PSEUDORADIAL SPACES AND COPIES OF $\omega_1 + 1$

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Abstract. In this paper we compare the concepts of pseudoradial spaces and the recently defined strongly pseudoradial spaces in the realm of compact spaces. We show that MA+$\mathfrak{c} = \omega_2$ implies that there is a compact pseudoradial space that is not strongly pseudoradial. We essentially construct a compact, sequentially compact space X and a continuous function $f : X \to \omega_1 + 1$ in such a way that there is no copy of $\omega_1 + 1$ in X that maps cofinally under f. We also give some conditions that imply the existence of copies of ω_1 in spaces. In particular, PFA implies that compact almost radial spaces of radial character ω_1 contain many copies of ω_1.

1. Introduction

All spaces are assumed to be Hausdorff.

Recall that a topological space X is *pseudoradial* if for every non-closed subset $A \subset X$ there is a point $x \in \overline{A} \setminus A$ and a transfinite sequence $\langle x_\alpha \rangle_{\alpha<\kappa}$ with range in A and converging to x.

The systematic investigation on the topological properties of pseudoradial spaces was initiated by Arhangel'skiĭ more than 40 years ago. Since then, several subclasses of pseudoradial spaces have been considered by many authors.

Recently the further notion of strongly pseudoradial spaces appeared in the literature [2]. All ordinals in this paper will have the order topology when considered as topological spaces.

1.1. Definition A topological space X is called strongly pseudoradial if for any non-closed subset $A \subset X$ there is a limit ordinal γ and a continuous map $f : \gamma+1 \to X$ such that $f[\gamma] \subset A$ and $f(\gamma) \notin A$.

In [2] the authors pointed out that, without any loss of generality, in the above definition the ordinal γ can be assumed to be a regular cardinal and the function f injective.

Roughly speaking, the difference between pseudoradial and strongly pseudoradial spaces consists in replacing transfinite converging sequences with compact ordinals.

As $\omega + 1$ is a compact ordinal, we immediately see that every sequential space is strongly pseudoradial, but this is the only case when we can easily determine if a space is of that kind. The passage from $\omega + 1$ to the successor of an uncountable cardinal appears much more difficult. However, a remarkable consequence of the proper forcing axiom (PFA) describes a possibility to do it for $\gamma = \omega_1$ (see also [14, Theorem 5.14]).

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1.2. Theorem [1] PFA implies that in every countably compact regular space of character at most ω_1, the closure of a subset A can be obtained by first adding all limits of convergent sequences and then adding to the resulting set $\hat{A}$ all points x for which there is a copy W of ω_1 in $\hat{A}$ such that $W \cup \{x\}$ is homeomorphic to $\omega_1 + 1$.

1.3. Corollary PFA implies that every countably compact regular space of character at most ω_1 is strongly pseudoradial.

On the other hand, the one-point compactification of Ostaszewski's space shows that a compact pseudoradial space may fail to be strongly pseudoradial. This follows from hereditary separability of Ostaszewki's space.

A natural question is then whether it is possible to obtain the conclusion of Corollary 1.3 by weakening the topological hypothesis on the character. First, by means of a counterexample under Martin's axiom, we were able to show that we cannot remove that hypothesis altogether.

1.4. Theorem $\mathfrak{b} = \mathfrak{c} = \omega_2$ implies there is a compact pseudoradial space that is not strongly pseudoradial.

The *radial character* of a pseudoradial space X is the smallest cardinal κ such that the definition of pseudoradiality for X works by taking only transfinite sequences of length not exceeding κ. Thus, the following question is natural after considering Corollary 1.3.

1.5. Question Assume PFA. Is every compact Hausdorff pseudoradial space of radial character at most ω_1 strongly pseudoradial?

We were unable to answer Question 1.5. However, we can achive a partial positive result for a special class of pseudoradial spaces, as we now explain.

A sequence $\{x_\alpha : \alpha < \kappa\}$ converging to a point x is called *thin* if for any $\beta < \kappa$ we have that $x \notin \overline{\{x_\alpha : \alpha < \beta\}}$. A space is called *almost radial* if in the usual definition of pseudoradiality we replace "sequence" with "thin sequence".

Our counterexample X from Theorem 1.4 is a compact, sequentially compact space X with a point $\rho \in X$ of character ω_1 such that there are no countable sequences or copies of ω_1 converging to ρ (see Theorem 5.2).

In Section 4 we include several results on existence of copies of ω_1. In particular, we highlight the following result that contrasts with our counterexample from Theorem 1.4.

1.6. Theorem Assume PFA. Let X be a compact almost radial space of radial character at most ω_1. Then every point of X is either the limit of a countable sequence or the limit of a copy of ω_1.

The search of copies of ω_1 goes back to [13, Problem 1.3] where Nyikos asks whether there exists a first countable, countably compact, non-compact space which does not contain a copy of ω_1. Nyikos himself proved that a consequence of $\diamondsuit$, which is in fact compatible with MA + $\mathfrak{c} > \omega_1$, gives a positive answer to his question (see [8, 19.1]). In [8] Fremlin, assuming PFA, gives sufficient conditions for the

existence of copies of ω_1 for spaces that can be mapped onto ω_1 and gives a series of applications. This question was also considered under PFA in [1], where other conditions for the existence of copies of ω_1 are given. Later, Eisworth and Nyikos in [7] give a model of CH in which every first countable, countably compact, non-compact space contains a copy of ω_1. Also, in [6] Eisworth shows that any perfect pre-image of ω_1 with countable tightness contains a closed copy of ω_1.

2. $\mathcal{T}$-ALGEBRAS

In order to construct the space from Theorem 1.4, we will use Koszmider's notion of $\mathcal{T}$-algebra from [12]. $\mathcal{T}$-algebras are special kinds of the minimally generated Boolean algebras first studied by Koppelberg [11]. All our Boolean algebras will be subalgebras of $\mathcal{P}(\omega)/\mathsf{fin}$ with the order relation $\subset^*$ of almost inclusion. Section 3 of [4] contains a a thorough analysis of the following discussion.

Given a Boolean algebra $A \subset \mathcal{P}(\omega)$ and $x \subset \omega$, the Boolean algebra generated by $A \cup \{x\}$ is

$$A(x) = \{(a_0 \cap x) \cup (a_1 \setminus x) : a_0, a_1 \in A\}.$$

Let $A \subset \mathcal{P}(\omega)$ be a Boolean algebra and u an ultrafilter of A. An element $x \subset \omega$ is called *minimal* for $\langle A, u\rangle$ if u is the only ultrafilter in A that does not generate an ultrafilter in the Boolean algebra $A(x)$. Notice that in this case, $\omega \setminus x$ is also minimal for $\langle A, u\rangle$.

Let $\lambda \leq \mathfrak{c}$ and $\mathcal{A} = \{a_{\alpha+1} : \alpha < \lambda\} \subset [\omega]^\omega$. For each $\alpha \leq \lambda$, define B_α to be the Boolean algebra generated by $\{a_{\beta+1} : \beta < \alpha\}$. We will say that $\mathcal{A}$ is a *coherent minimal sequence* if for every $\alpha < \lambda$ the filter u_α in B_α generated by $\{a_{\beta+1} : \beta < \alpha\}$ is an ultrafilter and $a_{\alpha+1}$ is minimal for $\langle B_\alpha, u_\alpha\rangle$.

Let us describe the Stone space of B_λ. Consider $\alpha < \lambda$. Let x_α the filter in B_λ generated by $u_\alpha \cup \{\omega \setminus a_{\alpha+1}\}$. Since $a_{\alpha+1}$ is minimal for $\langle B_\alpha, u_\alpha\rangle$, $x_\alpha \cap B_{\alpha+1}$ is an ultrafilter (thus, proper) in $B_{\alpha+1}$. By recursion it is possible to show that, $x_\alpha \cap B_\beta$ is in fact an ultrafilter in B_β for all $\alpha < \beta \leq \lambda$. It is not hard to conclude that the Stone space of B_λ is equal to $X_\lambda = \{u_\lambda\} \cup \{x_\alpha : \alpha < \lambda\}$.

Let $\alpha < \lambda$. Then the clopen set defined by $\omega \setminus a_{\alpha+1}$ in the Stone space misses $\{x_\beta : \alpha < \beta < \lambda\} \cup \{u_\lambda\}$. Thus, the segment $\{x_\beta : \beta \leq \alpha\}$ is open. This easily implies that the Stone space of B_λ is scattered.

Next, we define $\mathcal{T}$-algebras. First, given $t \in 2^{<\mathfrak{c}}$ such that $\mathsf{dom}(t) = \alpha + 1$ for some α, we define $t^\star = (t{\restriction}_\alpha)^\frown(1 - t(\alpha))$. Each $\mathcal{T}$-algebra will be defined by using a tree. A subtree T of $2^{<\mathfrak{c}}$ is called *acceptable* if the following two conditions hold:

(i) the domain of each member of T is a successor ordinal,
(ii) if $t \in T$ and $\alpha < \mathsf{dom}(t)$, then $t{\restriction}_{\alpha+1} \in T$, and
(iii) for all $t \in 2^{<\mathfrak{c}}$, $t \in T$ if and only if $t^\star \in T$.

For each $p \in 2^{\leq\mathfrak{c}}$, $o(p)$ will denote its order type. Given an acceptable tree $T \subset 2^{<\mathfrak{c}}$, a T-algebra is a Boolean algebra generated by a sequence $\{a_t : t \in T\} \subset [\omega]^\omega$ such that the following properties hold

(a) given $t \in T$, $\{a_{t{\restriction}\alpha+1} : \alpha + 1 < o(t)\}$ is a coherent minimal sequence, and
(b) given $t \in T$, $a_t = \omega \setminus a_{t^\star}$.

It turns out that the Stone space of a $\mathcal{T}$-algebra is easy to describe. In fact, the set of ultrafilters of a T-algebra $\mathcal{A} = \{a_t : t \in T\}$ is in one-to-one correspondence with the set bT of branches of T. Given $p \in bT$, $\{a_{p{\restriction}_{\alpha+1}} : \alpha < \mathsf{dom}(p)\}$ generates

the ultrafilter that corresponds to p. So without loss of generality, we will identify the Stone space of $\mathcal{A}$ with bT.

It also turns out that some of the topological properties of bT can be checked by looking at the topology generated by the branches. For every branch $p \in bT$, by property (a) in the definition of T-algebra, there is a topological space $X_p = \{x_{p\restriction\alpha} : \alpha < o(p)\} \cup \{p\}$ defined by the coherent minimal sequence $\{a_{p\restriction\alpha+1} : \alpha < o(p)\}$ as described above.

Given $p, q \in bT$ with $p \neq q$, we define $p \wedge q$ to be the largest common predecesor of p and q. Thus, $o(p \wedge q)$ is the ordinal α such that $p\restriction_\alpha = q\restriction_\alpha$ but $p(\alpha) \neq q(\alpha)$.

Fix $p \in bT$. Then there is a continuous function $\pi_p : bT \to X_p$ that projects the Stone space of the T-algebra onto the branch space. For every $q \in bT \setminus \{p\}$, $\pi_p(q) = x_{p\restriction o(p\wedge q)}$ and $\pi_p(p) = p$. Equivalently, for every $\alpha < o(p)$

$$\pi_p^{\leftarrow}(x_{p\restriction\alpha}) = \{q \in bT : o(p \wedge q) = \alpha\}.$$

The relation we are interested in is summarized with the following results.

2.1. Lemma [5, Proposition 3.4] Let $p \in bT$, $S \subset bT \setminus \{p\}$. Then $p \in \mathrm{cl}_{bT}(S)$ if and only if $p \in \mathrm{cl}_{X_p}(\{\pi_p(q) : q \in S\})$.

2.2. Lemma [5, Proposition 3.4] Let $p \in bT$ and $\{p_n : n < \omega\} \subset bT \setminus \{p\}$. Then $\{p_n : n < \omega\}$ converges to p in bT if and only if $\{\pi_p(p_n) : n < \omega\}$ converges to p in X_p.

Since we are interested in copies of ω_1, we would like to find a necessary condition for the existence of copies of ω_1 that can be checked in branches and avoided through careful construction of the T-algebra.

2.3. Lemma Let $f : \omega_1 + 1 \to bT$ be an embedding. Then there is a closed, unbounded set $S \subset \omega_1$ such that $(\pi_{f(\omega_1)} \circ f)\restriction S$ is one-to-one and increasing. Thus, $X_{f(\omega_1)}$ contains a copy of $\omega_1 + 1$.

Proof. Let $p = f(\omega_1)$. First, notice that the fibers of points of $X_p \setminus \{p\}$ under $\pi_p \circ f$ are all countable. Otherwise, there is an uncountable $R \subset \omega_1$ and $\beta < o(p)$ such that $(\pi_p \circ f)[R] = \{x_{p\restriction\beta}\}$. This would imply that $a_{p\restriction\beta+1}$ and $\omega \setminus a_{p\restriction\beta+1}$ define clopen sets that separate $p = (\pi_p \circ f)(\omega_1)$ from $(\pi_p \circ f)[R]$ in X_p, which contradicts the continuity of $\pi_p \circ f$.

Thus, there is an increasing enumeration $\{\xi(\alpha) : \alpha < \omega_1\}$ of the set $\{\beta < o(p) : \exists\alpha < \omega_1(\pi_p(f(\alpha)) = x_{p\restriction\beta})\}$. For each $\alpha < \omega_1$, let $R_\alpha = \{\beta < \omega_1 : \pi_p(f(\beta)) = x_{p\restriction\xi(\alpha)}\}$. We shall recursively define an increasing injective function $\sigma : \omega_1 \to \omega_1$ and an element $s_\alpha \in R_{\sigma(\alpha)}$ for each $\alpha < \omega_1$.

For every non-limit ordinal $\alpha < \omega_1$, let $\sigma(\alpha) = \min(\omega_1 \setminus \sigma[\alpha])$ and let s_α be the first element of $R_{\sigma(\alpha)}$. Now assume that $\alpha < \omega_1$ is a limit ordinal. Define $s_\alpha = \sup\{s_\beta : \beta < \alpha\} < \omega_1$, then there exists $\gamma < \omega_1$ such that $s_\alpha \in R_\gamma$. We then define $\sigma(\alpha) = \gamma$.

Assume that for some $\delta < \alpha$ we have that $\gamma \leq \sigma(\delta)$. By continuity, $\{(\pi_p \circ f)(s_\beta) : \beta < \alpha\}$ converges to $(\pi_p \circ f)(s_\alpha)$. However, initial segments are open in X_p so $\{x_{p\restriction\beta} : \beta \leq \xi(\sigma(\delta))\}$ is an open set that contains $(\pi_p \circ f)(s_\alpha)$ and misses $\{(\pi_p \circ f)(s_\beta) : \delta < \beta < \alpha\}$. This is a contradiction so we obtain that $\sigma(\alpha) > \sigma(\delta)$ for all $\delta < \alpha$.

This completes the construction and it easily follows that the set $S = \{s_\alpha : \alpha < \omega_1\}$ has the properties required. □

In our arguments below, we will construct our acceptable tree T by recursion. So we will have the typical situation where we have constructed a tree T' which will eventually be a subtree of T. Clearly, at this point of the construction, it is possible to consider the space of branches bT' of T'. We will use the notation we defined for the branch space on bT'. Also, notice that if $q \in bT'$ and $p \in bT$ is such that $q \subset p$, then there is a projection map $\pi^p_q : X_p \to X_q$ where $\pi^p_q(x) = x$ if $x \subset q$ and $\pi^p_q(x) = q$ if $q \subset x \subset p$.

3. Proof of Theorem 1.4

We will assume $\mathfrak{b} = \mathfrak{c} = \omega_2$ in this section. In order to construct the space required in Theorem 1.4, we will construct a sequence $\{t_\alpha : \alpha < \mathfrak{c}\} \subset 2^{<\mathfrak{c}}$ with the property that for all $\beta < \mathfrak{c}$ there is $\alpha < \beta$ and $i \in 2$ such that $t_\beta = (t_\alpha)^\frown i$. Then our acceptable tree will be

$$T = \{(t_\alpha)^\frown i : \alpha < \mathfrak{c}, i \in 2\}.$$

We will also recursively define a T-algebra $\{a_t : t \in T\}$ (with the notation of Section 2) along with the definition of the tree. Then the branch space bT will be the space we are looking for. Clearly, bT is compact.

For $\alpha < \omega_1$, define $t_\alpha \in 2^\alpha$ to be such that $t_\alpha(\beta) = 0$ for all $\beta < \alpha$ and let $\{a_{t_{\alpha+1}} : \alpha < \omega_1\}$ be any strictly $\subset^*$-decreasing sequence of infinite subsets of ω. We remark that by the discussion above, both $(t_\alpha)^\frown 0 = t_{\alpha+1}$ and $(t_\alpha)^\frown 1 = t^*_{\alpha+1}$ will be in T, and both $a_{t_{\alpha+1}}$ and $a_{t^*_{\alpha+1}} = \omega \setminus a_{t_{\alpha+1}}$ are defined, for every $\alpha < \omega_1$. Let $\rho \in 2^{\omega_1}$ be such that $\rho(\beta) = 0$ for all $\beta < \omega_1$.

Clearly, the topology defined on $X_\rho = \{x_{t_\alpha} : \alpha < \omega_1\} \cup \{\rho\}$ is the order topology. For sake of notational simplicity, for each $\alpha < \omega_1$, the filter x_{t_α} will be denoted by y_α. Once we have the tree T completely defined, if $p \in bT$ is such that $\rho \subset p$, then the function $\pi^p_\rho \circ \pi_p$ maps bT onto X_ρ. We make a record of this fact as follows.

3.1. Claim X_ρ is naturally homeomorphic to $\omega_1 + 1$.

In fact, in our construction, ρ will never be extended so $\rho \in bT$ (see Claim 3.5 below). However, even before we know that, we can already prove that bT has uncountable tightness.

3.2. Claim There exists $p \in bT$ such that $\rho \subset p$ and p has uncountable tightness in bT.

Proof. For each $\alpha < \omega_1$, choose any $q_\alpha \in bT$ such that $t_{\alpha+1} \subset q_\alpha$ and $q_\alpha(\alpha + 1) = 1 \neq t_{\alpha+2}(\alpha + 1)$. Define $A = \{q_\alpha : \alpha < \omega_1\}$ and let $p \in bT$ be any complete accumulation point of A. We claim that there is no countable subset of A with p in its closure.

First, notice that p extends ρ. Otherwise, there is some $\beta < \omega_1$ such that $t_{\alpha+1} \subset p$ for $\alpha < \beta$ and $p(\beta) \neq t_{\beta+1}(\beta)$. Then $\omega \setminus a_{t_{\beta+1}}$ is in the ultrafilter defined by p but not in the ultrafilter defined by q_α for all $\beta \leq \alpha < \omega_1$. Thus, we obtain a contradiction.

Now, let $N \subset \omega_1$ be countable, we will show that $\{q_\alpha : \alpha \in N\}$ does not have p in its closure. Let $\beta < \omega_1$ be an upper bound of N. If $\alpha \in N$, by the definition of q_α, $\omega \setminus a_{t_{\alpha+1}}$ is in the ultrafilter defined by q_α. Since $a_{t_{\beta+1}} \subset^* a_{t_{\alpha+1}}$, $\omega \setminus a_{t_{\beta+1}}$ is in the ultrafilter defined by q_α. Thus, $a_{t_{\beta+1}}$ gives a neighborhood of q that does not contain q_α, for all $\alpha \in N$. Thus, the conclusion follows. □

So we are left to define $\{t_\alpha : \omega_1 \leq \alpha < \mathfrak{c}\}$ and we will do this by recursion. Given $\omega_1 \leq \alpha < \mathfrak{c}$, let T_α be the nodes of the tree that have been defined before step α, that is, the set $\{(t_\beta)^\frown i : \beta < \alpha, i \in 2\}$. On step $\alpha \in \mathfrak{c} \setminus \omega_1$, we will choose t_α among bT_α, the branches of T_α (notice that all branches are of length $< \mathfrak{c}$), and define $a_{(t_\alpha)^\frown 0}$ and $a_{(t_\alpha)^\frown 1}$. Consider a surjective function $e = (e_0, e_1) : \mathfrak{c} \setminus \omega_1 \to (\mathfrak{c} \setminus \omega_1) \times \mathfrak{c}$ such that for all $\alpha < \mathfrak{c}$, $e_0(\alpha) < \alpha$.

It is known that under $\mathfrak{c} \leq \omega_2$, every sequentially compact, compact space is pseudoradial. This was proved by Šapirovskiĭ under CH [15] and by Juhász and Szentmiklóssy under $\mathfrak{c} = \omega_2$ [10]. Thus, in order to make the space pseudoradial, it is enough to make it sequentially compact.

Let $\omega_1 \leq \alpha < \mathfrak{c}$. Let $\{f_{\langle\alpha,\beta\rangle} : \beta < \mathfrak{c}\} \subset {}^\omega T_\alpha$ be an enumeration of all increasing ω-sequences of elements of T_α. In other words, $f_{\langle\alpha,\beta\rangle} : \omega \to T_\alpha$ is such that $f_{\langle\alpha,\beta\rangle}(n) \subset f_{\langle\alpha,\beta\rangle}(n+1)$ for all $n < \omega$. We will require the following inductive assumption.

> $(a)_\alpha$ For every $\omega_1 \leq \beta < \alpha$, there are $A_\beta \in [\omega]^\omega$ and $q \in bT_\alpha$ such that $f_{e(\beta)}[A_\beta] \subset q$ and $\{x_{f_{e(\beta)}(n)} : n \in A_\beta\}$ converges to q in X_q.

3.3. Claim $(a)_\alpha$ for all $\omega \leq \alpha < \omega_1$ implies that bT is sequentially compact.

Proof. By Lemma 2.2, bT is sequentially compact if and only if for every increasing ω-sequence f of elements of T, there is a branch p extending f such that some subsequence of f converges to p in X_p. Indeed, $f = f_{e(\beta)}$ for some $\beta < \omega_1$, and this implies that $\{x_{f_{e(\beta)}(n)} : n \in A_\beta\}$ converges to some $p \in bT$. To see this, notice that $(a)_\alpha$ is preserved under limits in the following sense: if $\lambda \leq \mathfrak{c}$ is a limit ordinal and $\{q_\alpha : \alpha < \lambda\} \subset T$ are such that $\{x_{f_{e(\beta)}(n)} : n \in A_\beta\}$ converges to q_α in X_{q_α} for all $\alpha < \lambda$, then $\{x_{f_{e(\beta)}(n)} : n \in A_\beta\}$ converges to $q = \bigcup\{q_\alpha : \alpha < \lambda\}$ in X_q. □

Next we would like to add some inductive hypothesis so that at the end of the construction, we obtain a point with no copies of ω_1 converging to it. Our strategy will be to never split the branch $\rho \in bT_{\omega_1}$ so that it remains a branch of bT at the end of the construction, and kill all possible copies of ω_1 converging to it. According to Lemma 2.3, we can test convergence to ρ just by looking at copies of ω_1 contained in X_ρ. Also, all copies of ω_1 contain copies of the ordinal

$$\omega^{\cdot 2} + 1 = \{\omega \cdot n + m : n, n < \omega\}$$

which is homeomorphic to the one-point compactification of the free union of countably many convergent sequences.

So the strategy will be to consider all copies of $\omega^{\cdot 2}+1$ contained in X_ρ and make sure that they are not lifted to copies of $\omega^{\cdot 2} + 1$ in bT. Thus, let $\{s(\alpha) : \alpha < \mathfrak{c}\}$ be the set of all continuous bijective functions with domain $\omega^{\cdot 2}+1$ and image contained in ω_1. We require the following inductive assumption.

$(b)_\alpha$ For every $\omega_1 \leq \beta < \alpha$, there are $C_\beta \in [\omega]^\omega$, a function $\varphi_\beta \in {}^\omega\omega$, and $t \in T_\alpha$ with $t_{s(\omega\cdot 2)} \subset t$ such that $a_t \in y_{s(\omega\cdot n)}$ and $a_{t^\star} \in y_{s(\omega\cdot n+\varphi_\beta(n))}$ for every $n \in C_\beta$.

So assume that we are in step $\alpha < \mathfrak{c}$ of the construction. We need to choose t_α, A_α, C_α, φ_α and define the partition $\omega = a_{(t_\alpha)^\frown 0} \cup a_{(t_\alpha)^\frown 1}$.

First, we explain how to choose A_α. Let $q_\emptyset = \bigcup f_{e(\alpha)}$ and $B_\emptyset = \omega$; then it is easy to see that in $X_{q_\emptyset}$, $\{x_{f_{e(\alpha)}(n)} : n \in B_\emptyset\}$ converges to $q_\emptyset$. If there is $q \in bT_\alpha$ such that $\{x_{f_{e(\alpha)}(n)} : n \in B_\emptyset\}$ converges to q in X_q, we define $A_\alpha = \omega$. Otherwise, there exists $q'_\emptyset \in T_\alpha$ such that $\{x_{f_{e(\alpha)}(n)} : n \in B_\emptyset\}$ converges to $q'_\emptyset$ in $X_{q'_\emptyset}$ but for $i \in 2$, $\{x_{f_{e(\alpha)}(n)} : n \in B_\emptyset\}$ does not converge to $(q'_\emptyset)^\frown i$ in $X_{(q'_\emptyset)^\frown i}$. Let $q_i = (q'_\emptyset)^\frown i$ for $i \in 2$. Then by the definition of $\mathcal{T}$-algebra, there is a partition $B_\emptyset = \omega = B_0 \cup B_1$ such that $\{x_{f_{e(\alpha)}(n)} : n \in B_i\}$ converges to q_i for $i \in 2$. Continuing in this fashion, by recursion on ${}^{<\omega}\omega$, we try to construct a sequence of nodes $\{q_s : s \in {}^{<\omega}\omega\} \subset T_\alpha$ and a sequence of sets $\{B_s : s \in {}^{<\omega}\omega\} \subset [\omega]^\omega$ such that $\{x_{f_{e(\alpha)}(n)} : n \in B_s\}$ converges to q_s in X_{q_s}. If there is some s such that there is a branch $q \in bT_\alpha$ with $\{x_{f_{e(\alpha)}(n)} : n \in B_s\}$ converging to q in X_q, we define $A_\alpha = B_s$. Otherwise, given $s \in {}^{<\omega}\omega$, we can always choose incompatible $q_{s^\frown 0}$, $q_{s^\frown 1}$ above q_s and a partition $B_s = B_{s^\frown 0} \cup B_{s^\frown 1}$ as required. Assume that we never stopped in the construction (otherwise, we are done). For each $\varphi \in {}^\omega\omega$, let $q_\varphi = \cup\{q_s : s \subset \varphi\}$ and let B_φ be any pseudointersection of $\{B_s : s \subset \varphi\}$; it easily follows that $\{x_{f_{e(\alpha)}(n)} : n \in B_\varphi\}$ converges to q_φ in X_{q_φ}. The set $\{B_\varphi : \varphi \in {}^\omega\omega\}$ is of size $\mathfrak{c}$ and $|T_\alpha| < \mathfrak{c}$, so this means that there is $\psi \in {}^\omega\omega$ such that $q_\psi \in bT_\alpha$. Define $A_\alpha = B_\psi$ and let $r = q_\psi$. Then $\{x_{f_{e(\alpha)}(n)} : n \in A_\alpha\}$ converges to $r \in bT_\alpha$ in X_r.

The next step is to choose $t_\alpha \in bT_\alpha$ and $C_\alpha \in [\omega]^\omega$. Consider the copy of $\omega^{\cdot 2}$ given by $\{y_{s(\alpha)(\xi)} : \xi \in \omega^{\cdot 2}\}$ in X_ρ. By an argument similar to the one in the previous paragraph, it is possible to find $C_\alpha \subset \omega$ and some $t_\alpha \in bT_\alpha$ such that $\bigcup s(\alpha) \subset t_\alpha$ and the sequence $\{y_{s(\alpha)(\omega\cdot n)} : n \in C_\alpha\}$ converges to t_α in X_α.

Recall that according to $(a)_{\alpha+1}$, we need to preserve the convergence of each sequence $f_{e(\beta)}{\restriction}_{A_\beta}$ for all $\beta \leq \alpha$. Since we are choosing to split t_α, we only need to worry about those sequences with $f_{e(\beta)}[A_\beta] \subset t_\alpha$ (and in fact only those that converge to t_α).

Thus, we have to define $\varphi_\alpha : \omega \to \omega \setminus \{0\}$ in such a way that the sequence $\{y_{s(\alpha)(\omega\cdot n+\varphi_\alpha(n))} : n \in B_\alpha\}$ is almost disjoint with each of $f_{e(\beta)}[A_\beta]$ for all $\beta \leq \alpha$. Consider the set

$$S = \big\{\beta \leq \alpha : \sup(f_{e(\beta)}[A_\beta]) = s(\alpha)(\omega^{\cdot 2})\big\}$$

Let $\beta \leq \alpha$. If $\beta \notin S$, there is nothing to worry about. Otherwise, there exists a function $\psi_\beta : \omega \to \omega$ such that for every $n < \omega$,

$$\big\{y_{s(\beta)(\omega\cdot n+m)} : \psi_\beta(n) \leq m\big\} \cap \big\{f_{e(\beta)}(n) : n \in A_\beta\big\} = \emptyset.$$

Consider also the following result the proof of which, being standard, we shall omit.

3.4. Lemma Assume that we have a topological space with underlying set $\omega^{\cdot 2}+1$ that satisfies the following properties.

(1) Every point of the form $\omega \cdot n + m$ with $n < \omega$ and $0 < m < \omega$ is isolated.
(2) For each $n < \omega$, $\{\omega \cdot n + m : m < \omega\}$ converges to $\omega \cdot (n+1)$.
(3) $\{\omega \cdot n : n < \omega\}$ converges to $\omega^{\cdot 2}$.

(4) $\omega^{\cdot 2}$ has character strictly less than $\mathfrak{b}$.

Then there exists $\varphi : \omega \to \omega \setminus \{0\}$ such that $\{\omega \cdot n + \varphi(n) : n < \omega\}$ converges to $\omega^{\cdot 2}$.

This implies that we can choose $\varphi_\alpha : C_\alpha \to \omega \setminus \{0\}$ such that $\{y_{s(\alpha)(\omega \cdot n + \varphi_\alpha(n))} : n \in C_\alpha\}$ converges to t_α in X_{t_α}.

By $\mathfrak{b} = \omega_2$ we can further assume that for every $\beta \in S$ we have that $\{n \in C_\alpha : \varphi_\alpha(n) \leq \psi_\beta(n)\}$ is finite. Thus, we obtain that $\{y_{s(\alpha)(\omega \cdot n + \varphi_\alpha(n))} : n \in C_\alpha\}$ is almost disjoint from $f_{e(\beta)}[A_\beta]$ for all $\beta \leq \alpha$.

So all that remains is to define the partition $\omega = a_{(t_\alpha)^\frown 0} \cup a_{(t_\alpha)^\frown 1}$ in such a way that $(a)_{\alpha+1}$ and $(b)_{\alpha+1}$ hold. For the sake of notational simplicity, let $z_n = y_{s(\alpha)(\omega \cdot n + \varphi_\alpha(n))}$ for all $n \in C_\alpha$. Given $a \in B_{t_\alpha} \setminus u_{t_\alpha}$, we will denote its associated clopen set as

$$a^* = \{x_t : t \subset t_\alpha, a \in x_t\}.$$

For each $n \in C_\alpha$, since z_n is a point of first countability of X_{t_α} so let $\{c(n,m) : m < \omega\} \subset B_{t_\alpha}$ define a local open base at z_n.

Assume $\beta < o(t_\alpha)$. The point $x_{t_\alpha \restriction \beta}$ is not a limit point of $\{z_n : n \in C_\alpha\}$ (in X_{t_α}). By normality, there are open sets U_β and V_β with $U_\beta \cap V_\beta = \emptyset$ such that $x_{t_\alpha \restriction \beta} \in U_\beta$ and $\{z_n : n \in C_\alpha\} \subset^* V_\beta$. Then there exists a function $g_\beta : C_\alpha \to \omega$ such that $\{n \in C_\alpha : c(n, g_\beta(n))^* \not\subset V_\beta\}$ is finite.

Now, let $\omega_1 \leq \beta \leq \alpha$. We know that $\{z_n : n \in C_\alpha\}$ is almost disjoint with $f_{e(\beta)}[A_\beta]$ so we can find a function $h_\beta : C_\alpha \to \omega$ such that $\{n \in C_\alpha : f_{e(\beta)}[A_\beta] \cap c(n, g_\beta(n))^* \neq \emptyset\}$ is finite. Also, there exists a function $h : C_\alpha \to \omega$ such that for all $n < \omega$, $y_{s(\alpha)(\omega \cdot n)} \notin c(n, h(n))$.

The set of functions

$$\{g_\beta : \beta < o(t_\alpha)\} \cup \{h_\beta : \omega_1 \leq \beta \leq \alpha\} \cup \{h\}$$

is of size $< \mathfrak{b}$ so there exists $g : C_\alpha \to \omega$ that bounds them all mod finite. We obtain an open set $W = \bigcup \{c(n, g(n))^* : n \in C_\alpha\}$ of X_{t_α} with the following properties:

(i) $\{z_n : n \in C_\alpha\} \subset W$,
(ii) the only limit point of W in X_{t_α} is t_α,
(iii) $\{y_{s(\alpha)(\omega \cdot n)} : n \in C_\alpha\} \cap W = \emptyset$, and
(iv) for every $\omega_1 \leq \beta < \omega_1$ such that $f_{e(\beta)}[A_\beta] \subset t_\alpha$ then one of the two following conditions holds:
  (a) $\{x_{f_{e(\beta)}(n)} : n < \omega\} \cap W$ is finite, or
  (b) $\{x_{f_{e(\beta)}(n)} : n < \omega\}$ does not converge to t_α in X_{t_α} .

Thus, we define

$$a_{(t_\alpha)^\frown 0} = \bigcup \{c(n, g(n)) : n \in C_\alpha\},$$

and $a_{(t_\alpha)^\frown 1} = \omega \setminus a_{(t_\alpha)^\frown 0}$. From properties (i) to (iv) above it is easy to see that both $(a)_{\alpha+1}$ and $(b)_{\alpha+1}$ will hold. Thus, we have finished our construction.

Notice that in our construction, the branches that we split in every step have subsequences of X_ρ converging to them. Thus, we can infer the following.

3.5. Claim The branch $\rho \in bT_{\omega_1}$ is never split, so $\rho \in bT_\alpha$.

By inductive hypothesis $(a)_\alpha$ for all $\alpha \in \mathfrak{c} \setminus \omega_1$ we obtain that bT is sequentially compact (Claim 3.3). Since $\mathfrak{c} = \omega_2$ we obtain, as discussed above, that bT is pseudoradial.

Finally, we prove that bT is not strongly pseudoradial. We will prove that this property fails at $\rho \in bT$. If bT were strongly pseudoradial, there would be an infinite cardinal κ and an embedding $f : \kappa+1 \to T$ such that $f[\kappa] \subset bT \setminus \{\rho\}$ and $f(\kappa) = \rho$. By Claims 3.2 and 3.5, and by the fact that ρ has character ω_1 in bT, we obtain that $\kappa = \omega_1$. By Lemma 2.3, we may assume that $\pi_\rho \circ f$ is injective. Since $\pi_\rho \circ f$ is continuous, it is an embedding so $(\pi_\rho \circ f)[\omega_1]$ is a copy of ω_1 contained in X_ρ. So in particular, there is a copy of $\omega^{\cdot 2} + 1$ contained in X_ρ, we may assume that the first one in our enumeration is $\{y_{s(\gamma)(\xi)} : \xi \in \omega^{\cdot 2} + 1\}$. According to Lemma 2.2, there is $q \in bT$ such that $y_{s(\gamma)(\omega^{\cdot 2})} \subset q$ and $\{y_{s(\gamma)(\xi)} : \xi \in \omega^{\cdot 2}\} \cup \{q\} \subset X_q$ is homeomorphic to $\omega^{\cdot 2} + 1$. But then, according to $(b)_{\gamma+1}$, it easily follows that the sets $\{y_{s(\omega\cdot n)} : n \in C_\gamma\}$ and $\{y_{s(\omega\cdot n+\varphi_\gamma(n))} : n \in C_\gamma\}$ are separated by clopen sets of X_q. This is a contradiction so indeed bT is not strongly pseudoradial.

4. Forcing copies of ω_1

Here we give a generalization of several results ([8], [1] and [6]) concerning the existence of a proper forcing that forces a copy of ω_1.

Let X be any completely regular countably compact non-compact space with a base $\mathcal{B}$ of open sets such that $X \in \mathcal{B}$ and $\emptyset \notin \mathcal{B}$. Given $x \in X$, $\mathcal{B}_x$ will denote $\{B \in \mathcal{B} : x \in B\}$. For a subset H of X, we consider the ω-closure of H

$$\mathrm{cl}_\omega(H) = \bigcup\{\overline{a} : a \in [H]^{\leq \aleph_0}\}.$$

Say that H is ω-closed if $\mathrm{cl}_\omega(H) = H$.

Suppose that $\mathcal{F}$ is a countably complete maximal free filter of ω-closed subsets of X. Choose any regular cardinal κ such that $X, \mathcal{B}, \mathcal{F}, \omega_1 \in H(\kappa)$.

4.1. Definition For any countable set M, we define the trace of the filter $\mathcal{F}$ as

$$\mathrm{Tr}(\mathcal{F}, M) = \bigcap\{\overline{F \cap M} : F \in M \cap \mathcal{F}\}.$$

We will need the following fact. We refer the reader to [3] for the use of elementary submodels in topology.

4.2. Lemma For any countable elementary submodel $M \prec H(\kappa)$, such that $\mathcal{F} \in M$, then for any subset H of X that is in M, then

(1) if $H \cap \mathrm{Tr}(\mathcal{F}, M)$ is not empty, then $H \in \mathcal{F}^+$,
(2) if $H \in \mathcal{F}^+$ then $\overline{H \cap M}$ contains $\mathrm{Tr}(\mathcal{F}, M)$.

Proof. Assume that $H \in M$ and that $H \cap \mathrm{Tr}(\mathcal{F}, M) \neq \emptyset$. Since $\mathrm{Tr}(\mathcal{F}, M) \subset (\bigcap \mathcal{F}) \cap M$, it follows that $H \cap F \cap M \neq \emptyset$ for all $F \in \mathcal{F} \cap M$. By elementarity, it follows that $H \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ so $H \in \mathcal{F}^+$. By the maximality of $\mathcal{F}$ and elementarity, $\mathrm{cl}_\omega(H) \in \mathcal{F} \cap M$. Again by elementarity, $\mathrm{cl}_\omega(H) \cap M$ is contained in $\overline{H \cap M}$. This completes the proof. □

We next define a poset $\mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$.

4.3. Definition A condition $p \in \mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$ is a function with domain $\mathcal{M}_p$ such that, for each $M, M_1 \in \mathcal{M}_p$,

(1) $\mathcal{M}_p$ is a finite $\in$-chain of countable elementary submodels of $H(\kappa)$, each containing ω_1, X, $\mathcal{B}$, and $\mathcal{F}$,

(2) $p(M) = \langle x_{p(M)}, \mathcal{U}_{p(M)}, \mathcal{F}_{p(M)} \rangle$ is an element of $X \times [\mathcal{B}]^{<\aleph_0} \times [\mathcal{F}]^{<\aleph_0}$,
(3) $x_{p(M)} \in \mathrm{Tr}(\mathcal{F}, M)$,
(4) if $M \in M_1$, then $p(M) \in M_1$.

We define $p \leq q$ provided:

(5) $\mathcal{M}_q \subset \mathcal{M}_p$,
(6) for each $M \in \mathcal{M}_q$, $x_{p(M)} = x_{q(M)}$, $\mathcal{F}_{q(M)} = \mathcal{F}_{p(M)}$, and $\mathcal{U}_{q(M)} \subset \mathcal{U}_{p(M)}$,
(7) for each $M_1 \in \mathcal{M}_q$ and $M \in (\mathcal{M}_p \setminus \mathcal{M}_q) \cap M_1$ such that $\mathcal{M}_q \cap M_1 \in M$, then $x_{p(M)} \in U$ for any U such that $x_{q(M_1)} \in U$ and $U \in \mathcal{U}_{q(M_2)}$ for some $M_2 \in \mathcal{M}_q$ with $M_1 \subset M_2$.

We can rephrase the complicated last condition with the help of the following definitions: for $q \in \mathcal{P}$ and $M \in \mathcal{M}_q$, let

$$\mathcal{U}(q, M) = \mathcal{B}_{x_{q(M)}} \cap \bigcup \{\mathcal{U}_{q(M_1)} : M \subset M_1 \in \mathcal{M}_q\}.$$

Then let $W(q, M) = \bigcap \mathcal{U}(q, M)$ if $\mathcal{U}(q, M) \neq \emptyset$ and $W(q, M) = X$ otherwise. An alternative way to state the last condition is that

(7') if $M \in \mathcal{M}_p \setminus \mathcal{M}_q$ and if M_1 is the $\in$-minimal element of $\mathcal{M}_q$ containing M, then $x_{p(M)} \in W(q, M_1)$.

We can notice that it then follows that $W(p, M) \subset W(q, M_1)$. This is a key property to have to ensure that $\mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$ is transitive.

We do not include a proof that $\mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$ is proper because we will give a proof of a stronger statement, Theorem 4.8 below. However, we do prove the following.

4.4. Proposition If $G \subset \mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$ is a generic filter, then:

(1) $\mathcal{M}_G = \bigcup \{\mathcal{M}_p : p \in G\}$ is an uncountable $\in$-chain,
(2) $C_G = \{M \cap \omega_1 : M \in \mathcal{M}_G\}$ is a cub subset of ω_1, and
(3) for each $M \in \mathcal{M}_G$ such that $M \cap \omega_1 = \delta$ is a limit point of C_G, $\mathcal{F} \cap M = \bigcup \{\mathcal{F} \cap M' : M' \in \mathcal{M}_G \cap M\}$.

Proof. Let us use $\mathcal{P}$ to denote $\mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$. We omit the easy proof that $\mathcal{M}_G$ and C_G are uncountable. For each countable $M \prec H(\kappa)$ such that $\mathcal{F} \in M$, let $\{F(M, n) : n \in \omega\}$ be an enumeration of $\mathcal{F} \cap M$.

Claim 1: For each $\delta \in \omega_1$ and $n \in \omega$, the following set is dense in $\mathcal{P}$:

$$\begin{aligned} D_{\delta,n} \quad = \quad & \{p \in \mathcal{P} \colon (\exists M \in \mathcal{M}_p) \text{ such that either} \\ & (\delta \subset M \text{ and } (\forall q \leq p)\ (\textstyle\bigcup (M \cap \mathcal{M}_q) = \bigcup (M \cap \mathcal{M}_p))) \text{ or} \\ & (M \cap \omega_1 = \delta \text{ and } (\exists M_1 \in \mathcal{M}_p \cap M)\ (F(M, n) \in \mathcal{F}_{p(M_1)}))\}. \end{aligned}$$

Informally speaking, the first condition asserts that δ is not a limit of C_G; the second condition asserts that $\delta \in C_G$ and that the element $F(M, n)$ of M will appear in every $M' \in \mathcal{M}_G$ with $M_1 \in M' \in M$.

Proof of Claim 1: Let $p_1 \in \mathcal{P}$ be arbitrary. It is easy to extend p_1 so as to ensure that $\delta \in \bigcup \mathcal{M}_{p_1}$. Let δ^* be the minimum of the set $\{M \cap \omega_1 : (\exists p^* < p_1)\ M \in \mathcal{M}_{p^*} \text{ and } \delta \leq M \cap \omega_1\}$. Choose $p_2 \leq p_1$ such that $\delta^* \in \{M \cap \omega_1 : M \in \mathcal{M}_{p_2}\}$.

Suppose first that $\delta^* = \delta$ and let $M \in \mathcal{M}_{p_2}$ be such that $\delta^* = M \cap \omega_1$. If $p_2 \notin D_{\delta,n}$ then the first clause in the definition must fail. Therefore we may choose some $q \leq p_2$ such that $M \cap \mathcal{M}_q \neq \emptyset$ and $M_1 = \bigcup (M \cap \mathcal{M}_q)$, the maximum element of $M \cap \mathcal{M}_q$, is not in $\mathcal{M}_{p_2}$. Simply define q^* where the only change from q is that

$F(M,n)$ is in $\mathcal{F}_{q^*(M_1)}$. Since M_1 is the maximum element of $\mathcal{M}_q \cap M$ and since $F(M,n)$ is evidently in M, it follows that $q^* \in \mathcal{P}$ and through routine checking that q^* is also below p_2. Therefore q^* is in $D_{\delta,n}$.

Now we assume that $\delta < \delta^*$ and again that p_2 is not in $D_{\delta,n}$. Let M be the element of $\mathcal{M}_{p_2}$ with $M \cap \omega_1 = \delta^* > \delta$ and note that (by the failure of the first clause) there is a $q \leq p_2$ such that $\bigcup(M \cap \mathcal{M}_q) \neq \bigcup(M \cap \mathcal{M}_{p_2})$. Choose such a q and again let M_1 be the maximum element of $M \cap \mathcal{M}_q$. Fix any strictly descending sequence $\{F_\alpha : \alpha \in \omega_1\} \subset \mathcal{F}$ that is an element of M_1. Note that $F_\delta \in M$ since $\{F_\alpha : \alpha \in \omega_1\} \in M$. We now define an extension p of p_2 that is in $D_{\delta,n}$. Set $\mathcal{M}_p$ equal to $\mathcal{M}_{p_2} \cup \{M_1\}$ and $p_2 \subset p$. To define p we just have to choose the value for $p(M_1)$. We let $p(M_1) = \langle x_{q(M_1)}, \mathcal{U}_{q(M_1)}, \{F_\delta\}\rangle$. Since we already have that q is an extension of p_2, it is routine to check that p is also an extension of p_2. We check that p satisfies the first condition of $D_{\delta,n}$. If $q < p$ and $M_1 \in M' \in \mathcal{M}_q \cap M$, then $p(M_1) \in M'$, implying that $\delta \in M'$, and this contradicts the definition of δ^*.

Claim 2: If $G \subset \mathcal{P}$ is a filter that meets $D_{\delta,0}$ for all $\delta \in \omega_1$, then $C = \{M \cap \omega_1 : (\exists p \in G)\ M \in \mathcal{M}_p\}$ is a closed and unbounded subset of ω_1.

Proof of Claim 2: For $\delta \in \omega_1$, the fact that G meets $D_{\delta,0}$ implies that $C \setminus \delta$ is not empty. To show that C is closed we assume $\delta \notin C$ and show that it is not a limit point of C by showing that $C \cap \delta$ has a maximum element. Choose $p \in G \cap D_{\delta,0}$ and let $M \in \mathcal{M}_p$ be as in the definition of $D_{\delta,0}$. Since δ is not in C, $\delta \in M$. If $\mathcal{M}_p \cap M$ is empty, let $\beta = 0$, otherwise let $\bar{M}$ be the maximum element of $\mathcal{M}_p \cap M$, and let $\beta = \bar{M} \cap \omega_1$. It now follows that for all $q \leq p$ in G and $M' \in \mathcal{M}_q \cap M$, then $M' \cap \omega_1$ is less than or equal to β. It thus follows that C is disjoint from the interval (β, δ).

Claim 3: If G is a filter that meets $D_{\delta,n}$ for all $\delta \in \omega_1$ and $n \in \omega$, then condition (3) will hold.

Let $M \in \mathcal{M}_G$ such that $\delta = M \cap \omega_1$ is a limit. We will prove that each element $F(M,n) \in \mathcal{F} \cap M$ is in some $M' \in \mathcal{M}_G \cap M$. We may choose $p \in G \cap D_{\delta,n}$ such that $M \in \mathcal{M}_p$.

First, we argue that the first clause in $D_{\delta,n}$ does not hold. Since δ is a limit of C_G, $M \cap \mathcal{M}_G$ is infinite. But $M \cap \mathcal{M}_p$ is finite so there must exist $q \in G$ such that $M \cap \mathcal{M}_q \not\subset M \cap \mathcal{M}_p$. Any common extension of p and q contradicts the first clause.

Thus, the second clause holds. Let $M_1 \in \mathcal{M}_p \cap M$ such that $F(M,n) \in \mathcal{F}_{p(M_1)}$. Since δ is a limit of C_G, there exists $M' \in \mathcal{M}_G$ be such that $M_1 \cap \omega_1 < M' \cap \omega_1 < \delta$. Let $q \in \mathcal{P}$ with $q \leq p$ and $M' \in \mathcal{M}_q$. Then $M_1 \in M'$ are elements of $\mathcal{M}_q$, by the last condition in the definition of $q \in \mathcal{P}$ it follows that $q(M_1) \in M'$. This implies that $\mathcal{F}_{p(M_1)} = \mathcal{F}_{q(M_1)} \in M'$ so $F(M,n) \in M'$. □

Now, assume that $G \subset \mathcal{P}$ is a filter. This implicitly defines a function

$$f : C_G \to \{x_{p(M)} : p \in G,\ M \in \mathcal{M}_p\}$$

where $M \cap \omega_1 \in C_G$ is sent to $x_{p(M)}$. Moreover, if G is generic, then C_G is homeomorphic to ω_1 by (b) in Proposition 4.4.

4.5. Lemma If $G \subset \mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$ is a generic filter, then X contains a copy of ω_1 with the subspace topology inherited from $\mathcal{B}$.

Proof. Given $x \in X$ and $U \in \mathcal{B}_x$, the set

$$E_{x,U} = \{p \in \mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa} : \exists M \in \mathcal{M}_p \ (x = x_{p(M)}) \Rightarrow \exists M' \in \mathcal{M}_p \setminus M \ (U \in \mathcal{U}_{p(M')})\}$$

is easily seen to be dense in $\mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$.

Now we prove that f is continuous, let $\delta \in C_G$ and $x_\delta = f(\delta)$. It is enough to prove that every time $U \in \mathcal{B}$ with $x_\delta \in U$ there exists $\beta < \delta$ such that $\{x_\alpha : \beta < \alpha \leq \delta\} \subset U$.

This is clearly true when δ is not a limit, so assume in the following that δ is a limit. Let $p \in G \cap E_{x,U}$ be such that there is $M \in \mathcal{M}_p$ with $x_\delta = x_{p(M)}$. We may assume that $\mathcal{M} \cap M \neq \emptyset$ and let $M_1 = \bigcup(\mathcal{M} \cap M)$. Define $\beta = M_1 \cap \omega_1$. If $\beta < \alpha < \delta$, there is $q \in G$ and $M_2 \in \mathcal{M}_q$ with $M_2 \cap \omega_1 = \alpha$. We may assume that $q \leq p$. Since $p \in E_{x,U}$, $U \in \mathcal{U}_{p(M')}$ for some $M' \in \mathcal{M}_p \setminus M$. By the last condition in the definition of $q \leq p$, it follows that $x_{q(M_2)} \in U$.

This shows that $f : C_G \to X$ is continuous. Recall that C_G is homeomorphic to ω_1. Even if f is not a homeomorphism, we claim that some restriction of f is an embedding of ω_1 to X. Indeed, consider the Čech-Stone extension $\beta f : \beta C_G \to \beta X$. The only case in which βf is not an embedding is if it is not injective and this can only happen if there is $\alpha \in C_G$ such that $\beta f(\omega_1) = f(\alpha)$. Such an α is unique because f is injective. So $\beta f \restriction (C_G \setminus (\alpha + 1)) = f \restriction (C_G \setminus (\alpha + 1))$ is an embedding. Since $C_G \setminus (\alpha + 1)$ is homeomorphic to ω_1, the statement of the lemma follows. □

Now we formulate a strong generalization that encompasses PFA results such those for first countable spaces ([8]), or spaces of countable tightness ([6]), or even spaces with character at most ω_1([1]).

4.6. Definition Let X be a countably compact space. A function φ will be called suitable if the domain of φ is the set of all closed subsets of X, and for all $B \in \mathrm{dom}(\varphi)$, $\overline{\varphi(B)} = B$.

4.7. Definition If X is countably compact, φ is a suitable function on X, and $\mathcal{F}$ is a maximal free filter of closed subsets of X, then the poset $\mathcal{P}^{\varphi}_{X,\mathcal{B},\mathcal{F},\kappa}$ is the subposet of $\mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$ consisting of all those $p \in \mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$ satisfying that $x_{p(M)} \in \varphi(\mathrm{Tr}(\mathcal{F}, M))$ for all $M \in \mathcal{M}_p$.

For example, if X is ω-bounded and of countable tightness, then (under PFA) φ may be all those points of relative countable character ([6]).

4.8. Theorem If φ is suitable, then $\mathcal{P}^{\varphi}_{X,\mathcal{B},\mathcal{F},\kappa}$ is proper.

Proof. Let $\mathcal{P} = \mathcal{P}_{X,\mathcal{B},\mathcal{F},\kappa}$ and let $\mathcal{P} \in H(\theta)$ for a regular cardinal θ. Also let M be a countable elementary submodel of $H(\theta)$ such that $\mathcal{P} \in M$. Then to prove that $\mathcal{P}$ is proper, it suffices to prove that for any $p \in \mathcal{P}$ with $M \cap H(\kappa) \in \mathcal{M}_p$, then p is an $(M, \mathcal{P})$-generic condition. Equivalently, if $D \in M$ is any dense open subset of $\mathcal{P}$, we must show there is an $r \in D \cap M$ that is compatible with p.

By extending p we may assume that $p \in D$. Let $\mathcal{M}_p \setminus M$ be enumerated in increasing order as $\{M^p_0, \cdots, M^p_{\ell-1}\}$. Note that $M^p_0 = M \cap H(\kappa)$. An important property of $\mathcal{P}$ is that $\bar{p} = p \restriction (\mathcal{M}_p \cap M)$ is itself an element of $\mathcal{P}$ and is in M.

Say that $q \in \mathcal{P}$ end-extends $\bar{p}$ if $q \leq \bar{p}$ and $\mathcal{M}_{\bar{p}}$ is an initial segment of $\mathcal{M}_q$. Now let D_ℓ be the set of $q \in D$ that end-extend $\bar{p}$ and satisfy that $|\mathcal{M}_q \setminus \mathcal{M}_{\bar{p}}| = \ell$. It

follows that $D_\ell \in M$. For $q \in D_\ell$, let $\mathcal{M}_q \setminus \mathcal{M}_{\bar{p}}$ be enumerated as $\{M_i^q : i < \ell\}$ and let $\{x_i^q : i < \ell\}$ enumerate $\{x_{q(M_i^q)} : i < \ell\}$. We leave the reader to verify that if $r \in D_\ell \cap M$ is such that $\{x_i^r : i < \ell\}$ is a subset of $W(p, M_0^p)$, then r is compatible with p. So, it becomes our task to show there is such an $r \in D_\ell \cap M$.

In order for this proof to work, we have to argue in M_0^p rather than in M. To this end, we work with the set

$$T_\ell = \{\vec{x}^q = \langle x_i^q : i < \ell\rangle : q \in D_\ell\}.$$

It is immediate that $T_\ell \in M \cap H(\kappa) = M_0^p$. For any $\vec{x}^q \in T_\ell$ and $0 < j < \ell$, let $\vec{x}^q \restriction j$ denote $\langle x_i^q : i < j\rangle$ and $\vec{x}^q \restriction 0$ is the empty sequence.

Of course $T_\ell \subset X^\ell$, we will recursively define a sequence $T_j \subset X^j$ for $j < \ell$. For any $j < \ell$ and tuple $\vec{x} \in T_j$, we let

$$H(\vec{x}, T_{j+1}) = \{y \in X : \vec{x}^\frown y \in T_{j+1}\}.$$

Then, by recursion,

$$T_j = \{\vec{x} \in X^j : H(\vec{x}, T_{j+1}) \in \mathcal{F}^+\}.$$

This recursion is definable in M_0^p, hence for any $j < \ell$ and $\vec{x} \in M_j^p$, $H(\vec{x}, T_{j+1})$ is an element of M_j^p. It recursively follows from Lemma 4.2 that $\vec{x}^p \restriction j \in T_j$ for each $j < \ell$. This means that the empty sequence is an element of $T_0 \cap M_0^p$, implying that $H(\emptyset, T_1) \in \mathcal{F}^+ \cap M_0^p$.

By Lemma 4.2, $\mathrm{Tr}(\mathcal{F}, M_0^p)$ is contained in the closure of $H(\emptyset, T_1) \cap M_0^p$. Choose any $x_0 \in W(p, M_0^p) \cap H(\emptyset, T_1) \cap M_0^p$. By recursion, suppose we have chosen $\vec{x}_j = \langle x_0, \dots, x_{j-1}\rangle \in T_j \cap M_0^p$ so that, for each $i < j$, $x_i \in W(p, M_0^p)$. At step j, there is $x_j \in H(\vec{x}_j, T_{j+1}) \cap W(p, M_0^p) \cap M_0^p$ because $H(\vec{x}_j, T_{j+1}) \in \mathcal{F}^+$. Once we have chosen $\vec{x}_\ell \in T_\ell \cap M_0^p$, we choose, by elementarity, $r \in D_\ell \cap M$ such that $\vec{x}_\ell = \vec{x}_\ell^r$. $\square$

4.9. Theorem Assume PFA. Let X be a completely regular, countably compact, non-compact space with the property that every time $Y \subset X$ is separable, C is closed in X and $C \subset Y$ then C has a dense set of points with character al most ω_1 in C. Then X contains a copy of ω_1.

Proof. Let $\mathcal{F}$ be any maximal free closed filter. Define the function φ as follows. For any countable $M \prec H(\kappa)$ such that $X, \mathcal{F} \in M$, $\varphi(\mathrm{Tr}(\mathcal{F}, M))$ is the set of points of $\mathrm{Tr}(\mathcal{F}, M)$ that have relative character at most ω_1. For any closed set $B \subset X$ not of this form, let $\varphi(B) = B$. It should be clear that $\varphi \in H(\kappa)$. By the assumption of the theorem, φ is a suitable function. Let $\mathcal{B}$ be the set of all non-empty open subsets of X.

Let $\mathcal{P} = \mathcal{P}^\varphi_{X,\mathcal{B},\mathcal{F},\kappa}$. We want to identify ω_1-many dense subsets of $\mathcal{P}$ so that any filter G meeting them is enough to ensure that C_G is a cub and that f is a homeomorphism. The proof in Proposition 4.4 shows that there is a family of ω_1-many dense sets that will guarantee that C_G is a cub. As noted in the proof of Lemma 4.5, we only need G to capture sufficiently many neighborhoods of each x_δ in order to ensure that f is continuous.

For each separable B in the domain of φ and each $x \in \varphi(B)$, let $\{U(B, x, \alpha) : \alpha \in \omega_1\} \subset \mathcal{U}_x$ be chosen so that $\{B \cap U(B, x, \alpha) : \alpha \in \omega_1\}$ is a local base for x in B. For each $\alpha \in \omega_1$, let

$$E_\alpha = \{p \in \mathcal{P} : (\forall M \in \mathcal{M}_p\ \exists M' \in \mathcal{M}_p \setminus M)\ U(\mathrm{Tr}(\mathcal{F}, M), x_{p(M)}, \alpha) \in \mathcal{U}_{p(M')}\}.$$

It is easy to see that E_α is a dense subset of $\mathcal{P}$.

Now we assume that G is a filter on $\mathcal{P}$ and that $G \cap E_\alpha$ and $G \cap D_{\delta,n}$ are not empty for all $\alpha, \delta \in \omega_1$ and $n \in \omega$. As usual, let $\{x_\alpha : \alpha \in C_G\}$ enumerate the image of C_G by the above mentioned generic function f.

As discussed in the proof of Lemma 4.5, it is enough to prove that f is continuous. Let $\delta \in C_G$ be a limit point of C_G and let I be any cofinal sequence of $C_G \cap \delta$. By the definition of $D_{\delta,n}$, it follows that $\{\beta \in I : x_\beta \in F(M,n)\}$ is a cofinite subset of I. Therefore the set of limit points of $\{x_\beta : \beta \in I\}$ is a subset of $B = \mathrm{Tr}(\mathcal{F}, M)$. In addition, for each $\alpha \in \omega_1$, since $G \cap E_\alpha \cap D_{\delta,0}$ is not empty, $\{x_\beta : \beta \in I\} \setminus U(B, x_\delta, \alpha)$ is finite. It then follows that x_δ is the unique accumulation point of $\{x_\beta : \beta \in I\}$. □

We also include the following which shows that in some cases, we can control the point of convergence of the copy of ω_1.

4.10. Corollary Assume PFA. Let K be a compact Hausdorff space, $X \subset K$ be with the hypothesis of Theorem 4.9 and assume that $S \in [X]^{\omega_1}$ has all its complete accumulation points in $K \setminus X$. Then there is a copy of ω_1 contained in X that converges to some complete accumulation point of S.

Proof. Let $S = \{x_\alpha : \alpha < \omega_1\}$ be an enumeration. For each $\beta < \omega_1$, let

$$F_\beta = X \cap \overline{\{x_\alpha : \beta \leq \alpha < \omega_1\}}.$$

Let $\mathcal{F}$ be a maximal filter of closed sets that extends $\{F_\alpha : \alpha < \omega_1\}$ and proceed with the proof of Theorem 4.9.

For each $\alpha < \omega_1$, it is easy to prove that the set

$$D^\alpha = \{p \in \mathcal{P} : \exists M \in \mathcal{M}_p \ (F_\alpha \in \mathcal{F}_{p(M)})\}$$

is dense in $\mathcal{P}$. So we may assume that $G \cap D^\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

If $p \in G \cap D^\alpha$ and $M \in \mathcal{M}_p$ is such that $F_\alpha \in \mathcal{F}_{p(M)}$, then it follows that for any $\gamma \in C_G$ with $M \cap \omega_1 < \gamma$, $x_\gamma \in F_\alpha$. Thus,

$$\bigcap_{\alpha<\omega_1} \{x_{p(M)} : \exists p \in G, \exists M \in \mathcal{M}_p \ (\alpha < M \cap \omega_1)\} \subset \bigcap_{\alpha<\omega_1} F_\alpha,$$

which implies the statement of this Corollary. □

Recall that according to Šapirovski's result [9, 3.20, p.71], any compact space of size $< 2^{\omega_1}$ has a dense set of points of character $\leq \omega_1$. A space is ω-bounded if every countable subset has compact closure. Thus, the following result follows from Theorem 4.9.

4.11. Corollary PFA implies that any ω-bounded non-compact space of cardinality at most $\mathfrak{c}$, contains a copy of ω_1.

4.12. Corollary PFA implies that any pseudoradial space non-sequential space with radial character at most $\aleph_1$ contains a copy of ω_1.

Proof. If K is such a space, then K has uncountable tightness. This implies it has a (converging) free ω_1-sequence. So K has a subspace X that has a perfect mapping onto ω_1. With no loss of generality, X has density equal to $\aleph_1$. Let X be an element of an elementary submodel M of $H(\theta)$ (some suitably large θ) such that $M^{\omega_1} \subset M$ and $|M| = 2^{\omega_1} = \omega_2$. Since K is pseudoradial and has radial character

ω_1, it follows that X is ω-bounded and is contained in M. Now apply the previous Corollary. □

Finally, we prove the result announced in the Introduction.

Proof of Theorem 1.6. Let $p \in X$ and assume that there is a no countable sequence converging to p. Since X is almost radial, there is a thin sequence $S = \{x_\alpha : \alpha \in \kappa\}$ converging to p. As observed in [13, Lemma 5.6], we may assume that S is free. Since X has radial character ω_1 then $\kappa = \omega_1$. After this, apply the arguments in Corollaries 4.10 and 4.12 to complete the proof. □

5. Some final comments and questions

Let us start this section by making an observation of the proof of Theorem 1.4. In every step $\omega_1 \leq \alpha < \mathfrak{c}$ we can inductively notice that for every $p \in bT$ there is some subset of $\{y_\beta : \beta < \omega_1\}$, below p that has p in its closure. From this, the following follows easily.

5.1. Proposition For every $\alpha < \omega_1$, let $p_\alpha \in bT$ such that $\pi_\rho(p_\alpha) = y_\alpha$. Then the set $\{p_{\alpha+1} : \alpha < \omega_1\}$ is a free ω_1-sequence dense in bT.

Notice also that we only used $\mathfrak{c} = \omega_2$ to show that the space is pseudoradial but $b = \mathfrak{c}$ is enough for the following.

5.2. Theorem Assume $\mathfrak{b} = \mathfrak{c}$. Then there exists a compact, sequentially compact space X and a continuous function $\pi : X \to \omega_1 + 1$ such that every time $e : \omega_1 + 1 \to X$ is an embedding there exists $\alpha < \omega_1$ such that $(\pi \circ e)[\omega_1 + 1] \subset \alpha \cup \{\omega_1\}$.

Assume PFA. Then the counterexample X from Theorem 1.4 can be constructed and Theorem 1.6 holds. This means that X cannot satisfy the hypothesis of Theorem 1.6: X is either of radial character $\mathfrak{c} = \omega_2$ or X is not almost radial. However, we don't know which one of the two conditions. The only thing we know is that because of Corollary 1.3, X has character ω_2. So besides Question 1.5 we can also ask the following.

5.3. Question Does it follow from MA + $\mathfrak{c} = \omega_2$ that there is a almost radial compact Hausdorff space that is not strongly pseudoradial?

At first, when the authors of this paper attempted the proof of Theorem 1.4, we intended to kill all copies of ω_1. However, we were not able to give this construction. As we can see from the proof, we were able to kill all copies of ω_1 that converge to the distinguished point ρ, but there might exist other copies of ω_1 in other branches of the tree T.

From these considerations, one may naturally ask whether it is consistent with MA that there exists a compact, sequentially compact space of uncountable tightness that contains no topological copies of ω_1.

Consider Nyikos' example of a first countable space X that maps onto ω_1 but has no copies of ω_1. The construction of this example can be found in [8, 19.1]. Essentially, Nyikos example has underlying set $\omega_1 \times \{0, 1\}$ and it can be easily checked that any subspace of the form $(\alpha + 1) \times \{0, 1\}$, $\alpha < \omega_1$, is a compact metric space open in X. Thus, the one-point compactification of X is sequentially

compact, has uncountable tightness and has no copies of $\omega_1 + 1$. Nyikos' example can be constructed from a statement that is a consequence of $\diamondsuit$ and that is preserved under ccc forcings. We conclude the following.

5.4. Corollary It is consistent with MA and $\mathfrak{c}$ of arbitrary large size that there is a compact, sequentially compact space of uncountable tightness that contains no topological copies of $\omega_1 + 1$.

Again, by considering the result by Juhász and Szentmiklóssy [10] that pseudoradiality follows from sequential compactness under $\mathfrak{c} = \omega_2$ we obtain the following.

5.5. Corollary It is consistent with MA + $\mathfrak{c} = \omega_2$ that there is a compact pseudoradial space of radial character ω_1 that is not strongly pseudoradial.

Besides our Main Question 1.5, we may also ask the following.

5.6. Question Does it follow from MA + $\mathfrak{c} = \omega_2$ that there exists a pseudoradial space (of radial character ω_1) that contains no topological copies of ω_1?

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