SMOOTH L^2 DISTANCES AND ZEROS OF APPROXIMATIONS OF DEDEKIND ZETA FUNCTIONS

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ABSTRACT. We consider a family of approximations of the Dedekind zeta function $\zeta_K(s)$ of a number field $K/\mathbb{Q}$. Weighted L^2-norms of the difference of two such approximations of $\zeta_K(s)$ are computed. We work with a weight which is a compactly supported smooth function. Mean square estimates for the difference of approximations of $\zeta_K(s)$ can be obtained from such weighted L^2-norms. Some results on the location of zeros of a family of approximations of Dedekind zeta functions are also derived. These results extend results of Gonek and Montgomery on families of approximations of the Riemann zeta-function.

1. Introduction

In [4], Hardy and Littlewood provided an approximate functional equation for the Riemann zeta-function

$$\zeta(s) = \sum_{n\le X}\frac{1}{n^s} + \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\sum_{n\le Y}\frac{1}{n^{1-s}} + O(X^{-\sigma}) + O(Y^{\sigma-1}|t|^{-\sigma+1/2}),$$

where $s=\sigma+it$, $0\le\sigma\le 1$, $X>H>0$, $Y>H>0$, and $2\pi XY=|t|$, with the constant implied by the big-O term depending on H only. Define

$$\zeta_X(s) := F_X(s) + \chi(s)F_X(1-s),$$

where $F_X(s) := \sum_{n\le X} n^{-s}$ and $\chi(s) := \pi^{s-1/2}\Gamma((1-s)/2)/\Gamma(s/2)$. Denote $s=\sigma+it$ throughout the paper. Then for $X=\sqrt{|t|/2\pi}$, one form of the above approximate functional equation asserts that

$$\zeta(s) = \zeta_X(s) + O(|t|^{-\sigma/2}), \tag{1.1}$$

where $|t|\ge 1$ and $|\sigma-1/2|<1/2$. Spira [12, 13] independently studied the family of approximations $\zeta_X(s)$. Similar to $\zeta(s)$, the approximations $\zeta_X(s)$ satisfy the functional equation

$$\zeta_X(s) = \chi(s)\zeta_X(1-s).$$

More interestingly, in [12], Spira proved that all the complex zeros of $\zeta_1(s)$ and $\zeta_2(s)$ lie on the line $\sigma=1/2$. In other words $\zeta_1(s)$ and $\zeta_2(s)$ satisfy the Riemann hypothesis. Spira [13] also showed that if $X<\sqrt{t/2\pi}<X+1$, then $\zeta_X(s)$ approximates $\zeta(s)$. This can be shown easily from (1.1). In the same article, Spira also made an assertion that $2\zeta(s)$ can be approximated well by $\zeta_X(s)$ in the region $\sqrt{2\pi X}\le t\le 2\pi X$. Gonek and Montgomery [3] gave a clever proof of Spira's statement. They showed that the condition $\sqrt{2\pi X}\le t\le 2\pi X$ is not adequate to prove Spira's assertion, it is also needed that σ be near $1/2$ in order

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to complete the proof. More precisely they showed that for a fixed $\lambda \in (0,1)$, $C > 1$, $0 < \sigma_0 \le \sigma < 1$, and $X^\lambda \le |t| \le (2\pi/C)X$,

$$\zeta_X(s) = 2\zeta(s) + o(1),$$

where

$$\max(\sigma_0, 1-\lambda) < \sigma < \min\left(1, \frac{\lambda}{2(1-\lambda)}\right).$$

The above inequality is nonempty if and only if $\sigma > 1/2$. They also deduced that for $\sigma = 1/2$ and $1 < |t| \le (2\pi/C)X$,

$$\zeta_X\left(\frac{1}{2}+it\right) = 2\zeta\left(\frac{1}{2}+it\right) + O(X^{1/2}|t|^{-1}) + O(|t|^{-1/2}),$$

which shows how the transition to a good approximation takes place as $|t|$ increases past $X^{1/2}$. These motivate one to investigate such approximations in more generality. A natural question that arises is how the sequence $\zeta_N(1/2+it)$ converges in the L^2-norm. In particular we are interested in studying the integral

$$\int_0^T \left|\zeta_N\left(\frac{1}{2}+it\right) - \zeta_M\left(\frac{1}{2}+it\right)\right|^2 dt. \tag{1.2}$$

We will study such questions for a Dedekind zeta function of a number field. Since the case of the Riemann zeta-function follows from the case of a Dedekind zeta function, we will focus on Dedekind zeta functions throughout the paper.

Let $K/\mathbb{Q}$ be a number field. For $\mathrm{Re}(s) > 1$ the Dedekind zeta function $\zeta_K(s)$ of K is given by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \tag{1.3}$$

where the first sum is taken over all nonzero integral ideals $\mathfrak{a}$ of K and where $N(\mathfrak{a})$ denotes the absolute norm of $\mathfrak{a}$. In the second sum, $a(n)$ denotes the number of integral ideals $\mathfrak{a}$ with norm $N(\mathfrak{a}) = n$.

The function $\zeta_K(s)$ is analytic everywhere except for a simple pole at $s = 1$ (see Neukrich [9]). The residue at this pole is given by

$$H(K) = \mathrm{Res}_{s=1}\left(\zeta_K(s)\right) = \frac{2^r \pi^{n_0 - r} R_K h_K}{w_K\sqrt{|d_K|}},$$

where $r = r_1 + r_2$ (with r_1 the number of real embeddings and r_2 the number of pairs of complex embeddings of K), $n_0 = [K\colon \mathbb{Q}]$ denotes the degree of $K/\mathbb{Q}$, R_K denotes the regulator, h_K denotes the class number, w_K denotes the number of roots of unity in K, and d_K denotes the discriminant of K (see [9, p. 467]). The function $\zeta_K(s)$ satisfies the functional equation

$$\zeta_K(s) = B(s)\zeta_K(1-s),$$

where

$$B(s) = \left(\frac{2^{2r_2}\pi^{n_0}}{|d_K|}\right)^{s-\frac{1}{2}} \frac{\Gamma^{r_1}(\frac{1-s}{2})\Gamma^{r_2}(1-s)}{\Gamma^{r_1}(\frac{s}{2})\Gamma^{r_2}(s)}. \tag{1.4}$$

The coefficients of the Dirichlet series of $\zeta_K(s)$ satisfy the bound

$$a(n) \le (d(n))^{n_0-1} \ll n^{\epsilon}, \tag{1.5}$$

where $\epsilon > 0$ and $d(n)$ is the divisor function. This can be seen through the Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} (1 - N(\mathfrak{p})^{-s})^{-1} = \prod_{p} \prod_{\mathfrak{p}|p} (1 - N(\mathfrak{p})^{-s})^{-1} \tag{1.6}$$

for $\mathrm{Re}(s) > 1$ (see Chandrasekharan and Narasimhan [1, Lemma 9]). One also has the estimate

$$\sum_{n \le x} a(n) = H(K)x + O(x^{1-1/n_0}) \tag{1.7}$$

(see Marcus [8, p. 159]). For the second discrete moments we have

$$\sum_{n \le x} a(n)^2 \ll x \log^{n_0-1} x$$

for any number field K. If $K/\mathbb{Q}$ is a Galois extension, then we have

$$\sum_{n \le x} a(n)^2 \sim C(K) x \log^{n_0-1} x, \tag{1.8}$$

where $C(K)$ is a constant that depends on the field K (see [1, Theorem 3]).

In view of (1.3), we define a partial sum of $\zeta_K(s)$ by

$$F_{K,X}(s) := \sum_{n \le X} \frac{a(n)}{n^s}.$$

If we denote

$$\zeta_{K,X}(s) := F_{K,X}(s) + B(s) F_{K,X}(1-s), \tag{1.9}$$

then one form of the approximate functional equation of $\zeta_K(s)$ gives

$$\zeta_K(s) = \zeta_{K,X}(s) + O_K\left(|t|^{\frac{n_0}{2}(1-\sigma-\frac{1}{n_0})} \log t \right),$$

where $X = \sqrt{|d_K|}(|t|/2\pi)^{n_0/2}$, $|t| \ge 1$ and $|\sigma - 1/2| < 1/2$ (see [1]). For a given number field K, it is easy to see that $\zeta_K(s)$ can be approximated by $\zeta_{K,X}(s)$ in the region $X < \sqrt{|d_K|}(|t|/2\pi)^{n_0/2} < X + 1$ and $\sigma > 1 - 1/n_0$. We wish to obtain an asymptotic of the moment integral (1.2) for the family of approximations $\zeta_{K,X}(s)$. We will obtain this in a slightly different way. In particular, we will estimate the L^2 distance between $\zeta_{K,M}(s)$ and $\zeta_{K,N}(s)$, weighted by a smooth function which satisfies certain conditions.

More specifically, let $h(t)$ be a smooth function with the following properties:

(1) $0 \le h(t) \le 1$ for all $t \in \mathbb{R}$,
(2) $h(t)$ is compactly supported in a subset of $(0, \infty)$,
(3) $\|h^{(j)}(t)\|_\infty \ll_j 1$ for each $j = 0, 1, 2, \ldots$.

The Fourier transform of $h(t)$ is denoted by $\hat{h}(s)$. Our first result is as follows.

Theorem 1.1. *Let $K/\mathbb{Q}$ be a Galois extension of degree n_0. Let h be a smooth function satisfying (1)-(3). Then for any fixed $\epsilon_0 > 0$ and $T^{\epsilon_0} \le N \le M \le T^{n_0/2-\epsilon_0}$, we have*

$$\int_{-\infty}^{\infty} h\left(\frac{t}{T}\right)\left|\zeta_{K,N}\left(\frac{1}{2}+it\right) - \zeta_{K,M}\left(\frac{1}{2}+it\right)\right|^2 dt \sim 2T\frac{\hat{h}(0)}{n_0}C(K)(\log^{n_0} M - \log^{n_0} N),$$

where $C(K)$ is defined in (1.8).

Remark: In the special case $K = \mathbb{Q}$ it is known that $C(K) = 1$. Hence for the above family of approximations of the Riemann zeta-function one has

$$\int_{-\infty}^{\infty} h\left(\frac{t}{T}\right)\left|\zeta_{N}\left(\frac{1}{2}+it\right) - \zeta_{M}\left(\frac{1}{2}+it\right)\right|^2 dt \sim 2T\hat{h}(0)\log\frac{M}{N}$$

for $T^{\epsilon_0} \le N \le M \le T^{1/2-\epsilon_0}$.

Although $\zeta_1(s)$ and $\zeta_2(s)$ satisfy the Riemann hypothesis, in [13] Spira numerically showed that there are infinitely many zeros off the critical line for $\zeta_3(s)$. In their paper [3], Gonek and Montgomery studied the behavior of zeros of $\zeta_X(s)$ for larger X. They established a zero free region for $\zeta_X(s)$. In the same paper they gave an asymptotic for the number of zeros of $\zeta_X(s)$ in a rectangular box and provided a lower bound for the number of zeros on a segment of the critical line. More strikingly, they showed that almost all the zeros of $\zeta_X(s)$ lie on the critical line and are simple. Such results are sensitive to the length X of the truncated sum $F_X(s)$. In [7], the first and the last two authors provide analogues of some results from [3] to an L-function associated with a cusp form. In that case the length N can be taken to be shorter. We now present some similar results for $\zeta_{K,X}(s)$.

By the reflection and duplication formulas for the gamma function one obtains

$$B(s)B(1-s) = 1. \tag{1.10}$$

Combining (1.9) and (1.10) we have the functional equation

$$\zeta_{K,X}(s) = B(s)\zeta_{K.X}(1-s). \tag{1.11}$$

Note that $\Gamma(s)$ has simple poles at $s = 0, -1, -2, -3, \dots$. Therefore $\zeta_{K,X}(s)$ has zeros at negative odd integers with order r_2 and at non-positive even integers with order $r_1 + r_2$. From the functional equation (1.11), we find that the nontrivial zeros are symmetric about the critical line $\sigma = \frac{1}{2}$. Moreover $\zeta_{K,X}(s)$ is real on the real line, so the zeros of $\zeta_{K,X}(s)$ are symmetric with respect to the real axis as well. Our next results are concerned with properties of the nontrivial zeros of $\zeta_{K,X}(s)$. One result in this direction is an inequality which gives an equivalent condition for the zeros on the critical line.

Theorem 1.2. *Let $K/\mathbb{Q}$ be a number field and X be a positive integer. Then $|\zeta_{K,X}(1-s)| > |\zeta_{K,X}(s)|$ holds for all s with $|t| > 40$ and $1/2 < \sigma < 1$, if and only if all the zeros $\beta + i\gamma$ of $\zeta_{K,X}(s)$ with $\beta \in (0,1)$ and $|\gamma| > 40$ lie on the critical line.*

Our next result concerns the cases for which $X = 1, 2$.

Theorem 1.3. *Let $K/\mathbb{Q}$ be a number field. Then all the zeros of $\zeta_{K,1}(s)$ for $|t| > 40$ lie on the critical line. Moreover, if $|a_2| \le 1$ then there exists a constant $t(K)$ such that all the zeros of $\zeta_{K,2}(s)$ lie on the critical line for $|t| > t(K)$.*

We now provide a zero free region for $\zeta_{K,X}(s)$.

Theorem 1.4. *Let $K/\mathbb{Q}$ be a Galois extension of degree n_0 and $\lambda > 1/n_0$. Let $\rho_{K,X} = \beta_{K,X} + i\gamma_{K,X}$ be a zero of $\zeta_{K,X}(s)$. There exists a constant X_0 such that if $X > X_0$ and $|\gamma_{K,X}| \geq 2\pi X^{\lambda}$, then*

$$\left|\beta_{K,X} - \frac{1}{2}\right| < \frac{1}{n_0\lambda - 1}\left(\frac{1}{2} + \frac{n_0^2\lambda \log\log X}{\log X}\right) \quad \text{for } \ 1/n_0 < \lambda < 2/n_0$$

and

$$\left|\beta_{K,X} - \frac{1}{2}\right| < \frac{1}{2} + \frac{2n_0 \log\log X}{\log X} \quad \text{for } \ \lambda \geq 2/n_0.$$

Moreover, there exists a constant T_0 such that if $X \geq 1$ and $\gamma_{K,X} \geq \max(2\pi X^{2/n_0}, T_0)$, then

$$\left|\beta_{K,X} - \frac{1}{2}\right| < n_0 + 2.$$

Next, we introduce some notations to study the number of zeros of $\zeta_{K,X}(s)$ in a rectangular box containing the critical strip. We denote

$$N_{K,X}(T) := \#\{\rho = \sigma + i\gamma : \ \zeta_{K,X}(\rho) = 0 \ \text{ and } \ 0 < \gamma \leq T\}$$

and

$$N_{K,X}^0(T) = \#\{\rho = \frac{1}{2} + i\gamma : \ \zeta_{K,X}(\rho) = 0 \ \text{ and } \ 0 < \gamma \leq T\}.$$

We have the following asymptotic result.

Theorem 1.5. *Let $K/\mathbb{Q}$ be a Galois extension of degree n_0 and d_K be the discriminant. Let λ be a constant with $\lambda > 1/n_0$. There exists a constant X_0 such that if $X > X_0$, $T \geq 2\pi X^{\lambda}$, and $U \geq 2$, then*

$$\begin{aligned} N_{K,X}(T+U) - N_{K,X}(T) =& n_0\frac{T+U}{2\pi}\log\frac{T+U}{2\pi} - n_0\frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{U}{2\pi}(n_0 - \log|d_k|) \\ &+ O_K\left(\frac{\lambda}{n_0\lambda - 1}\log(T+U)\right). \end{aligned} \tag{1.12}$$

Furthermore, there exists a constant T_0 such that if $X \geq 1$ with $\gamma_{K,X} \geq \max(2\pi X^{2/n_0}, T_0)$, then (1.12) holds with λ replaced by 1 in the big-O term.

We also have the following lower bound.

Theorem 1.6. *The number of zeros of $F_{K,X}$ with imaginary part in $(0, T]$ and real part greater than or equal to $\frac{1}{2}$ is*

$$\leq \frac{aT}{2\pi}\log M + O_K(X),$$

where $0 \leq a \leq 1$ and M is the largest integer less than or equal to X such that $a(M) \neq 0$. There exists a constant T_0 such that if $X \geq 1$, $T \geq \max(2\pi X^{2/n_0}, T_0)$, and $U \geq 2$, then

$$\begin{aligned} N_{K,X}^0(T+U) - N_{K,X}^0(T) \geq n_0\frac{T+U}{2\pi}\log\frac{T+U}{2\pi M^{2a/n_0}} - n_0\frac{T}{2\pi}\log\frac{T}{2\pi M^{2a/n_0}} \\ - \frac{U}{2\pi}(n_0 - \log|d_K|) + O_K(X). \end{aligned}$$

Furthermore the quantity on the right-hand side is a lower bound for the number of simple zeros on the corresponding segment of the critical line.

If one chooses $1 \le X \le T^{o(1)}$ and $T > T_0$, with T_0 the same constant as in Theorem 1.4, then by Theorems 1.5 and 1.6 we have

$$N^0_{K,X}(T+U) - N^0_{K,X}(T) \ge N_{K,X}(T+U) - N_{K,X}(T) + O_K(U \log M) + O_K(X).$$

Choosing $U \ge T^\beta$ for some positive constant β and $X \le T^{o(1)}$, one obtains

$$\liminf_{T\to\infty} \frac{N^0_{K,X}(T+U) - N^0_{K,X}(T)}{N_{K,X}(T+U) - N_{K,X}(T)} = 1.$$

This means that as $T \to \infty$, almost all the zeros of $\zeta_{K,X}(s)$ with imaginary part in $(0,T]$ lie on the critical line for $X \le T^{o(1)}$. Also by the last part of Theorem 1.6, one finds that almost all such zeros are simple.

2. Preliminary Results

In this section we gather all the necessary ingredients to prove our results. The following inequality is due to Gonek and Montgomery [3, Lemma 2.1].

Lemma 2.1. *Let $\sigma > 1$. Then we have*

$$\frac{\sigma-1}{\sigma} < |\zeta(s)| < \frac{\sigma}{\sigma-1}.$$

These results are also required and important in their own right.

Lemma 2.2. *Let $K/\mathbb{Q}$ be a Galois extension of degree n_0. Then for $\sigma > 1$ we have*

$$\left(\frac{\sigma-1}{\sigma}\right)^{n_0} < |\zeta_K(s)| < \left(\frac{\sigma}{\sigma-1}\right)^{n_0}.$$

Proof. Since $K/\mathbb{Q}$ is a Galois extension, for each rational prime p we have the decomposition

$$p\mathcal{O}_K = (\mathfrak{p}_1 \ldots \mathfrak{p}_\mathfrak{r})^\mathfrak{e} \quad \text{with} \quad N(\mathfrak{p}_\mathfrak{i}) = p_i^\mathfrak{f},$$

where $\mathfrak{e}$ is the ramification index, $\mathfrak{f}$ is the inertial degree, and $\mathfrak{r}$ is the decomposition index. Also $\mathfrak{efr} = n_0$. Thus, for each prime p we have

$$\prod_{\mathfrak{p}|p}(1-N(\mathfrak{p})^{-\sigma})^{-1} = (1-p^{-\mathfrak{f}\sigma})^{-\mathfrak{r}} = \left(\sum_{k=0}^\infty p^{-k\mathfrak{f}\sigma}\right)^\mathfrak{r} \ge \sum_{k=0}^\infty p^{-k\mathfrak{fr}\sigma} \ge \sum_{k=0}^\infty p^{-kn_0\sigma} = (1-p^{-n_0\sigma})^{-1}$$

and

$$\prod_{\mathfrak{p}|p}(1-N(\mathfrak{p})^{-\sigma})^{-1} = \left(\sum_{k=0}^\infty p^{-k\mathfrak{f}\sigma}\right)^\mathfrak{r} \le \left(\sum_{k=0}^\infty p^{-k\sigma}\right)^{\mathfrak{fr}} \le \left(\sum_{k=0}^\infty p^{-k\sigma}\right)^{n_0} = (1-p^{-\sigma})^{-n_0}.$$

Combining the above two inequalities with the Euler product (1.6) we have

$$\zeta(n_0\sigma) \le \zeta_K(\sigma) \le \zeta(\sigma)^{n_0}. \tag{2.1}$$

Also by the triangle inequality one has

$$\left|\prod_{\mathfrak{p}}(1-N(\mathfrak{p})^{-s})^{-1}\right| \ge \prod_{p}(1+N(\mathfrak{p})^{-\sigma})^{-1} \ge \prod_{p} \frac{(1-N(\mathfrak{p})^{-2\sigma})^{-1}}{(1-N(\mathfrak{p})^{-\sigma})^{-1}} = \frac{\zeta_K(2\sigma)}{\zeta_K(\sigma)}. \tag{2.2}$$

Combining (2.1) and (2.2) we find

$$\frac{\zeta(2n_0\sigma)}{\zeta(\sigma)^{n_0}} \le |\zeta_K(s)| \le \zeta(\sigma)^{n_0}.$$

Using Lemma 2.1 we see that

$$\left(\frac{\sigma-1}{\sigma}\right)^{n_0} \leq |\zeta_K(s)| \leq \left(\frac{\sigma}{\sigma-1}\right)^{n_0},$$

which completes the proof. □

Lemma 2.3. *Let $K/\mathbb{Q}$ be a number field of degree n_0. We have the following estimates:*

a) Let $\sigma > 1$. Then

$$\left|\sum_{n>X}\frac{a(n)}{n^s}\right| \leq \frac{H(K)}{\sigma-1}X^{1-\sigma} + O_K\left(X^{1-\sigma-1/n_0}\right).$$

b) Let $\sigma \leq 0$. Then

$$\left|\sum_{n\leq X}\frac{a(n)}{n^s}\right| \leq \frac{H(K)}{1-\sigma}X^{1-\sigma} + O_K\left(X^{1-\sigma-1/n_0}\right).$$

Proof. Let $\sigma > 1$ and define $I(x) := \sum_{n\leq x} a(n)$. From (1.7) and by partial summation we have

$$\begin{aligned}\left|\sum_{n>X}\frac{a(n)}{n^s}\right| \leq \sum_{n>X}\frac{a(n)}{n^\sigma} &= \sigma\int_X^\infty I(t)t^{-1-\sigma}\,dt - I(X)X^{-\sigma}\\ &= \frac{H(K)}{\sigma-1}X^{1-\sigma} + O_K\left(X^{1-\sigma-1/n_0}\right).\end{aligned}$$

For the second part of the lemma we take $\sigma \leq 0$. Note that

$$\left|\sum_{n\leq X}\frac{a(n)}{n^s}\right| \leq \sum_{n\leq X}\frac{a(n)}{n^\sigma}.$$

By using (1.7) one can see that

$$\begin{aligned}\sum_{n\leq X}\frac{a(n)}{n^\sigma} &= \sigma\int_1^X I(t)t^{-1-\sigma}\,dt + I(X)X^{-\sigma} - 1\\ &= \frac{H(K)}{-\sigma+1}X^{-\sigma+1} + O_K\left(X^{1-\sigma-1/n_0}\right),\end{aligned}$$

which completes the proof of the lemma. □

Lemma 2.4. *Let $K/\mathbb{Q}$ be a number field of degree n_0. Then for $|t| > 10$ and $\sigma > 1/2$ we have*

$$\frac{\partial}{\partial\sigma}\left(\log\frac{1}{|B(s)|}\right) > n_0(\log|s| - 2.55468).$$

Proof. By Stirling's formula (see [2]) we have

$$\log\Gamma(s) = (s-1/2)\log s - s + \frac{1}{2}\log 2\pi + \frac{1}{12s} - 2\int_0^\infty \frac{P_3(x)}{(s+x)^3}\,dx, \tag{2.3}$$

with $P_3(x)$ a periodic function of period 1. For $x \in [0,1]$ it is given by

$$P_3(x) = \frac{x}{12}(2x^2 - 3x + 1).$$

It is straightforward that for $x \in [0,1]$,

$$|6P_3(x)| \leq \frac{\sqrt{3}}{36}. \tag{2.4}$$

From (1.4) and (2.3), one has

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\bigg(\log\frac{1}{|B(s)|}\bigg) &= \mathrm{Re}\left(\frac{\partial}{\partial s}\log\frac{1}{B(s)}\right)\\
&= \mathrm{Re}\bigg(-\frac{r_1}{2s}+\frac{r_1}{2(s-1)}-\frac{r_1}{6s^2}-\frac{r_1}{6(s-1)^2}+\frac{r_1}{2}\log\frac{s}{2}+\frac{r_1}{2}\log\frac{(1-s)}{2}\\
&\quad -\frac{r_2}{2s}+\frac{r_2}{2(s-1)}-\frac{r_2}{12s^2}-\frac{r_2}{12(s-1)^2}+r_2\log s+r_2\log(1-s)\\
&\quad -\log\left(\frac{2^{2r_2}\pi^{n_0}}{d_K}\right)+3r_1\int_0^\infty\frac{P_3(x)}{(s/2+x)^4}\,dx\\
&\quad +3r_1\int_0^\infty\frac{P_3(x)}{((1-s)/2+x)^4}\,dx\\
&\quad +6r_2\int_0^\infty\frac{P_3(x)}{(s+x)^4}\,dx+6r_2\int_0^\infty\frac{P_3(x)}{(1-s+x)^4}\,dx\bigg)\\
&= \mathrm{Re}\bigg(\frac{r_1+r_2}{2s(s-1)}-\frac{2r_1+r_2}{12s^2}-\frac{2r_1+r_2}{12(s-1)^2}+\frac{n_0}{2}\log s(1-s)\\
&\quad -\log\left(\frac{(2\pi)^{n_0}}{d_K}\right)+3r_1\int_0^\infty\frac{P_3(x)}{(s/2+x)^4}\,dx+3r_1\int_0^\infty\frac{P_3(x)}{((1-s)/2+x)^4}\,dx\\
&\quad +6r_2\int_0^\infty\frac{P_3(x)}{(s+x)^4}\,dx+6r_2\int_0^\infty\frac{P_3(x)}{(1-s+x)^4}\,dx\bigg).
\end{aligned}$$

Thus we have

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\bigg(\log\frac{1}{|B(s)|}\bigg) &= \mathrm{Re}\left(\frac{\partial}{\partial s}\log\frac{1}{B(s)}\right)\\
&\geq \frac{n_0}{2}\log|s(1-s)|-\frac{r_1+r_2}{2|s(s-1)|}-\frac{2r_1+r_2}{12|s|^2}-\frac{2r_1+r_2}{12|s-1|^2}\\
&\quad -\log\left(\frac{(2\pi)^{n_0}}{|d_K|}\right)-\frac{(4r_1+r_2)\sqrt{3}}{108}\bigg(\frac{1}{|s|^3}+\frac{1}{|1-s|^3}\bigg).
\end{aligned} \tag{2.5}$$

Choose $|s|>2$. Then clearly $|1-s|>|s|-1>|s|/2$. Applying this to (2.5), one finds

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\bigg(\log\frac{1}{|B(s)|}\bigg) &\geq \frac{n_0}{2}\log\frac{|s|^2}{2}-\frac{r_1+r_2}{|s|^2}-\frac{2r_1+r_2}{3|s|^2}-\frac{2r_1+r_2}{3|s|^2}\\
&\quad -\log\left(\frac{(2\pi)^{n_0}}{|d_K|}\right)-\frac{(4r_1+r_2)\sqrt{3}}{108}\bigg(\frac{5}{|s|^3}\bigg)\\
&\geq n_0\left(\log|s|-\frac{7}{3|s|^2}-\frac{5\sqrt{3}}{27|s|^3}-\log(4\pi)\right)+\log|d_K|.
\end{aligned}$$

Using the Minkowski bound

$$|d_K|^{\frac{1}{2}} \geq \frac{n_0^{n_0}}{n_0!}\left(\frac{\pi}{4}\right)^{n_0} \tag{2.6}$$

and $|s| > 10$, we conclude that

$$\frac{\partial}{\partial\sigma}\left(\log\frac{1}{|B(s)|}\right) \geq n_0(\log|s| - 2.55468).$$

This proves the lemma. □

Lemma 2.5. *Let $K/\mathbb{Q}$ be a number field of degree n_0 and let d_K be the discriminant. If $|t| > 20n_0$ and $\sigma > 1/2$, then*

$$|B(s)| < 1.03\left(\frac{|s|}{2\pi e}\right)^{n_0(\frac{1}{2}-\sigma)}|d_K|^{\frac{1}{2}-\sigma}.$$

Proof. From [14] we have

$$|\Gamma(s)| = (2\pi)^{1/2}e^{-\sigma}|s|^{\sigma-1/2}e^{-t\arg s}|\exp(R_1(s) + 1/12s)|, \tag{2.7}$$

with $R_1(s) < \frac{1}{6|s|}$. Using (2.7), we find

$$\begin{aligned}\left|\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\right| &= (2e)^{\sigma-\frac{1}{2}}|1-s|^{-\frac{\sigma}{2}}|s|^{-\frac{\sigma}{2}+\frac{1}{2}} \\ &\times\exp\left\{\frac{t}{2}\arg s(1-s) + R_1\left(\frac{1-s}{2}\right) - R_1\left(\frac{s}{2}\right) + \frac{1}{6(1-s)} - \frac{1}{6s}\right\}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\left|\frac{\Gamma(1-s)}{\Gamma(s)}\right| &= e^{2\sigma-1}|1-s|^{\frac{1}{2}-\sigma}|s|^{\frac{1}{2}-\sigma} \\ &\times\exp\left\{t\arg s(1-s) + R_1(1-s) - R_1(s) + \frac{1}{12(1-s)} - \frac{1}{12s}\right\}.\end{aligned} \tag{2.9}$$

Hence by (1.4), (2.8) and (2.9) we deduce

$$\begin{aligned}|B(s)| = |d_K|^{\frac{1}{2}-\sigma}&\left(\frac{|s|}{2\pi e}\right)^{n_0(\frac{1}{2}-\sigma)}\exp\left(\frac{n_0 t}{2}\arg(s(1-s))\right)\times \\ &\left|1-\frac{1}{s}\right|^{\frac{r_2-n_0\sigma}{2}}\frac{|\exp(r_1R_1(\frac{1-s}{2}) + r_2R_1(1-s) + \frac{2r_1+r_2}{12(1-s)}|}{|\exp(r_1R_1(\frac{s}{2}) + r_2R_1(s) + \frac{2r_1+r_2}{12s}|}.\end{aligned} \tag{2.10}$$

Next we denote

$$z := r_1R_1\left(\frac{1-s}{2}\right) + r_2R_1(1-s) + \frac{2r_1+r_2}{12(1-s)} - r_1R_1\left(\frac{s}{2}\right) + r_2R_1(s) + \frac{2r_1+r_2}{12s}.$$

Therefore

$$|z| \leq \frac{2r_1+r_2}{6|1-s|} + \frac{2r_1+r_2}{6|s|} + \frac{2r_1+r_2}{12|1-s|} + \frac{2r_1+r_2}{12|s|} \leq \frac{n_0}{2|t|} \leq \frac{1}{40}.$$

Since $|z| \leq 1/40 < 1$, we have

$$|e^z| \leq \left(1 - |z|\left(\frac{1}{1-|z|}\right)\right)^{-1} \leq 39/38. \tag{2.11}$$

For $|t| > 20n_0$, one can see that $|1 - 1/s| > 1$. Therefore, for $r_2/n_0 \leq 1/2 \leq \sigma$ we have

$$\left|1 - \frac{1}{s}\right|^{r_2 - n_0\sigma} < 1. \tag{2.12}$$

Clearly

$$t\arg(s/2) + \arg((1-s)/2)) < 0. \tag{2.13}$$

Combining (2.10), (2.11), (2.12) and (2.13) we obtain

$$|B(s)| < 1.03\left(\frac{|s|}{2\pi e}\right)^{n_0(\frac{1}{2}-\sigma)} |d_K|^{\frac{1}{2}-\sigma}.$$

Hence the lemma is proved. □

We also need the following estimate for the number of zeros of $F_{K,X}(s)$ up to height T, which can be found in [5, 6].

Lemma 2.6. *Let $K/\mathbb{Q}$ be a number field. Let $X, T \geq 2$ and let $N_{K,X,F}(s)$ denote the number of zeros of $F_{K,X}$ whose imaginary parts are in the interval $(0, T]$, and let M be the largest integer less than or equal to X such that $a(M) \neq 0$. Then*

$$\left|N_{K,X,F}(T) - \frac{T}{2\pi}\log M\right| \ll_K X.$$

3. Proof of theorem 1.1

For the sake of simplicity we define

$$h_T(t) := h\left(\frac{t}{T}\right).$$

Now we compute the weighed second moment of the difference between two approximations $\zeta_{K,N}(s)$ and $\zeta_{K,M}(s)$ on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. From the definition (1.9) one can write

$$\begin{aligned} I &:= \int_{-\infty}^{\infty} h_T(t)\left|\zeta_{K,N}\left(\frac{1}{2}+it\right) - \zeta_{K,M}\left(\frac{1}{2}+it\right)\right|^2 dt \\ &= \int_{-\infty}^{\infty} h_T(t)\left(\sum_{N\leq n\leq M} a(n)n^{-\frac{1}{2}-it} + B\left(\frac{1}{2}+it\right)\sum_{N\leq n\leq M} a(n)n^{-\frac{1}{2}+it}\right) \\ &\qquad \times \left(\sum_{N\leq m\leq M} a(m)m^{-\frac{1}{2}+it} + B\left(\frac{1}{2}-it\right)\sum_{N\leq m\leq M} a(m)m^{-\frac{1}{2}-it}\right) dt. \end{aligned} \tag{3.1}$$

Invoking (1.10) in (3.1) and exchanging the summation and integration we have

$$I = \sum_{N\leq m,n\leq M} \frac{1}{\sqrt{mn}} \int_{-\infty}^{\infty} h_T(t)\left(a(n)n^{-it} + B\left(\frac{1}{2}+it\right)a(n)n^{it}\right)\left(a(m)m^{it} + B\left(\frac{1}{2}-it\right)a(m)m^{-it}\right)dt$$

$$= \sum_{N\le m,n\le M} \frac{a(n)a(m)}{\sqrt{mn}}\Bigg(\int_{-\infty}^{\infty} h_T(t)\Bigg(\Big(\frac{n}{m}\Big)^{it} + \Big(\frac{n}{m}\Big)^{-it}\Bigg)dt$$

$$+ \int_{-\infty}^{\infty} h_T(t)\Bigg(B\Big(\frac{1}{2}+it\Big)(nm)^{it} + B\Big(\frac{1}{2}-it\Big)(nm)^{-it}\Bigg)dt\Bigg)$$

$$=: I_1 + I_2,$$

where

$$I_1 := \sum_{N\le m,n\le M} \frac{a(n)a(m)}{\sqrt{mn}} \int_{-\infty}^{\infty} h_T(t)\Bigg(\Big(\frac{n}{m}\Big)^{it} + \Big(\frac{n}{m}\Big)^{-it}\Bigg)dt$$

and

$$I_2 := \sum_{N\le m,n\le M} \frac{a(n)a(m)}{\sqrt{mn}} \int_{-\infty}^{\infty} h_T(t)\Bigg(B\Big(\frac{1}{2}+it\Big)(nm)^{it} + B\Big(\frac{1}{2}-it\Big)(nm)^{-it}\Bigg)dt.$$

The diagonal terms $m = n$ of I_1 contribute

$$\sum_{N\le m\le M} \frac{2a(m)^2}{m} \int_{-\infty}^{\infty} h\Big(\frac{t}{T}\Big)dt = 2T\hat{h}(0) \sum_{N\le m\le M} \frac{a(m)^2}{m}. \tag{3.2}$$

The off-diagonal terms $m \ne n$ of I_1 can be written as

$$\sum_{N\le m\ne n\le M} \frac{a(m)a(n)}{\sqrt{mn}} \int_{-\infty}^{\infty} h_T(t)\Bigg(\Big(\frac{n}{m}\Big)^{it} + \Big(\frac{n}{m}\Big)^{-it}\Bigg)dt$$

$$= \sum_{N\le m\ne n\le M} \frac{a(m)a(n)}{\sqrt{mn}} \int_{-\infty}^{\infty} h_T(t)\Big(e^{it\log\frac{n}{m}} + e^{-it\log\frac{n}{m}}\Big)dt$$

$$= \sum_{N\le m< n\le M} \frac{2a(m)a(n)}{\sqrt{mn}} \int_{-\infty}^{\infty} h_T(t)\Big(e^{it\log\frac{n}{m}} + e^{-it\log\frac{n}{m}}\Big)dt$$

$$= \sum_{N\le m< n\le M} \frac{2a(m)a(n)}{\sqrt{mn}} (S_{11}(m,n) + S_{12}(m,n)), \tag{3.3}$$

where

$$S_{11}(m,n) := \int_{-\infty}^{\infty} h_T(t)e^{it\log\frac{n}{m}}dt$$

and

$$S_{12}(m,n) := \int_{-\infty}^{\infty} h_T(t)e^{-it\log\frac{n}{m}}dt.$$

Integrating by parts one obtains

$$S_{11}(m,n) = \int_{-\infty}^{\infty} h_T(t) e^{it\log\frac{n}{m}}dt = \frac{(-1)^r}{T^r}\int_{-\infty}^{\infty} h^{(r)}\left(\frac{t}{T}\right)\frac{e^{it\log\frac{n}{m}}}{(i\log\frac{n}{m})^r}dt \tag{3.4}$$

for any positive integer r. Note that

$$\log\left(1+\frac{n-m}{m}\right) \geq \log\left(1+\frac{1}{m}\right) \geq \frac{1}{2m} \tag{3.5}$$

for large m. Using (3.5) in (3.4) we find

$$S_{11}(m,n) \ll \left(\frac{2m}{T}\right)^r \int_{-\infty}^{\infty}\left|h^{(r)}\left(\frac{t}{T}\right)\right|dt \ll \|h^{(r)}\|_\infty\left(\frac{(2m)^r}{T^{r-1}}\right).$$

Similarly

$$S_{12}(m,n) \ll \|h^{(r)}\|_\infty\left(\frac{(2m)^r}{T^{r-1}}\right).$$

Therefore

$$\begin{aligned}\sum_{N\leq m<n\leq M}\frac{2a(m)a(n)}{\sqrt{mn}}(S_{11}(m,n)+S_{12}(m,n)) &\ll \|h^{(r)}\|_\infty\left(\sum_{N\leq m<n\leq M}\frac{2a(m)a(n)}{\sqrt{mn}}\frac{(2m)^r}{T^{r-1}}\right)\\ &\ll_K \left(\frac{M^{r+3}}{T^{r-1}}\right)\end{aligned} \tag{3.6}$$

for any positive integer r. Combining (3.2), (3.3), and (3.6) we see

$$I_1 = 2T\hat{h}(0)\sum_{N\leq m\leq M}\frac{a(m)^2}{m} + O_K\left(\frac{M^{r+3}}{T^{r-1}}\right). \tag{3.7}$$

We now estimate I_2. Let

$$I_2 = \sum_{N\leq m,n\leq M}\frac{a(n)a(m)}{\sqrt{mn}}(S_{21}(m,n)+S_{22}(m,n)),$$

where

$$S_{21}(m,n) := \int_{-\infty}^{\infty} h_T(t)B\left(\frac{1}{2}+it\right)e^{it\log(nm)}dt \tag{3.8}$$

and

$$S_{22}(m,n) := \int_{-\infty}^{\infty} h_T(t)B\left(\frac{1}{2}-it\right)e^{-it\log(nm)}dt.$$

From Stirling's formula we have

$$B(s) = \left(\frac{(2\pi)^{n_0}}{d_K}\right)^{s-\frac{1}{2}}\exp\left(\frac{\pi}{4}ir_1\right)\exp\left(n_0\left(\frac{1}{2}-s\right)\log(1-s)i + n_0(s-1)\right)\left(1+O\left(\frac{1}{|s|}\right)\right).$$

Now in a bounded vertical strip one can write

$$B(\sigma+it)=\left(\frac{(2\pi)^{n_0}}{t^{n_0}|d_K|}\right)^{\sigma-\frac{1}{2}+it}e^{n_0 it+\frac{\pi}{4}ir_1}\left(1+O\left(\frac{1}{|t|}\right)\right)$$

for $a\le\sigma\le b$ and $t\ge 1$. In particular

$$B\left(\frac{1}{2}+it\right)=\exp\{-n_0t\log t+it(n_0\log(2e\pi)-\log(|d_K|)+\frac{\pi}{4}ir_1\}\left(1+O\left(\frac{1}{|t|}\right)\right) \tag{3.9}$$

for $t\ge 1$. Invoking (3.9) in (3.8) we have

$$\begin{aligned} S_{21}(m,n)&=\int_0^\infty\left(h_T(t)e^{-in_0t\log t+it(n_0\log(2e\pi)-\log(|d_K|)+\log(nm))+\frac{\pi}{4}ir_1}\left\{1+O\left(\frac{1}{t}\right)\right\}\right)dt\\ &=\int_0^\infty h_T(t)e^{iF(t)+\frac{\pi}{4}ir_1}dt+O(\|h\|_\infty\log T), \end{aligned} \tag{3.10}$$

where $F(t)=-n_0t\log t+t(n_0\log(2e\pi)-\log(|d_K|)+\log(nm))$. Note that

$$|F'(t)|=\left|n_0\log\frac{2\pi}{t}+\log\frac{nm}{d_K}\right|\gg_{K,\epsilon_0}\log T \tag{3.11}$$

for all t in the support of the function h_T and $m,n\le T^{\frac{n_0}{2}-\epsilon_0}$. Then from (3.11) and by integrating by parts we have

$$\begin{aligned} \int_0^\infty h_T(t)e^{iF(t)+\frac{i\pi r_1}{4}}dt&=\int_{T^\epsilon}^\infty\frac{h_T(t)}{iF'(t)}d\left(e^{iF(t)+\frac{i\pi r_1}{4}}\right)\\ &\le\int_0^\infty\left(\frac{|h'(t/T)|}{T|F'(t)|}+\frac{n_0|h(t/T)|}{t|F'(t)|^2}\right)dt\\ &\ll_{K,\epsilon_0}\frac{1}{\log T}\max(\|h\|_\infty,\|h'\|_\infty). \end{aligned} \tag{3.12}$$

Combining (3.10) and (3.12) we have

$$S_{21}(m,n)\ll_{K,\epsilon_0}\log T.$$

By an similar argument we obtain

$$S_{22}(m,n)\ll_{K,\epsilon_0}\log T.$$

Putting these together we arrive at

$$\begin{aligned} I_2&=\sum_{N\le m,n\le M}\frac{a(m)a(n)}{\sqrt{mn}}(S_{21}(m,n)+S_{22}(m,n))\\ &\ll_{K,\epsilon_0,h}\log T\sum_{N\le m,n\le M}\frac{a(m)a(n)}{\sqrt{mn}}\\ &\ll_{K,\epsilon_0}M^{1+\epsilon}\log T. \end{aligned} \tag{3.13}$$

Hence from (3.7), (3.13), and using that $M \leq T^{1-\epsilon_0}$ we have

$$I = 2T\hat{h}(0) \sum_{N \leq m \leq M} \frac{a(m)^2}{m} + O_{K,\epsilon_0}\left(T^{4-\epsilon_0(r+3)}\right) + O_{K,\epsilon_0}\left(T^{(1-\epsilon_0)(1+\epsilon)} \log T\right).$$

Thus by choosing $\epsilon < \epsilon_0$ and r large enough we deduce

$$I = 2T\hat{h}(0) \sum_{N \leq m \leq M} \frac{a(m)^2}{m}(1 + o_K(1))$$

for $T \to \infty$. Finally by partial summation and (1.8) we conclude

$$I = 2T\frac{\hat{h}(0)}{n_0}C(K)(\log^{n_0} M - \log^{n_0} N)(1 + o_K(1))$$

for $M, N \geq T^{\epsilon_0}$ and $T \to \infty$. This completes the proof of the theorem.

4. Proof of Theorem 1.2

From (1.4) one can see that $B(s)$ is analytic for all s with $t \neq 0$. Let us now define $h(s) := -\log|B(s)|$ for $t \neq 0$. We will show that $h(s) > 0$ for $1/2 < \sigma < 1$ and $|t| \geq 40$. Here we should mention that by using the bound in Lemma 2.5 one can obtain $h(s) > 0$ for $1/2 < \sigma < 1$ and sufficient large $|t|$. Using the fact that $n_0 = r_1 + 2r_2$, one has

$$\begin{aligned}
h(s) &= \log \left| \left(\frac{2^{2r_2}\pi^{n_0}}{|d_K|} \right)^{-s+\frac{1}{2}} \frac{\Gamma^{r_1}\left(\frac{s}{2}\right)\Gamma^{r_2}(s)}{\Gamma^{r_1}\left(\frac{1-s}{2}\right)\Gamma^{r_2}(1-s)} \right| \\
&= -r_1\left(\sigma - \frac{1}{2}\right)\log \pi + r_1 \log\left|\Gamma\left(\frac{\sigma+it}{2}\right)\right| - r_1 \log\left|\Gamma\left(\frac{1-\sigma+it}{2}\right)\right| \\
&\quad - 2r_2\left(\sigma - \frac{1}{2}\right)\log 2\pi + r_2 \log|\Gamma(\sigma+it)| - r_2 \log|\Gamma(1-\sigma+it)| \\
&\quad + \left(\sigma - \frac{1}{2}\right)\log|d_K| \\
&= \left(\sigma - \frac{1}{2}\right)\Bigg(-2r_2\left(\log 2\pi - \frac{\partial}{\partial\sigma}\log|\Gamma(\sigma+it)|\Big|_{\sigma=\sigma_1}\right) \\
&\quad - r_1\left(\log \pi - \frac{\partial}{\partial\sigma}\log\left|\Gamma\left(\frac{\sigma+it}{2}\right)\right|\Big|_{\sigma=\sigma_2}\right) + \log|d_K|\Bigg),
\end{aligned}$$

with $0 \leq \sigma_1, \sigma_2 \leq 1$. In the penultimate step we also used that $|\Gamma(s)| = |\overline{\Gamma(s)}|$. Next we prove that

$$\frac{\partial}{\partial\sigma}\log|\Gamma(\sigma+it)|\Big|_{\sigma=u} - 2\log 2\pi > 0 \tag{4.1}$$

for all $0 \leq u \leq 1$. From (2.3) we have

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\log|\Gamma(s)| &= \frac{\partial}{\partial\sigma}\,\mathrm{Re}\,\log\Gamma(s) \\
&= \mathrm{Re}\,\frac{\partial}{\partial s}\log\Gamma(s)
\end{aligned}$$

$$= \text{Re}\left(\log s - \frac{1}{2s} - \frac{1}{12s^2} + 6\int_0^\infty \frac{P_3(x)}{(s+x)^4}\,dx\right)$$
$$= \log\sqrt{\sigma^2+t^2} - \frac{\sigma}{2(\sigma^2+t^2)} - \frac{\sigma^2-t^2}{12(\sigma^2+t^2)^2}$$
$$+ 6\int_0^\infty \frac{P_3(x)((\sigma+x)^4 - 6(\sigma+x)^2t^2 + t^4)}{\left((\sigma+x)^2+t^2\right)^4}\,dx. \tag{4.2}$$

Invoking the inequalities (2.4) and $(\sigma+x)^4 - 6(\sigma+x)^2t^2 + t^4 \le \left((\sigma+x)^2+t^2\right)^2$ in (4.2) we derive

$$\frac{\partial}{\partial\sigma}\log|\Gamma(\sigma+it)| \ge \log\sqrt{\sigma^2+t^2} - \frac{\sigma}{2(\sigma^2+t^2)} - \frac{\sigma^2-t^2}{12(\sigma^2+t^2)^2} - \frac{\sqrt{3}}{36}\int_0^\infty \frac{dx}{\left((\sigma+x)^2+t^2\right)^2}. \tag{4.3}$$

For $\sigma > 0$ the integral part of (4.3) can be bounded by

$$\frac{\sqrt{3}}{36}\int_0^\infty \frac{dx}{(x^2+t^2)^2} = \frac{\sqrt{3}\pi}{144t^3}. \tag{4.4}$$

Combining (4.3) and (4.4) we have

$$\frac{\partial}{\partial\sigma}\log|\Gamma(\sigma+it)| \ge \log\sqrt{\sigma^2+t^2} - \frac{\sigma}{2(\sigma^2+t^2)} - \frac{\sigma^2-t^2}{12(\sigma^2+t^2)^2} - \frac{\sqrt{3}\pi}{144t^3} \ge 2\log 2\pi$$

for $0 \le \sigma \le 1$ and $|t| > 40$. Using the Minkowski bound (2.6) and (4.1), we conclude that $h(s) > 0$ for $1/2 < \sigma < 1$ and $|t| > 40$. Therefore, for $|t| > 40$, the inequality $|\zeta_{K,X}(1-s)| > |\zeta_{K,X}(s)|$ holds true if and only if $\zeta_{K,X}(s) \neq 0$ in the vertical strip $1/2 < \sigma < 1$. By using (1.11), one can make the similar statement in the strip $0 < \sigma < 1/2$, which concludes the proof of the theorem.

5. Proof of Theorem 1.3

Let $n_0 > 2$. An argument similar to the argument in [11, 15] will work for $n_0 = 1$. In the previous section we proved $h(s) = -\log|B(s)| > 0$ for $1/2 < \sigma < 1$ and $|t| > 40$. Hence for $X = 1$ we see trivially

$$|\zeta_{K,1}(s)| = |1 + B(s)| \ge 1 - |B(s)| > 0$$

for $1/2 < \sigma < 1$ and $|t| > 40$. The same holds for $0 < \sigma < 1/2$.

Next we will prove the desired result in the case when $X = 2$. From (1.9) we can write

$$|\zeta_{K,2}(s)| = \left|1 + \frac{a(2)}{2^s} + B(s)\left(1 + \frac{a(2)}{2^{1-s}}\right)\right| \ge \left|1 + \frac{a(2)}{2^s}\right| - |B(s)|\left|\left(1 + \frac{a(2)}{2^{1-s}}\right)\right|.$$

So it suffices to prove that

$$1/|B(s)| > \left|\frac{1 + \frac{a(2)}{2^{1-s}}}{1 + \frac{a(2)}{2^s}}\right| \tag{5.1}$$

for large enough t and $\sigma > 1/2$.

First we consider the case $3/4 < \sigma < 1$. Using the fact that $|a(2)| \le 1$, one obtains

$$\left|\frac{1+\frac{a(2)}{2^{1-s}}}{1+\frac{a(2)}{2^s}}\right| \le \left|\frac{1+2^{\sigma-1}}{1-2^{-\sigma}}\right| \le \frac{1+2}{1-2^{-3/4}} < 5. \tag{5.2}$$

Then from (5.2) and Lemma 2.5, inequality (5.1) holds true if

$$\frac{1}{|B(s)|} > 0.97\left(\frac{|s|}{2\pi e}\right)^{n_0(-\frac{1}{2}+\sigma)} |d_K|^{-\frac{1}{2}+\sigma} > 5. \tag{5.3}$$

For $\sigma \ge 1$, one can see that $1+2^{\sigma-1} \le 2^\sigma$. Following the same logic as in (5.2) and (5.3), inequality (5.1) holds true if

$$0.97\left(\frac{|s|}{2\sqrt{2}\pi e}\right)^{n_0(-\frac{1}{2}+\sigma)} |d_K|^{-\frac{1}{2}+\sigma} > \frac{\sqrt{2}}{1-2^{-3/4}} \tag{5.4}$$

for $\sigma \ge 1$. Finally, we consider the case $1/2 < \sigma \le 3/4$. Let

$$g_1(s) := B(s)\frac{1+\frac{a(2)}{2^{1-s}}}{1+\frac{a(2)}{2^s}} \quad \text{and} \quad l(s) := \log\left|\frac{g_1(s)}{g_1(1/2+it)}\right|.$$

From (1.10) it is clear that $|g_1(1/2+it)| = 1$. Proceeding as in the previous section, one derives that

$$l(s) = \left(\sigma - \frac{1}{2}\right)\frac{\partial}{\partial\sigma}\left(\log\frac{1}{|B(s)|} - \log\left|\frac{1+\frac{a(2)}{2^{1-s}}}{1+\frac{a(2)}{2^s}}\right|\right)\Bigg|_{\sigma=\sigma_1},$$

for some $\sigma_1 > 1/2$. Therefore $l(s) > 0$ only when

$$\frac{\partial}{\partial\sigma}\left(\log\frac{1}{|B(s)|}\right)\Bigg|_{\sigma=\sigma_1} > \frac{\partial}{\partial\sigma}\left(\log\left|\frac{1+\frac{a(2)}{2^{1-s}}}{1+\frac{a(2)}{2^s}}\right|\right)\Bigg|_{\sigma=\sigma_1}, \tag{5.5}$$

for some $\sigma_1 \in (1/2, 3/4]$. Note that

$$\begin{aligned}\frac{\partial}{\partial\sigma}\left(\log\left|\frac{1+\frac{a(2)}{2^{1-s}}}{1+\frac{a(2)}{2^s}}\right|\right) &= \operatorname{Re}\frac{\partial}{\partial s}\left(\log\frac{1+\frac{a(2)}{2^{1-s}}}{1+\frac{a(2)}{2^s}}\right)\\ &= a(2)\log 2 \operatorname{Re}\left(\frac{a(2)+2^{s-1}+2^{-s}}{(1+a(2)2^{s-1})(1+a(2)2^{-s})}\right)\\ &\le \log 2\left(\frac{1+2^{\sigma-1}+2^{-\sigma}}{(1-2^{\sigma-1})(1-2^{-\sigma})}\right)\\ &< 27. \end{aligned} \tag{5.6}$$

Using Lemma 2.4 and (5.6), inequality (5.5) holds for

$$n_0(\log|t| - 2.55468) > 27. \tag{5.7}$$

Clearly (5.3), (5.4), and (5.7) hold true when $t > t(K)$ for some large constant $t(K)$ depending on K. The functional equation (1.11) gives the case $\sigma < 1/2$. This completes the proof of the theorem.

6. Proof of Theorem 1.4

Let $\rho = \beta + i\gamma$ be a complex zero of $\zeta_{K,X}(s)$ with $|\gamma| \geq 2\pi e X^{\lambda}$. We will show that $\zeta_{K,X}(s)$ never vanishes for

$$\beta > \frac{n_0\lambda}{2(n_0\lambda - 1)}\left(1 + \frac{2n_0 \log\log X}{\log X}\right),$$

when $1/n_0 < \lambda \leq 2/n_0$, and is nonzero for

$$\beta > 1 + \frac{2n_0 \log\log X}{\log X},$$

when $\lambda > 2/n_0$. Let $\lambda > 1/n_0$, $|t| \geq 2\pi e X^{\lambda}$, and

$$\sigma > \max\left(1, \frac{\lambda n_0}{2(\lambda n_0 - 1)}\right)\left(1 + \frac{c \log\log X}{\log X}\right) \tag{6.1}$$

for some positive constant c which will be determined later. From (1.9) we have

$$|\zeta_{K,X}(s)| \geq \left|\sum_{n\leq X} \frac{a(n)}{n^s}\right| - |B(s)| \left|\sum_{n\leq X} \frac{a(n)}{n^{1-s}}\right|.$$

By Lemmas 2.2 and 2.3 we see that

$$\begin{aligned}\left|\sum_{n\leq X} \frac{a(n)}{n^s}\right| &\geq |\zeta_{K,X}(s)| - \left|\sum_{n> X} \frac{a(n)}{n^s}\right| \\ &> \left(\frac{\sigma-1}{\sigma}\right)^{n_0} - \frac{H(K)X^{1-\sigma}}{\sigma - 1} + O_X\left(X^{1-\sigma-1/n_0}\right). \end{aligned} \tag{6.2}$$

By (6.1) we always have

$$\sigma > 1 + \frac{c \log\log X}{\log X}.$$

Hence from (6.2) we have

$$\left|\sum_{n\leq X} \frac{a(n)}{n^s}\right| > \left(\frac{c\log\log X}{\log X + c \log\log X}\right)^{n_0} - \frac{1}{\log^c X}\left(\frac{\log X}{c \log\log X}\right)\left(H(K) + O\left(X^{-1/n_0}\right)\right).$$

Therefore for $c \geq 2n_0$ and sufficiently large X we have

$$\left|\sum_{n\leq X} \frac{a(n)}{n^s}\right| > \left(\frac{c \log\log X}{2\log X}\right)^{n_0}. \tag{6.3}$$

Using Lemmas 2.3 and 2.5 we have

$$|B(s)| \left|\sum_{n\leq X} \frac{a(n)}{n^{1-s}}\right| < 1.03 \left(\frac{|s|}{\pi e}\right)^{n_0(\frac{1}{2}-\sigma)} |d_K|^{\frac{1}{2}-\sigma}\left(H(K)X^{\sigma} + O\left(X^{\sigma-1/n_0}\right)\right)$$

for $|t| > \max\{2\pi e X^{\lambda}, 20n_0\}$. For a large value of X,

$$|B(s)| \left|\sum_{n\leq X} \frac{a(n)}{n^{1-s}}\right| < 1.03 |d_K|^{\frac{1}{2}-\sigma}\left(\frac{|s|}{\pi e}\right)^{n_0(\frac{1}{2}-\sigma)} H(K)X^{\sigma}$$

$$< 1.03|d_K|^{\frac{1}{2}-\sigma} H(K) X^{\lambda n_0(\frac{1}{2}-\sigma)+\sigma}. \tag{6.4}$$

Now consider $1/n_0 < \lambda < 2/n_0$. By the aid of (6.1), the exponent of X in (6.4) can be written as

$$n_0\lambda\left(\frac{1}{2}-\sigma\right)+\sigma = \frac{\lambda n_0}{2} - \sigma(n_0\lambda - 1) < -n_0\lambda \frac{c\log\log X}{2\log X} \leq -\frac{c\log\log X}{2\log X}.$$

If $\lambda \geq \frac{2}{n_0}$, then the exponent of X in (6.4) is

$$n_0\lambda\left(\frac{1}{2}-\sigma\right)+\sigma \leq (1-2\sigma)+\sigma \leq -\frac{c\log\log X}{\log X}.$$

Combining the above two cases and using (6.4), we derive

$$\left|B(s)\sum_{n\leq X}\frac{a(n)}{n^{1-s}}\right| < \frac{1.03|d_K|^{\frac{1}{2}-\sigma}H(K)}{\log^{c/2} X}. \tag{6.5}$$

Therefore from (6.3) and (6.5) we have

$$|\zeta_{K,X}(s)| > \frac{c\log\log X}{2\log X} - \frac{1.03|d_K|^{\frac{1}{2}-\sigma}H(K)}{\log^{\frac{c}{2}} X} > 0,$$

with $X > X_0$ for some X_0.

For the last part of the theorem we will use the inequality

$$\begin{aligned}|\zeta_{K,X}(s)| &\geq \zeta_K(s) - \left|\sum_{n>X}\frac{a(n)}{n^s}\right| - |B(s)|\left|\sum_{n\leq X}\frac{a(n)}{n^{1-s}}\right| \\ &\geq \left(\frac{\sigma-1}{\sigma}\right)^{n_0} - \left|\sum_{n>X}\frac{a(n)}{n^s}\right| - |B(s)|\left|\sum_{n\leq X}\frac{a(n)}{n^{1-s}}\right|.\end{aligned}$$

We recall the trivial bound $a(n) \leq (d(n))^{n_0-1} \leq n^{n_0-1}$. Using this fact together with Lemma 2.5 one can show that there exists a large T_0 (depending on K) such that if $|t| > \max(2\pi X^{2/n_0}, T_0)$ and $\sigma > n_0+2$, then $\zeta_{K,X}(s) \neq 0$.

Using the functional equation (1.11) one gets the zero free region on the left side of the vertical axis. This completes the proof of Theorem 1.4.

7. Proof of Theorem 1.5

Let $T > 2\pi e X^{\lambda}$ be a large number for $\lambda > \frac{1}{n_0}$. Then by Theorem 1.4 we conclude that the zeros of $\zeta_{K,X}(s)$ with ordinates $T < \gamma < T+U$, for some positive constant U, must lie in a rectangle of width $2w-1$, with $w = \max\{n_0+1, n_0\lambda/(n_0\lambda-1)\}$.

Let R be a positively oriented rectangle with vertices $w+iT$, $w+i(T+U)$, $1-w+i(T+U)$ and $1-w+iT$. From Theorem 1.4, we observe that the complex zeros will be inside the rectangle R for sufficiently large X. Without loss of generality we assume that the edges of the rectangle do not pass through any zeros of $\zeta_{K,X}(s)$. Then by the argument principle we have

$$N(T+U) - N(T) = \frac{1}{2\pi}\triangle_R(\zeta_{K,X}(s)).$$

From (1.9) we have

$$\zeta_{K,X}(s) = 1 + \sum_{2\le n\le X} \frac{a(n)}{n^s} + B(s) \sum_{1\le n\le X} \frac{a(n)}{n^{1-s}}. \tag{7.1}$$

Then from (1.5) and (7.1) we write

$$\begin{aligned}|\zeta_{K,X}(s) - 1| &\le \sum_{2\le n\le X} \frac{a(n)}{n^\sigma} + |B(s)| \sum_{1\le n\le X} \frac{a(n)}{n^{1-\sigma}} \\ &\le \sum_{2\le n\le X} \frac{(d(n))^{n_0-1}}{n^\sigma} + |B(s)| \sum_{1\le n\le X} \frac{a(n)}{n^{1-\sigma}}.\end{aligned}$$

Since $t \ge T \ge 2\pi e X^\lambda$ and $d(n) \le n$, applying (6.4) we find that

$$\begin{aligned}|\zeta_{K,X}(s) - 1| &\le \frac{1}{2^{\sigma+1-n_0}} + \int_2^X \frac{1}{x^{\sigma+1-n_0}}\, dx + O_K\left(|d_K|^{\frac{1}{2}-\sigma} X^{\lambda n_0(\frac{1}{2}-\sigma)+\sigma}\right) \\ &\le \frac{4}{5},\end{aligned} \tag{7.2}$$

for $\sigma \ge w$ and large X. Therefore $\operatorname{Re}\zeta_{K,X}(s) > 0$ for $\sigma \ge w$ and $t \ge T$. Then the change of argument of $\zeta_{K,X}(s)$ along the right vertical segment of R is bounded by π.

Using the functional equation (1.11) we may write

$$\arg(\zeta_{K,X}(1 - w + it)) = \arg(\zeta_{K,X}(w + it)) - \arg(B(w + it)) \tag{7.3}$$

Recall Stirling's formula in the form

$$\log\Gamma(s) = \left(s - \frac{1}{2}\right)\log s - s + \frac{1}{2}\log 2\pi + O\left(\frac{1}{|s|}\right), \tag{7.4}$$

for $|s| \to \infty$ and $|\arg s| \le \pi - \epsilon$. Then from (1.4) and (7.4) we have

$$\log B(s) = \left(s - \frac{1}{2}\right)\log\frac{(2\pi)^{n_0}}{|d_K|} + \frac{\pi}{4} i r_1 + n_0\left(\frac{1}{2} - s\right)\log(1 - s)i + n_0(s - 1) + O\left(\frac{1}{|s|}\right)$$

as $|s| \to \infty$ and $\arg \le \pi - \epsilon$. Also for $t \to \infty$

$$\operatorname{Re}(\log s) = \log t + O\left(\frac{\sigma^2}{t^2}\right) \quad \text{and} \quad \operatorname{Im}(\log s) = \left(\frac{\pi}{2} - \frac{\sigma}{t}\right) + O\left(\frac{\sigma^3}{t^3}\right).$$

Note that

$$\begin{aligned}\arg B(s) &= \operatorname{Im}(\log B(s)) \\ &= \operatorname{Im}\left(\left(s - \frac{1}{2}\right)(n_0\log 2\pi - \log|d_K|)\right. \\ &\quad \left.+ \frac{\pi}{4} i r_1 + n_0\left(\frac{1}{2} - s\right)\log(1 - s)i + n_0(s - 1) + O\left(\frac{1}{|s|}\right)\right) \\ &= -n_0 t\log\frac{t}{2\pi} + t(n_0 - \log|d_K|) + \frac{\pi r_1}{4} + n_0\left(\frac{1}{2} - \sigma\right)\pi + O\left(\frac{\sigma^2}{t}\right).\end{aligned} \tag{7.5}$$

Therefore, from (7.3) and (7.5), the change of the argument on the left vertical segment of R is

$$n_0(T+U)\log\frac{T+U}{2\pi} - n_0 T\log\frac{T}{2\pi} - U(n_0 - \log|d_K|) + O\left(\frac{w^2}{T}\right).$$

Next we consider the change in $\arg\zeta_{K,X}(s)$ along the bottom edge of R. Let q be the number of zeros of $\mathrm{Re}(\zeta_{K,X}(\sigma+iT))$ on the interval $(1-w, w)$. Then there are at most $q+1$ subintervals of $(1-w,w)$ in which Re $(\zeta_{K,X}(\sigma+iT))$ is of constant sign. Therefore the variation of $\arg\zeta_{K,X}(\sigma+iT)$ is at most π in each subinterval. So we have

$$\arg\zeta_{K,X}(\sigma+iT)|_{1-w}^{w} \leq (q+1)\pi. \tag{7.6}$$

To estimate q, first we define

$$g(z) := \zeta_{K,X}(z+iT) + \overline{\zeta_{K,X}(\bar{z}+iT)}. \tag{7.7}$$

If $z=\sigma$ is a real number then we have

$$g(\sigma) = \text{ Re } (\zeta_{K,X}(\sigma+iT)).$$

Let $R = 2(2w-1)$ and consider the disk $|z-w| < R$ centered at w. Choose T large so that

$$\text{Im } (z+iT) > T - R > 0.$$

Thus, $\zeta_{K,X}(z+iT)$, and hence also $g(z)$, are analytic in the disk $|z-w|<R$. Let $n(r)$ be the number of zeros of $g(z)$ in the disk $|z-w|<r$ and $R_1 = R/2$. Then we have

$$\int_0^R \frac{n(r)}{r}\,dr \geq n(R_1)\int_{R_1}^R \frac{dr}{r} = n(R_1)\log 2. \tag{7.8}$$

By Jensen's theorem,

$$\int_0^R \frac{n(r)}{r}\,dr = \frac{1}{2\pi}\int_0^{2\pi}\log\frac{|g(w+Re^{i\theta})|}{|g(w)|}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log|g(w+Re^{i\theta})|\,d\theta - \log|g(w)|. \tag{7.9}$$

From (7.2) we have

$$|\zeta_{K,X}(w+iT)| \geq \frac{1}{5},$$

for $T \geq 2\pi e X^{\lambda}$ and $\lambda > 1/n_0$, which shows that $\log|g(w)|$ is well defined. From the definition (1.9) we have

$$|\zeta_{K,X}(s)| \leq \sum_{n\leq X}\frac{a(n)}{n^{\sigma}} + |B(s)|\sum_{n\leq X}\frac{a(n)}{n^{1-\sigma}}.$$

By Lemma 2.5, we have

$$B(s) \ll |s|^{n_0(\frac{1}{2}-\sigma)}.$$

One can show (similarly to Lemma 2.3) that

$$\sum_{n\leq X}\frac{a(n)}{n^{\sigma}} \ll_K \begin{cases} X^{1-\sigma} & \text{if } \sigma\neq 1 \\ \log X & \text{if } \sigma = 1 \end{cases}.$$

Thus,

$$|\zeta_{K,X}(s+iT)| \ll_K (X^{1-\sigma} + \log X + T^{n_0(\frac{1}{2}-\sigma)}X^{\sigma}).$$

Therefore from (7.7) we have

$$|g(s)| \le |\zeta_{K,X}(s+iT)| + |\zeta_{K,X}(s-iT)| \ll (X^{1-\sigma} + \log X + T^{n_0(\frac{1}{2}-\sigma)}X^{\sigma}). \tag{7.10}$$

Since $|s-w| < R = 2(2w-1)$, then $2-3w < \sigma < 5w-2$. Also $T \ge 2\pi e X^{\lambda}$ for $\lambda > 1/n_0$. Thus the expression on the right-hand side of (7.10) is largest when $\sigma = -3w+2$. Therefore

$$\begin{aligned} |g(s)| &\ll_K (X^{3w-1} + \log X + T^{n_0(-\frac{3}{2}+3w)}X^{2-3w}) \\ &\ll_K (T^{(3w-1)/\lambda} + T^{n_0(-\frac{3}{2}+3w)+(-3w+2)/\lambda}) \\ &\ll_K (T^{n_0(3w-1)} + T^{n_0(-\frac{3}{2}+3w)+n_0(-3w+2)}) \\ &\ll_K T^{3n_0 w}. \end{aligned}$$

Finally

$$|g(w+Re^{i\theta}| \ll_K T^{3wn_0}.$$

Hence, from (7.8) and (7.9), it follows that $n(R_1) \ll_K w\log T$. The zeros of $\zeta_{K,X}(\sigma+iT)$ for $1-w<\sigma<w$ correspond to the zeros of $g(\sigma)$ in the same interval. Since the interval $(1-w,w)$ is contained in the disk $|s-w| < R_1$ we have $q \le n(R_1)$. Therefore from (7.6) we conclude that the change of the argument on the lower horizontal segment of R is $\ll_K w\log T$. Similarly, the change of the argument on the top vertical segment of R is $\ll_K w\log(T+U)$.

Combing the four sides of R, we see that

$$\triangle_R(\zeta_{K,X}(s)) = n_0(T+U)\log\frac{T+U}{2\pi} - n_0 T\log\frac{T}{2\pi} - U(n_0 - \log|d_K|) + O_K(w\log(T+U)),$$

which concludes the first part of Theorem 1.5. Note that $w \ll_K \lambda/(n_0\lambda - 1)$. For the last part of the theorem, we will follow the same argument as in the proof above, on the positively oriented rectangle $[n_0+2+iT, n_0+2+i(T+U), -1-n_0+i(T+U), -1-n_0+iT]$. This proves the theorem.

8. Proof of Theorem 1.6

Let $N^{+}_{K,X,F}(T)$ denote the number of zeros of $F_{K,X}(s)$ with imaginary part in $(0,T]$ and real part $\ge \frac{1}{2}$. From Lemma 2.6 we have

$$N^{+}_{K,X,F}(T) \le \frac{aT}{2\pi}\log M + O_K(X) \tag{8.1}$$

for some $0 \le a \le 1$. We re-write (1.9) as

$$\begin{aligned} \zeta_{K,X}(s) &= F_{K,X}(s)\left(1 + B(s)\frac{F_{K,X}(1-s)}{F_{K,X}(s)}\right) \\ &= F_{K,X}(s) Z_{K,X}(s). \end{aligned}$$

Then $\zeta_{K,X}(\frac{1}{2}+it) = 0$ if and only if

(i) $F_{K,X}(\frac{1}{2}+it) = 0$
(ii) $Z_{K,X}(\frac{1}{2}+it) = 0$.

We are going to estimate the number of zeros of $Z_{K,X}(s)$ on the critical line $\mathrm{Re}\,(s) = \frac{1}{2}$. Note that

$$\begin{aligned} & Z_{K,X}(\frac{1}{2} + it) = 0 \\ \iff & B(s)\frac{F_{K,X}(1-s)}{F_{K,X}(s)} = -1 \\ \iff & \arg(B(s)\frac{F_{K,X}(1-s)}{F_{K,X}(s)}) = (2k+1)\pi \\ \iff & \arg B(s) - 2\arg F_{K,X}(s) = (2k+1)\pi. \end{aligned}$$

We estimate the number of zeros of $\arg B(s) - 2\arg F_{K,X}(s)$ on the line segment $(\frac{1}{2}+iT, \frac{1}{2}+i(T+U)]$. Let $\mathcal{L}(\epsilon)$ be the curve defined in [3, p. 20]. Let $m(g)$ be the multiplicity of the zero $\frac{1}{2}+ig$ of $F_{K,X}(s)$. An argument similar to [3, p. 20] shows that a lower bound for the number of zeros of $Z_{K,X}(s)$ is

$$= \lim_{\epsilon\to 0^+} \frac{1}{2\pi}|\triangle_{\mathcal{L}(\epsilon)}(\arg B(s) - 2\arg F_{K,X}(s))| - \sum_{T\le g\le T+U} m(g) + O_K(1).$$

This also gives a lower bound for the number of distinct zeros of $Z_{K,X}(s)$ on the line segment $(\frac{1}{2}+iT, \frac{1}{2}+i(T+U)]$.

Note that a similar computation to (7.5) gives

$$\triangle_{\mathcal{L}(\epsilon)} \arg B(s) = -n_0(T+U)\log\frac{T+U}{2\pi} + n_0(T)\log\frac{T}{2\pi} + U(n_0 - \log|d_K|) + O\left(\frac{w^2}{T}\right).$$

Next we estimate $\triangle_{\mathcal{L}(\epsilon)}F_{K,X}(s)$. Consider the contour $C(\epsilon)$ consisting of $\mathcal{L}(\epsilon)$ and three line segments: top $(\frac{1}{2}+i(T+U), w+i(T+U)]$, right $[w+i(T), w+i(T+U)]$, bottom $(\frac{1}{2}+i(T+U), w+i(T)]$, with counter-clockwise orientation. If ϵ is small enough, we have

$$\triangle_{C(\epsilon)} \arg F_{K,X}(s) = -2\pi(N^+_{K,X,F}(T+U) - N^+_{K,X,F}(T)).$$

From the definition of $F_{K,X}(s)$ and an argument similar to (7.2) we find

$$|F_{K,X}(s) - 1| < 1.$$

Hence

$$\arg F_{K,X}(w+it)|_T^{T+U} \ll 1.$$

Note that

$$\mathrm{Im}\ (F_{K,X}(\sigma+iT)) = -\sum_{n\le X}\frac{a(n)\sin(T\log n)}{n^\sigma}.$$

By a generalization of Descartes's Rule of Signs (see Pólya and Szegö [10], Part V, Chapter 1, No. 77), the number of real zeros of $\mathrm{Im}\ (F_{K,X}(\sigma+iT))$ in the interval $1/2 \le \sigma \le 3$ is less than or equal to the number of sign changes in the sequence $\{a(n)\sin(T\log n)\}$, for $1 \le n \le X$, which in turn is less than or equal to the number of nonzero coefficients of $a(n)\sin(T\log n)$. Therefore

$$\arg F_{K,X}(\sigma+iT)|_{1/2}^{w} \ll_K X.$$

Similarly

$$\arg F_{K,X}(\sigma+i(T+U))|_{1/2}^{w} \ll_K X.$$

Thus

$$\triangle_{\mathcal{L}(\epsilon)} \arg F_{K,X}(s) = -2\pi(N^+_{K,X,F}(T+U) - N^+_{K,X,F}(T)) + O_K(X).$$

Thus we have

$$\begin{aligned}\lim_{\epsilon\to 0^+} \frac{1}{2\pi}|\triangle_{\mathcal{L}(\epsilon)}(\arg B(s) - 2\arg F_{K,X}(s))| - \sum_{T\le g\le T+U} m(g) \\ \ge n_0\frac{T+U}{2\pi}\log\frac{T+U}{2\pi} - n_0\frac{T}{2\pi}\log\frac{T}{2\pi} - U(n_0 - \log|d_K|) - \sum_{T\le g\le T+U} m(g) \\ - 2(N^+_{K,X,F}(T+U) - N^+_{K,X,F}(T)) + O_K(\log X).\end{aligned}$$

Using Lemma (8.1) we have

$$N^+_{K,X,F}(T+U) - N^+_{K,X,F}(T) \le \frac{aU}{2\pi}\log M + O_K(X)$$

for some $0 \le a \le 1$. Note that

$$\sum_{T\le g\le T+U} m(g)$$

is the number of zeros of $F_{K,X}$ on the line segment of $(\frac{1}{2}+iT, \frac{1}{2}+i(T+U)]$. Combining this with the estimate of the number of zeros of $Z_{K,X}(s)$, we have

$$\begin{aligned}N^0_{K,X}(T+U) - N^0_{K,X}(T) \ge n_0\frac{T+U}{2\pi}\log\frac{T+U}{2\pi M^{2a/n_0}} - n_0\frac{T}{2\pi}\log\frac{T}{2\pi M^{2a/n_0}} \\ - \frac{U}{2\pi}(n_0 - \log|d_K|) + O_K(X).\end{aligned}$$

This completes the proof of the theorem.

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