_{L=1}^{K} \sum_{n=1}^{\infty} \int_{c+iT_0}^{c+iT} \frac{(-2)^L a_L(n)}{\log^L(s/2\pi)} \left(\frac{x}{n}\right)^s \, ds + O(f(m)x) + O\left(KxT \left(\frac{f(m)\log\log T}{\log T}\right)^{\frac{K+1}{3}}\right). \tag{3.33}$$

Let n' be the nearest integer to x. Then,

$$\int_{c+iT_0}^{c+iT} \frac{1}{\log^L(s/2\pi)} \left(\frac{x}{n'}\right)^s \, ds \ll \left(\frac{x}{n'}\right)^c \int_{T_0}^{T} \frac{1}{\log^L t} \, dt \ll \frac{T}{\log^L T}.$$

If x is not an integer, then by integrating by parts, we obtain

$$\int_{c+iT_0}^{c+iT} \frac{1}{\log^L(s/2\pi)} \left(\frac{x}{n}\right)^s \, ds \ll \frac{x}{C^L n^c \log(x/n)},$$

where C is an absolute constant. Therefore,

$$\sum_{n=1}^{\infty} a_L(n) \int_{c+iT_0}^{c+iT} \frac{1}{\log^L(s/2\pi)} \left(\frac{x}{n}\right)^s \, ds \ll \frac{a_L(n')T}{\log^L T} + \sum_{\substack{n=1 \\ n\ne n'}}^{\infty} \frac{x^c a_L(n)}{C^L n^c \log(x/n)}. \tag{3.34}$$

Also,

$$\sum_{1\le n\le n'/2} \frac{x^c a_L(n)}{C^L n^c \log(x/n)} + \sum_{n\ge 2n'} \frac{x^c a_L(n)}{C^L n^c \log(x/n)} \le \sum_{n=1}^{\infty} \frac{a_L(n)}{C^L n^c} \ll \frac{xL2^{mL}}{C^L(c-1)^{L+1}} = \frac{xL2^{mL}\log^{L+1} x}{C^L}.$$

For the remaining terms in the sum on the right side of (3.34), we have

$$\sum_{\substack{n'/2\le n\le 2n' \\ n\ne n'}} \frac{x^c a_L(n)}{C^L n^c \log(x/n)} \ll \frac{L2^{mL}\log^{L+1} x}{C^L} \sum_{\substack{n'/2\le n\le 2n' \\ n\ne n'}} \frac{1}{\log(x/n)}.$$

Since

$$\log\frac{x}{n} \ge \log\frac{n'}{n} = -\log\left(1 - \frac{n'-n}{n'}\right) \ge \frac{|n-n'|}{n'},$$

we have

$$\sum_{\substack{n'/2\le n\le 2n'\\ n\neq n'}} \frac{1}{\log(x/n)} \le \sum_{\substack{n'/2\le n\le 2n'\\ n\neq n'}} \frac{n'}{|n-n'|} \ll x\log 2x.$$

Therefore, from (3.33),

$$I_{21} \ll T\sum_{k=1}^{K} \frac{k2^{(m+1)k}\log^{k+1} x}{\log^k T} + x\log 4x \sum_{k=1}^{K} \frac{k2^{(m+1)k}\log^{k+1} x}{C^k} + xT\left(\frac{f(m)\log^{\frac{1}{3}}\log T}{\log^{\frac{1}{3}} T}\right)^{K+1}.$$

Using (3.25) and the above estimates in (3.23), we obtain

$$I_{2} \ll T\sum_{k=1}^{K} \frac{k2^{(m+1)k}\log^{k+1} x}{\log^k T} + x\log 4x \sum_{k=1}^{K} \frac{k2^{(m+1)k}\log^{k+1} x}{C^k} + xT\left(\frac{f(m)\log^{\frac{1}{3}}\log T}{\log^{\frac{1}{3}} T}\right)^{K+1}. \tag{3.35}$$

Let n_x be the nearest prime power to x. Then by Lemma 2.3

$$\begin{aligned} I_1 = \int_{c+iT_0}^{c+iT} \frac{\zeta'}{\zeta}(s)x^s\, ds &= -\int_{c+iT_0}^{c+iT} \sum_{n=2}^{\infty} \Lambda(n)\left(\frac{x}{n}\right)^s\, ds \\ &= -i\Lambda(n_x)\int_{T_0}^{T}\left(\frac{x}{n_x}\right)^{it} dt + O\left(x^c \sum_{\substack{n=2\\ n\neq n_x}}^{\infty} \frac{\Lambda(n)}{n^c\log(x/n_x)}\right) \\ &= -i\Lambda(n_x)\delta_{x,T} + O\left(x\log(2x)\log\log(2x)\right), \end{aligned} \tag{3.36}$$

where

$$\delta_{x,T} = \int_0^T \left(\frac{x}{n_x}\right)^{it} dt.$$

Clearly $\delta_{x,T} \ll T$. If $x = n_x$ then

$$\delta_{x,T} = T$$

otherwise

$$\delta_{x,T} = \frac{\left(\frac{x}{n_x}\right)^{iT} - 1}{i\log\frac{x}{n_x}} \ll \left|\log\frac{x}{n_x}\right|^{-1}.$$

Notice that the first term on the right side of (3.36) disappears if x is not an integer. Combining (3.15), (3.22), (3.35), and (3.36) for the contribution from the integral along the right vertical side of the rectangle $\mathcal{R}$ in (3.6), we arrive at

$$\begin{aligned} \int_{c+iT_0}^{c+iT} \frac{F_m'}{F_m}(s)x^s ds = -i\Lambda(n_x)\delta_{x,T} &+ O\left(x\log(2x)\log\log(2x)\right) + O\left(xT\left(\frac{f(m)\log\log T}{\log T}\right)^{\frac{K+1}{3}}\right) \\ &+ O\left(T\sum_{k=1}^{K}\frac{k2^{mk}\log^{k+1}x}{\log^k T}\right) + O\left(x\log^2 4x\sum_{k=1}^{K}(f(m)\log x)^k\right) \end{aligned} \tag{3.37}$$

Now, we move on to estimate the integral along the left vertical side of the rectangle $\mathcal{R}$ in (3.6). From the functional equation (1.10) one can derive

$$\frac{F_{m+1}}{F_m}(s) = (-1)^{m+1}\frac{F_{m+1}}{F_m}(1-s) + (-1)^{m+1}\frac{H^{(m+1)}}{H^{(m)}}(1-s) - \frac{H^{(m+1)}}{H^{(m)}}(s).$$

Thus, for the integral along the left vertical line, we have

$$\begin{aligned}\int_{1-c+iT_0}^{1-c+iT} \frac{F_{m+1}}{F_m}(s)x^s ds &= (-1)^{m+1}\int_{1-c+iT_0}^{1-c+iT} \frac{F_{m+1}}{F_m}(1-s)x^s ds \\ &\quad + \int_{1-c+iT_0}^{1-c+iT}\left((-1)^{m+1}\frac{H^{(m+1)}}{H^{(m)}}(1-s) - \frac{H^{(m+1)}}{H^{(m)}}(s)\right)x^s\ ds \\ &=: I_4 + I_5. \end{aligned} \tag{3.38}$$

Integrating by parts and employing (3.18) and (3.19), we find that

$$I_5 \ll \frac{\log 2T}{\log x}. \tag{3.39}$$

Also, trivially we have

$$I_5 \ll T\log 2T.$$

We rewrite the integral I_4 above as

$$\begin{aligned} I_4 &= \int_{1-c+iT_0}^{1-c+iT} \frac{\zeta'}{\zeta}(1-s)x^s\ ds + \sum_{k=0}^{K-1}\int_{1-c+iT_0}^{1-c+iT} E'_{m1}(1-s)(E_m(1-s)-1)^k x^s\ ds \\ &\quad + \int_{1-c+iT_0}^{1-c+iT} \frac{E'_{m1}(1-s)(E_m(1-s)-1)^K}{E_m(1-s)}x^s\ ds + \int_{1-c+iT_0}^{1-c+iT}\frac{E'_{m2}}{E_m}(1-s)x^s\ ds \\ &=: I_{41} + I_{42} + I_{43} + I_{44}. \end{aligned} \tag{3.40}$$

Now, we compute I_{41} defined above as follows.

$$\begin{aligned} I_{41} = \int_{1-c+iT_0}^{1-c+iT} \frac{\zeta'}{\zeta}(1-s)x^s\ ds &= ix^{1-c}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^c}\int_{10}^{T}(nx)^{it}\ dt \\ &\ll \left(x^{1-c}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^c\log(xn)}\right) \\ &\ll \left(\frac{x^{1-c}}{c-1}\right) \\ &\ll \log x. \end{aligned} \tag{3.41}$$

Proceeding in a similar fashion as for I_{21} earlier and using (3.33), we have

$$I_{42} = \sum_{L=1}^{K}\sum_{n=1}^{\infty}(-2)^L\frac{a_L(n)}{n}\int_{1-c+iT_0}^{1-c+iT}\frac{1}{\log^L(s/2\pi)}(nx)^s\ ds + O\left(KT\left(\frac{f(m)\log^{\frac{1}{3}}\log T}{\log^{\frac{1}{3}} T}\right)^{K+1}\right), \tag{3.42}$$

where

$$\sum_{n=1}^{\infty}\frac{a_L(n)}{n}\int_{1-c+iT_0}^{1-c+iT}\frac{1}{\log^L(s/2\pi)}(nx)^s\ ds \ll \sum_{n=1}^{\infty}\frac{x^{1-c}a_L(n)}{C^L n^c\log(nx)} \ll \frac{L2^{mL}\log^{L+1}x}{C^L}.$$

Proceeding similarly as we did for I_{22} and I_3, we arrive at

$$I_{43} \ll T\left(\frac{f(m)\log^{\frac{1}{3}}\log T}{\log^{\frac{1}{3}} T}\right)^{K+1} \quad \text{and} \quad I_{44} \ll f(m). \tag{3.43}$$

Thus, from (3.38), (3.39), (3.40), (3.41), (3.43), and (3.42), the contribution from the integral along the left vertical side of the rectangle $\mathcal{R}$ in (3.6) becomes

$$\int_{1-c+iT_0}^{1-c+iT} \frac{F_m'}{F_m}(s)x^s ds = O\left(T\left(\frac{f(m)\log\log T}{\log T}\right)^{\frac{K+1}{3}}\right) + O\left(\log^2 4x \sum_{k=1}^{K}(f(m)\log x)k\right) + O\left(\min\left(T\log 2T, \frac{\log 2T}{\log x}\right)\right). \tag{3.44}$$

Using the estimates from (3.13), (3.14), (3.37), and (3.44) in (3.6), we now complete the proof of Theorem 1.3.

4. Proof of the zero density estimates: Theorem 1.5

As discussed earlier in the previous section, since the complex zeros of $\xi^{(m)}(s)$ are identical to those of $F_m(s)$, we prove the theorem for $F_m(s)$ instead. Let

$$f(s) := M_X(s)F_m(s) - 1, \tag{4.1}$$

where M_X is defined by (2.1). Consider

$$h(s) := 1 - f^2(s). \tag{4.2}$$

Here $h(s)$ is analytic except for the pole at $s = 1$. Let $P(x) = x$ in Lemma 2.2. Then, for $0 < \theta < 1$ and $X = T^\theta$, we have

$$\begin{aligned} M_X(s) &= \sum_{n\le X} \frac{\mu(n)}{n^s}\left(1 + O\left(\frac{\log n}{\log T}\right)\right)\left(1 - \frac{\log n}{\log X}\right) \\ &= \sum_{n\le X} \frac{\mu(n)}{n^s}\left(1 + O\left(\frac{\log n}{\log T}\right)\right). \end{aligned} \tag{4.3}$$

Let $\sigma \ge 2$. Then, from (3.2), (3.18), (4.1), and (4.3)

$$f(s) \ll \left|\zeta(s)\sum_{n\le X}\frac{\mu(n)}{n^s} - 1\right| + \frac{r(m)}{\log T} \ll \sum_{n\ge X}\frac{d(n)}{n^\sigma} + \frac{r(m)}{\log T} \ll \frac{1}{\sqrt{X}} + \frac{r(m)}{\log T}$$

for $\frac{T}{2} \le t < T$ and $\log r(m) \ll m$. From now on $r(m)$ denotes a function of m, not necessarily same at each occurence, and $\log r(m) \ll m$. Therefore, for some $X > X_0, T > T_0$, $m \le g(T)$, and $\sigma \ge 2$

$$|f(s)| < \frac{1}{2}. \tag{4.4}$$

Combining (4.2) and (4.4), we find that $h(2+it) \ne 0$ for $t > T_0$ and $X \ge X_0$. Let $\nu(\sigma', T)$ denote the number of zeros of $h(s)$ in the rectangle $\sigma > \sigma'$ and $0 < t \le T$. By the Hardy-Littlewood Lemma (see [31, p. 221]), one has

$$2\pi\int_{\sigma_0}^{2}\nu\left(\sigma, \frac{T}{2}, T\right)\, d\sigma = \int_{T/2}^{T}\log|h(\sigma_0+it)|\, dt - \int_{T/2}^{T}\log|h(2+it)|\, dt + \int_{\sigma_0}^{2}\arg h(\sigma_0+iT)\, d\sigma - \int_{\sigma_0}^{2}\arg h(\sigma_0+iT/2)\, d\sigma, \tag{4.5}$$

where $\nu\left(\sigma, \frac{T}{2}, T\right) = \nu(\sigma, T) - \nu\left(\sigma, \frac{T}{2}\right)$ and $\sigma_0 \ge \frac{1}{2}$ is fixed. From (4.2) and (4.4), we deduce that

$$\mathrm{Re}(h(2+it)) \ge \frac{1}{2}$$

for $t \geq T_0$ and $x \geq X_0$. Since $\zeta^{(k)}(s) \leq t^A$ for some constant A

$$h(\sigma + it) \ll r(m) X^A t^A$$

for $\sigma \geq 0$ and sufficiently large t. Therefore, from Lemma 2.7, we have

$$\arg h(\sigma + iT) - \arg h\left(\sigma + i\frac{T}{2}\right) \ll \log X + \log T + \log r(M)$$

for $\sigma \geq \sigma_0$. This gives

$$\int_{\sigma_0}^{2} \arg h(\sigma + iT)\ d\sigma - \int_{\sigma_0}^{2} \arg h\left(\sigma + i\frac{T}{2}\right)\ d\sigma \ll \log X + \log T + \log r(m) \ll \log T \tag{4.6}$$

for $0 < \theta < 1$, $m \leq g(T)$ and $X = T^\theta$. From (3.2), (4.3), and for $\operatorname{Re} s > 1$

$$\begin{aligned} M_X(s) F_m(s) &= \zeta(s) M_X(s) + \sum_{j=1}^{m} c_j \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \zeta^{(j)}(s) M_X(s) \\ &= \sum_{n=1}^{\infty} \frac{a_X(n)}{n^s} + \sum_{j=1}^{m} c_j \frac{H^{(m-j)}(s)}{H^{(m)}(s)} \sum_{n=2}^{\infty} \frac{b_{j,X}(n)}{n^s}, \end{aligned} \tag{4.7}$$

where $a_X(1) = 1$,

$$a_X(n) = \sum_{d|n} \mu(d) \left(1 + O\left(\frac{\log d}{\log T}\right)\right) \ll \begin{cases} \frac{b(n)}{\log T}, & \text{if } 2 \leq n < X, \\ d(n) + \frac{b(n)}{\log T}, & \text{if } n \geq X, \end{cases} \tag{4.8}$$

and

$$b_{j,X}(n) = \sum_{d|n} \log^j\left(\frac{n}{d}\right) \mu(d) \left(1 + O\left(\frac{\log d}{\log T}\right)\right) \ll \begin{cases} \Lambda_j(n) + \frac{c(n)}{\log T}, & \text{if } 2 \leq n < X, \\ c_1(n) + \frac{c(n)}{\log T}, & \text{if } n \geq X. \end{cases} \tag{4.9}$$

Here, $d(n)$ denotes the divisor function,

$$b(n) = \sum_{d|n} \mu^2(d) \log d, \quad c_1(n) = \sum_{d|n} \log^j\left(\frac{n}{d}\right) \mu^2(d), \quad \text{and} \quad c(n) = \sum_{d|n} \log^j\left(\frac{n}{d}\right) \mu^2(d) \log d. \tag{4.10}$$

Therefore, for $\operatorname{Re} s > 1$,

$$\sum_{n=1}^{\infty} \frac{b(n)}{n^s} = \zeta(s) \left(\frac{\zeta(s)}{\zeta(2s)}\right)', \quad \sum_{n=1}^{\infty} \frac{c_1(n)}{n^s} = \zeta^{(j)}(s) \frac{\zeta(s)}{\zeta(2s)}, \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{c(n)}{n^s} = \zeta^{(j)}(s) \left(\frac{\zeta(s)}{\zeta(2s)}\right)'.$$

Since $h(s)$ is analytic for $\sigma \geq 2$ and $h(s) \to 1$ as $\sigma \to \infty$, by the residue theorem

$$\int_{T/2}^{T} \log h(2 + it)\ dt = \int_{2}^{\infty} \log h\left(\sigma + i\frac{T}{2}\right)\ d\sigma - \int_{2}^{\infty} \log h\left(\sigma + iT\right)\ d\sigma. \tag{4.11}$$

Also,

$$\log|h(s)| \leq \log\left(1 + |f(s)|^2\right) \leq |f(s)|^2 \tag{4.12}$$

and

$$\log|h(s)| = \operatorname{Re}(\log h(s)).$$

Using this along with (3.18), (4.7), (4.11), and (4.12) we have

$$\int_{T/2}^{T} \log|h(2 + it)|\ dt \ll \int_{2}^{\infty} |f(\sigma)|^2\ d\sigma \ll r(m). \tag{4.13}$$

Thus, it remains to estimate only the first integral in (4.5), which is done by using the convexity theorem. From (4.1), we find that

$$I_1 := \int_{T/2}^{T} \left| f\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 \, dt \ll \int_{T/2}^{T} \left| M_X F_m\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 \, dt + T.$$

From (3.18) and integrating by parts, we have

$$\int_{T/2}^{T} \left| M_X F_m\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 \, dt \sim \int_{T/2}^{T} \left| M_X V\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 \, dt,$$

where

$$V(s) = \zeta(s) + \sum_{k=1}^{m} \frac{2^k c_k}{\log^k T} \zeta^k(s).$$

Also, by Lemma 2.2,

$$\int_{T/2}^{T} \left| M_X V\left(\frac{1}{2} - \frac{R}{\log T} + it\right)\right|^2 \, dt \sim cT.$$

From [6] it can be seen that $\log c \ll m$. Hence $I_1 \ll r(m)T$.

Next, we compute the integral

$$I_2 := \int_{T/2}^{T} \left| f\left(1 + \delta + it\right) - 1\right|^2 \, dt = \int_{T/2}^{T} \left| M_X F_m\left(1 + \delta + it\right) - 1\right|^2 \, dt.$$

From (3.18) and (4.7)

$$I_2 \ll \int_{T/2}^{T} \left| \sum_{n=2}^{\infty} \frac{a_X(n)}{n^{1+\delta+it}}\right|^2 \, dt + \sum_{j=1}^{m} \frac{1}{\log^{2j} T} \int_{T/2}^{T} \left| \sum_{n=2}^{\infty} \frac{b_{j,X}(n)}{n^{1+\delta+it}}\right|^2 \, dt.$$

Employing Lemma 2.6, (4.8), (4.9), and (4.10), we have

$$I_2 \ll \frac{r(m)T}{\log^2 T}.$$

From an easy modification of the classical convexity theorem (see [31, p. 233]), one can deduce that

$$\int_{T/2}^{T} |f(\sigma_0 + it)|^2 \, dt \ll r(m) T \log^{1-2\sigma_0} T, \tag{4.14}$$

uniformly for $\frac{1}{2} - \frac{R}{\log T} \le \sigma_0 \le 1 + \delta$. From (4.12) and (4.14), we find that

$$\int_{T/2}^{T} \log |h(\sigma_0 + it)| \, dt \ll r(m) T \log^{1-2\sigma_0} T. \tag{4.15}$$

Combining (4.5), (4.6), (4.13), (4.15), and the inequality

$$\int_{\sigma_0}^{2} \nu\left(\sigma, \frac{T}{2}, T\right) \, d\sigma \ge \int_{\sigma_0}^{1} N\left(\sigma, \frac{T}{2}, T\right) \, d\sigma,$$

which follows from (4.2), we obtain

$$\int_{\sigma_0}^{1} N\left(\sigma, T\right) \, d\sigma - \int_{\sigma_0}^{1} N\left(\sigma, \frac{T}{2}\right) \, d\sigma \ll r(m) T \log^{1-2\sigma_0} T$$

uniformly for $\frac{1}{2} \le \sigma_0 \le 1$. Now, we replace T by $T/2^n, n \ge 0$, in the above estimate, and sum over n for $0 \le n \le \infty$ to complete the proof of Theorem 1.5.

5. Uniform distribution and Discrepancy Bounds: Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. We start with the identity

$$\sum_{0\le\gamma\le T} x^{i\gamma} = \sum_{0\le\gamma\le T} x^{\rho-1/2} + \sum_{0\le\gamma\le T}\left(x^{i\gamma} - x^{\rho-1/2}\right), \tag{5.1}$$

which holds for any x. Let $x = e^{2\pi\alpha}$, where $\alpha > 0$ is any fixed real number. From (1.10), it can be shown that the non-trivial zeros of $\xi^{(m)}(s)$ are symmetric respect to the line $\sigma = 1/2$. Therefore,

$$\begin{aligned}\sum_{0\le\gamma\le T}\left(x^{i\gamma} - x^{\rho-1/2}\right) &\ll \sum_{\substack{0\le\gamma\le T\\ \beta>1/2}}\left|1 - x^{\beta-1/2}\right|\\ &\ll \sqrt{x}\log x \sum_{\substack{0\le\gamma\le T\\ \beta>1/2}}(\beta - 1/2)\\ &= \sqrt{x}\log x\int_{\frac{1}{2}}^{1} N_m(\sigma, T)\; d\sigma,\end{aligned}$$

where in the penultimate step, we use the mean value theorem. Combining this with Theorem 1.5, we find that

$$\sum_{0\le\gamma\le T}\left(x^{i\gamma} - x^{\rho-1/2}\right) \ll C\sqrt{x}T\log x, \tag{5.2}$$

where $\log C \ll m$. Let T be large enough such that

$$\log x \le \frac{\log T}{\log\log T}.$$

For $x > 1$, from Theorem 1.3, we have

$$\begin{aligned}\sum_{0\le\gamma\le T} x^{\rho-1/2} \ll \frac{T\log x}{\sqrt{x}} + \sqrt{x}\log(2xT)\log x + C\sqrt{x}T\log x + \sqrt{x}(C\log x)^{K+2}\\ + \sqrt{x}T\left(\frac{C\log\log T}{\log T}\right)^{\frac{K+1}{3}},\end{aligned} \tag{5.3}$$

where $\log C \ll m$. Combining the above estimates along with (1.6), (5.1), and (5.2), we have

$$\frac{1}{N_m(T)}\sum_{0\le\gamma\le T} x^{i\gamma} = o(1)$$

as $T \to \infty$ and uniformly for $m \le g(T)$. A similar result also holds for $0 < x < 1$. In this case, we first use (1.11) on the left side of (5.3), and then apply Theorem 1.3.

Invoking the Weyl criterion, Lemma 2.4, we conclude that the sequence $(\alpha\gamma)$ is uniformly distributed modulo one. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. case i): Unconditional bound

From Lemma 2.5, (1.6), (5.1), (5.2), and (5.3), we have

$$\begin{aligned}
D^*(\alpha;T) &\ll \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \frac{1}{k} \left| \sum_{0\le\gamma\le T} x^{ik\gamma} \right| \\
&\ll \frac{1}{M+1} + \frac{1}{T\log T} \sum_{k=1}^{M} \Bigg(T\frac{\log x}{x^{k/2}} + x^{k/2}\log T \log x + T\frac{k\log^2 x}{x^{k/2}\log T} + \frac{x^{k/2}}{k}(Ck\log x)^{K+2} \\
&\qquad + \frac{x^{k/2}}{k} T \left(\frac{C\log\log T}{\log T} \right)^{K+1} + Cx^{k/2} T\log x \Bigg) \\
&\ll \frac{1}{M+1} + \frac{1}{T\log T} \Bigg(T\frac{\log x}{x^{1/2}} + Mx^{M/2}\log T\log x + T\frac{M\log^2 x}{x^{1/2}\log T} + x^{M/2}(CM\log x)^{K+2} \\
&\qquad + x^{M/2} T\log M \left(\frac{C\log\log T}{\log T} \right)^{K+1} + CMx^{M/2}T\log x \Bigg),
\end{aligned}$$

where K is any fixed positive integer and $\log C \ll m$. Now, we set

$$M = \left\lfloor \frac{\log\log T}{\log x} \right\rfloor.$$

Hence, we deduce that

$$D^*(\alpha;T) \le \frac{a_1 \log x}{\log\log T} + \frac{e^{a_2 m}\log\log T}{\sqrt{\log T}},$$

where a_1 and a_2 are absolute constants, holds uniformly for $0 \le m \le g(T)$.

case ii): Assuming the Riemann hypothesis

Let $\beta = \frac{1}{2}$. Then, from Lemma 2.5 and Theorem 1.3, we have

$$\begin{aligned}
D^*(\alpha;T) &\ll \frac{1}{M+1} + \frac{1}{N_m(T)} \sum_{k=1}^{M} \frac{1}{k} \left| \sum_{0\le\gamma\le T} x^{ik\gamma} \right| \\
&\ll \frac{1}{M+1} + \frac{1}{T\log T} \sum_{k=1}^{M} \Bigg(T\frac{\log x}{x^{k/2}} + x^{k/2}\log T \log x + T\frac{k\log^2 x}{x^{k/2}\log T} + \frac{x^{k/2}}{k}(Ck\log x)^{K+2} \\
&\qquad + \frac{x^{k/2}}{k} T \left(\frac{C\log\log T}{\log T} \right)^{K+1} \Bigg) \\
&\ll \frac{1}{M+1} + \frac{1}{T\log T} \Bigg(T\frac{\log x}{x^{1/2}} + Mx^{M/2}\log T\log x + T\frac{M\log^2 x}{x^{1/2}\log T} + x^{M/2}(CM\log x)^{K+2} \\
&\qquad + x^{M/2} T\log M \left(\frac{C\log\log T}{\log T} \right)^{K+1} \Bigg),
\end{aligned}$$

where $\log C \ll m$. Set

$$M = \left\lfloor \frac{\log T}{\log x} \right\rfloor \quad \text{and} \quad K = \left\lfloor \frac{\log T}{\log\log T} \right\rfloor.$$

Therefore, we obtain

$$D^*(\alpha;T) \le \frac{c_1 \log x}{\log T} + \exp\left(\frac{c_2 m \log T}{\log\log T} \right) \frac{(\log\log T)^2}{T^{1/3}(\log T)^2},$$

where c_1 and c_2 are absolute constants, holds uniformly for $0 \le m \le g(T)$. This completes the proof of Theorem 1.2. $\square$

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