AN EFIMOV SPACE WITH CHARACTER LESS THAN $\mathfrak{s}$

ALAN DOW

ABSTRACT. It is consistent that there is a compact space of character less than the splitting number in which there are no converging sequences. Such a space is an Efimov space.

An infinite compact space is an Efimov space if it contains no converging sequences and no topological copies of $\beta\omega$. It remains a central open problem to determine if such a space exists in ZFC. The splitting number $\mathfrak{s}$, one of the basic and well-studied cardinal invariants of the continuum, is defined as the minimum cardinality of a splitting family (defined below). The splitting number has been shown to equal the minimum weight of a compact space in which there is an infinite sequence with no converging subsequence. A related cardinal invariant, $\mathfrak{z}$, is defined as the minimum weight of an infinite compact space containing no converging sequences. This was introduced and studied by D. Sobota and also studied in [3]. Evidently $\mathfrak{s} \leq \mathfrak{z}$ and the weight of every Efimov space is at least $\mathfrak{z}$. It was shown by Hausdorff that $\beta\omega$ has character $\mathfrak{c}$ and so a compact space of character less than $\mathfrak{s}$ will not contain a copy of $\beta\omega$. Other connections between the splitting number and Efimov's problem have been established. It was shown in [4] that barring large cardinals, if $\mathfrak{s} < \mathfrak{c}$, then there is an Efimov space and it was observed in [3] that under the same hypotheses an Efimov space can be constructed to have character equal to $\mathfrak{s}$. The space constructed in [7] was the first to be constructed in a model of $\mathfrak{s} = \mathfrak{c} > \aleph_1$. We use the same basic method for the construction of the space.

Another line of inquiry, concerning pseudointersection numbers of ultrafilters on ω, is related and led us to consider the possibility that it may be possible for an Efimov space to have character less than $\mathfrak{s}$. The pseudointersection number of an ultrafilter on ω is the minimum cardinality of a subset of the filter for which there is no infinite set that is mod finite contained in every member, i.e. a pseudointersection. Every infinite closed subspace of an Efimov space is again Efimov and so can be assumed to have a countable dense discrete subset. The trace of every neighborhood filter on that countable dense is a filter with no pseudointersection and it follows that every free ultrafilter on ω contains such a filter. Therefore, the example we construct in this paper produces a model in which the pseudointersection is less than the splitting number. This was already established in [2]. This was explored further in [8] to show that the splitting number can be much larger than the pseudointersection number of every ultrafilter. A common feature of these papers is that in order to make the value of $\mathfrak{s}$ large it is necessary to introduce a pseudointersection for at least one filter while preserving that the many filters witnessing small pseudointersection numbers are preserved to have no pseudointersection.

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1. Preliminaries and outline

In this section we review the techniques and background used throughout the paper. The set-theoretic aspects include finite-support iterated forcing of σ-centered (hence ccc) posets, the Laver version of a poset for adding a pseudointersection to a filter, and preservation results about not adding uncountable branches to trees. The topological aspects include Stone duality with Boolean algebras, recursive constructions of Ostaszewski style spaces, the theory of minimally generated Boolean algebras and the generalization to $\mathbb{T}$-algebras.

We begin with the promised definition of the splitting number. A family $\mathcal{S}$ of subset of ω is a splitting family if, for every infinite $a \subset \omega$, there is an $S \in \mathcal{S}$ such that each of $a \cap S$ and $a \setminus S$ are infinite; we say that S *splits* the set a. The splitting number $\mathfrak{s}$ is the minimum cardinality of a splitting family. The connection to pseudointersections is the following easily proven proposition.

Proposition 1. *If $\mathcal{B}$ is a Boolean subalgebra of $\mathcal{P}(\omega)$ of cardinality less than $\mathfrak{s}$, then there is an ultrafilter of $\mathcal{B}$ that has an infinite pseudointersection.*

Readers familiar with Mathias forcing will know that with $\mathcal{B} = \mathcal{P}(\omega)$, the forcing will add a new free ultrafilter on ω (i.e. $\mathcal{B}$) and a pseudointersection for it, while preserving that none of the ground model ultrafilters have a pseudointersection. We will be using the following variant in the style of Laver forcing ([16]) that was introduced in [1, 10]. This variant is σ-centered meaning that the poset can be written as a countable union of centered subsets.

The poset is a set of subtrees of $\omega^{<\omega}$. A subtree T_1 of a tree T means that all predecessors in T of every element of T_1 is also in T_1 and that T_1 is ordered with the inherited order. We refer to elements $t \in \omega^{<\omega}$ as nodes and use the notation $t^\frown j$ for the node $t \cup \{(\mathrm{dom}(t), j)\}$.

Definition 2. *For a filter $\mathcal{D}$ on ω, the poset $\mathbb{L}(\mathcal{D})$ denotes the set of sub-trees T of $\omega^{<\omega}$ satisfying the following properties:*

1. *for each $t \in T$ and $s \subset t$ in $\omega^{<\omega}$, $s \in T$,*
2. *a node $t \in T$ is said to be a branching node of T if the set $\{j \in \omega : t^\frown j\}$ has more than one element,*
3. *the minimum branching node of T is called the stem of T and is denoted with $\mathsf{stem}(T)$,*
4. *T is everywhere $\mathcal{D}$-branching above the stem in the sense that for all $\mathsf{stem}(T) \subseteq t$, $\{j \in \omega : t^\frown j \in T\} \in \mathcal{D}$.*

$\mathbb{L}(\mathcal{D})$ is ordered by inclusion. We use the standard notation $T_1 <_0 T_0$ to indicate that $T_1 \subset T_0$ and $\mathsf{stem}(T_1) = \mathsf{stem}(T_0)$.

It is well-known that if $\mathcal{D}$ is a free filter then $\mathbb{L}(\mathcal{D})$ adds a dominating real and if $\mathcal{D}$ is a free ultrafilter then $\mathbb{L}(\mathcal{D})$ adds a subset of ω that is not split by any set from the ground model (see [1, Theorem 8]). We will assume without mention that ultrafilter on ω refers to a free ultrafilter.

For any ordinal μ, we use the following conventions: a finite support iteration of σ-centered posets, $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \mu\rangle$ with limit P_μ, means that for all $\alpha \leq \mu$

1. P_0 is the trivial poset $\{\varnothing\}$;
2. P_α is the limit of the sequence $\langle P_\beta, \dot{Q}_\beta : \beta < \alpha\rangle$;
3. for $\alpha < \mu$, $\dot{Q}_\alpha$ is a P_α-name of a σ-centered poset;

(4) for each $p \in P_\alpha$, p is a function with finite domain, $\mathrm{dom}(p)$, contained in α, and $p(\beta) \in \dot{Q}_\alpha$ for each $\beta \in \mathrm{dom}(p)$;
(5) for $p, q \in P_\alpha$, $p < q$ providing $\mathrm{dom}(p) \supset \mathrm{dom}(q)$, and for all $\beta \in \mathrm{dom}(q)$, $p \restriction \beta \Vdash p(\beta) \leq_{\dot{Q}_\beta} q(\beta)$.

Another tree that will feature prominently in our construction is the full binary tree of height ω_1: $2^{<\omega_1}$. Utilizing the theory of $\mathbb{T}$-algebras, our main construction will be of a Boolean algebra with a generating family indexed by the nodes of the tree $2^{<\omega_1}$ in such a way that the ultrafilters canonically correspond to maximal (hence uncountable) branches of $2^{<\omega_1}$. More importantly, the family of generators indexed by the nodes of the corresponding branch will generate a base for the ultrafilter. This next result is critical for our goal of ensuring that no such filters acquire a pseudointersection. It will be our convention in this paper to let Γ refer to subtrees of $2^{<\omega_1}$. In this next proposition, Γ refers to the members of $2^{<\omega_1}$ that are in the ground model.

Proposition 3. [15] *Let $\Gamma = 2^{<\omega_1}$. In a forcing extension by a finite support iteration of σ-centered posets, all uncountable branches of Γ belong to the ground model.*

Now we turn to the topology background. An Ostaszewski style space will simply mean a locally countable and locally compact topology on ω_1 in which the set ω is open and dense. In fact, we also intend that every initial segment α is an open subset. Such spaces are scattered and the topology can be specified by presenting a sequence $\langle U_\alpha : \alpha \in \omega_1\rangle$ where each U_α is a subset of the initial segment $\alpha+1$ and is a compact clopen neighborhood of the point α. Furthermore, since ω is dense, we can set $a_\alpha = U_\alpha \cap \omega$ and consider the Stone space of the Boolean subalgebra of $\mathcal{P}(\omega)$ generated by the family $\{a_\alpha : \alpha \in \omega_1\}$. We identify the point α with the ultrafilter generated by the family $\{a_\alpha\} \cup \{\omega \setminus a_\beta : \beta < \alpha\}$, and we have constructed the same space. The one-point compactification of ω_1 with this topology is the Stone space of this same Boolean algebra.

We also wish to connect this presentation of Ostaszewski style spaces to Koppelberg's work on minimally generated Boolean algebras ([12]) where (minimal) generating sequences for Boolean algebras are introduced and discussed. An important connection to Efimov's problem proven in [12] is that the Stone space of a minimally generated Boolean algebra will not contain a copy of $\beta\mathbb{N}$ (in fact she proved that the Boolean algebra does not contain an uncountable free algebra).

As in [5], this motivates the terminology in this next definition and the adjective *coherent* refers to the optional coherence in the generation of the family of ultrafilters. The family of sets $\{a_\alpha : \alpha \in \omega_1\}$ discussed in the previous paragraph. By abstracting the properties it makes it easier to verify the properties of our construction.

Definition 4. *A sequence $\langle a_\alpha : \alpha \in \lambda\rangle$ of subsets of ω is coherent if for each $\alpha \leq \beta < \lambda$ there is a finite $F \subseteq \alpha$ such that either $a_\alpha \cap a_\beta \subseteq \bigcup_{\gamma \in F} a_\gamma = a_F$ or $a_\alpha \subseteq a_\beta \cup a_F$. The sequence is proper if for all β, a_β is not in the ideal generated by $\{a_\xi : \xi < \beta\}$.*

In our motivating example, a_α is defined to be $U_\alpha \cap \omega$, and now we work in reverse and use $\widehat{a}_\alpha$ as notation for U_α.

Definition 5. *If $\langle a_\alpha : \alpha \in \lambda\rangle$ is a proper coherent sequence of subsets of ω and if $\beta < \lambda$, $\widehat{a}_\beta$ is defined to be the set of $\alpha \leq \beta$ such that $a_\alpha \setminus a_\beta$ is contained in a_F for some finite $F \subseteq \alpha$. Observe that $\widehat{a}_\beta \subseteq \beta + 1$. For each non-empty finite $F \subset \omega_1$, let $\widehat{a}_F = \bigcup_{\alpha \in F} \widehat{a}_\alpha$.*

Note that every initial segment of a coherent sequence is also a coherent sequence, and that the value of $\widehat{a}_\alpha$ depends only on the initial sequence up to $\alpha + 1$.

two **Proposition 6** ([12]). *If $\mathcal{A} = \langle a_\alpha : \alpha \in \lambda\rangle$ is a proper coherent sequence, then the family $\{\widehat{a}_\beta : \beta \in \lambda\}$ generates a compact scattered Hausdorff topology, $\tau(\mathcal{A})$, on $\lambda + 1$ in which each $\widehat{a}_\beta$ is compact and open.*

Proof. The family $\mathcal{S} = \{\widehat{a}_\beta : \beta < \lambda\} \cup \{\lambda + 1 \setminus \widehat{a}_\beta : \beta < \lambda\}$ forms a subbase for the topology. The topology is Hausdorf since $\alpha \in \widehat{a}_\alpha$ and $\beta \in \lambda + 1 \setminus \widehat{a}_\alpha$ for all $\alpha < \beta \leq \lambda$. Similarly, the space is scattered since $\min(Y)$ is isolated in Y for all $Y \subset \lambda+1$. We use the Alexander Subbase Theorem to prove that the space is compact. Let $\mathcal{B} \subset \mathcal{S}$ be a cover of $\lambda + 1$. Let H be the set of $\gamma \leq \lambda$ such that the interval $[\gamma, \lambda]$ is covered by a finite subset of $\mathcal{B}$. Since $\lambda \in H$, H is not empty and we let γ_0 be the minimum element of H. Assume that $\gamma_0 > 0$. By the minimality of γ_0 it follows that $\lambda + 1 \setminus \widehat{a}_\beta \notin \mathcal{B}$ for all $\beta < \gamma_0$ and so there must be a $\beta < \lambda$ such that $\gamma_0 \in \widehat{a}_\beta \in \mathcal{B}$. Since the sequence is coherent, there is a finite $F \subset \gamma_0$ satisfying that $\widehat{a}_{\gamma_0} \subset \widehat{a}_\beta \cup \widehat{a}_F$. But now it follows easily that $\max F \in H$, contradicting the minimality of γ_0. $\square$

Finally, we introduce the remarkable generalization of linear coherence to a coherence structure based on binary trees. These are the $\mathbb{T}$-algebras introduced in [13]. However for this paper we restrict to subalgebras of $\mathcal{P}(\omega)$ and subtrees of $2^{<\omega_1}$. Also, our notation will more closely follow those of [6, 7] but especially [5]. For a node $\sigma \in 2^{<\lambda}$ for any ordinal λ, define $\sigma^\dagger$ to be σ if $\mathrm{dom}(\sigma)$ is not a successor ordinal, and otherwise $dom(\sigma^\dagger) = \mathrm{dom}(\sigma)$ and $\sigma^\dagger$ agrees with σ except at the maximum value in $dom(\sigma)$. For each $x \in 2^{\leq\omega_1}$, let λ_x denote the domain of x.

fifteen **Definition 7.** *A $\mathbb{T}$-algebra is a family $\mathcal{A}_\Gamma = \langle a_\sigma : \sigma \in \Gamma\rangle$ such that, for some ordinal λ*

1. *$\Gamma \subset 2^{<\lambda}$ is a subtree of $2^{<\lambda}$ that is downward closed (i.e. $\tau \in \Gamma$ for every $\sigma \in \Gamma$ and $\tau \subset \sigma$ in $2^{<\lambda}$),*
2. *if $\sigma \in \Gamma$, then $\sigma^\dagger \in \Gamma$ (say that Γ is twinned),*
3. *a_σ is empty for each $\sigma \in \Gamma$ that is not on a successor level,*
4. *for σ on a successor level of Γ, $a_{\sigma^\dagger} = \omega \setminus a_\sigma$,*
5. *for each $x \in 2^{\leq\lambda}$, the family $\mathcal{A}_{\Gamma,x} = \{a^x_\alpha = a_{x\restriction\alpha+1} : x \restriction \alpha{+}1 \in \Gamma\}$ is a proper coherent sequence.*

Henceforth Γ will always denote a subtree of $2^{<\omega_1}$ that is downward closed, has no maximal elements, and is closed under the $\dagger$ operation. For each such Γ, $X(\Gamma)$ will denote the set of minimal elements of $2^{\leq\omega_1} \setminus \Gamma$ (i.e. the maximal branches of Γ). We will let $X(\Gamma, \aleph_0)$ denote $\{x \in X(\Gamma) : \lambda_x < \omega_1\}$ (the countable elements of $X(\Gamma)$).

Lemma 8. [13] *Let $\mathcal{A}_\Gamma = \langle a_\sigma : \sigma \in \Gamma\rangle$ be a $\mathbb{T}$-algebra. Let $x \in X(\Gamma)$ and let $\mathcal{U}$ be any ultrafilter on the Boolean subalgebra, $\mathcal{B}_\Gamma$, of $\mathcal{P}(\omega)$ generated by $\mathcal{A}_\Gamma$.*

1. *The family of finite intersections from the set*

$$\{\omega \setminus a_{x\restriction\alpha} : \alpha \in \mathrm{dom}(x)\} \quad \textit{i.e. } \{\omega \setminus a^x_\alpha : a^x_\alpha \in \mathcal{A}_{\Gamma,x}\}$$

is a filter base for an ultrafilter $\mathcal{U}_x$ *on* $\mathcal{B}_\Gamma$.

(2) *There is a* $y \in X(\Gamma)$ *such that* $\mathcal{U} = \mathcal{U}_y$.

Proof. We first prove item (1). Since the family $\mathcal{A}_{\Gamma,x}$ is assumed to be a proper coherent sequence, it follows that no finite union from $\mathcal{A}_{\Gamma,x}$ is cofinite. To show that the filter $\mathcal{U}_x$ with the family from (1) as a base is an ultrafilter, it suffices to show that for every $\sigma \in \Gamma$, one of a_σ or $a_{\sigma^\dagger}$ is in $\mathcal{U}_x$. This is immediate for all $\sigma \subset x$ so we may suppose there is an $\alpha \in \mathrm{dom}(x)$ such that $(x \restriction \alpha) \not\subset \sigma$ and $(x \restriction \alpha)^\dagger \subseteq \sigma$. If $\sigma = (x \restriction \alpha)^\dagger$, then $a_\sigma = \omega \setminus a_{x\restriction\alpha}$ is an element of $\mathcal{U}_x$. Otherwise, using that each of $\mathcal{A}_{\sigma^\frown 0}$ and $\mathcal{A}_{\sigma^\dagger{}^\frown 0}$ are coherent, and that $a_{\sigma^\dagger} = \omega \setminus a_\sigma$, there is a finite $F \subset \alpha$ such that one of $\{a_{(x\restriction\alpha)^\dagger} \setminus a_\sigma \quad a_{(x\restriction\alpha)^\dagger} \setminus a_{\sigma^\dagger}\}$ is contained in $\bigcup\{a_{x\restriction\xi+1} : \xi{+}1 \in F\}$. By symmetry, assume that $a_{(x\restriction\alpha)^\dagger} \subset a_\sigma \cup \bigcup\{a_{x\restriction\xi+1} : \xi{+}1 \in F\}$. In other words, a_σ contains $(\omega \setminus a_{x\restriction\alpha}) \cap \bigcap\{\omega \setminus a_{x\restriction\xi+1} : \xi{+}1 \in F\}$, which is a member of $\mathcal{U}_x$.

Now we prove (2). We define $y \in X(\Gamma)$ by recursively defining an increasing chain in Γ. Let $y \restriction 1 = \sigma_1$ be chosen so that $a_{\sigma_1} \notin \mathcal{U}$. Assume that an increasing chain $\{\sigma_\beta \in 2^\beta : \beta < \alpha\} \subset \Gamma$ has been chosen so that $a_{\sigma_\beta} \notin \mathcal{U}$ for all $\beta < \alpha$. If α is a limit ordinal, then $\sigma_\alpha = \bigcup\{\sigma_\beta : \beta < \alpha\}$. If $\sigma_\alpha \notin \Gamma$, then set $y = \sigma_\alpha \in X(\Gamma)$. By part (1), it follows that $\mathcal{U}_y = \mathcal{U}$. Assume that $\alpha = \beta + 1$ and again, simply choose $\sigma_\alpha \supset \sigma_\beta$ so that $a_{\sigma_\alpha} \notin \mathcal{U}$. □

One of the features of a $\mathbb{T}$-algebra is that the Stone space can be analyzed by using the subalgebras (i.e. the Ostaszewski style spaces) generated by the generators indexed by a given branch. The space associated with $\mathcal{A}_{\Gamma,x}$, for $x \in X(\Gamma)$, was described in Proposition 6.

Lemma 9. *Let* $\mathcal{A}_\Gamma$ *be* $\mathbb{T}$*-algebra and let* $x \in X(\Gamma)$*. the mapping* $\varphi_x : X(\Gamma) \mapsto$ sixteen
$(\lambda_x + 1, \tau(\mathcal{A}_{\Gamma,x}))$ *is continuous where* $\varphi_x(x) = \lambda_x$ *and for all* $x \neq y \in X(\Gamma)$, $\varphi_x(y)$ *is the unique* $\alpha < \lambda_x$ *such that* $(y \restriction \alpha{+}1)^\dagger = x \restriction \alpha{+}1$*. In particular,* $\varphi_x(x) \neq \varphi_x(y)$ *for all* $y \in X(\Gamma) \setminus \{x\}$.

Proposition 10. *If* $\mathcal{A}_\Gamma$ *is a* $\mathbb{T}$*-algebra with the ScP then any converging sequence*
(ω*-sequence) in* $(X(\Gamma), \tau(\mathcal{A}_\Gamma))$ *converges to a point in* $X(\Gamma, \aleph_0)$*. Every* $x \in X(\Gamma, \aleph_0)$ 18
has countable character.

2. The stationary covering property of coherent sequences

However the stationary covering property was discovered in [6] as a tool for preserving countable compactness in what might usefully and informally be called Ostaszewski style spaces. A connection to the similarly important Scarborough-Stone problem was pointed out in [7] and in the last section of this paper we briefly explore the connections to the splitting number in that setting.

This next property was introduced in [6] under the name SP for the same purpose.

Definition 11. *Say that a coherent sequence* $\mathcal{A} = \{a_\alpha : \alpha \in \lambda\}$ *has the ScP (stationary covering property) if either,* $\lambda \in \omega_1$ *or for each stationary* $S \subset \omega_1$*, there is a finite* $F \subset \omega_1$ *such that* $\{\widehat{a}_F\} \cup \{\widehat{a}_\gamma : \gamma \in S\}$ *is a cover of* ω_1*. Say that* $S \cup F$ *is a cover and that* S *is an almost cover.*

We use the standard notation NS_{ω_1} to denote the ideal of non-stationary subsets of ω_1 and $NS^+_{\omega_1}$ denotes the set of stationary subsets of ω_1.

Lemma 12. *If a proper coherent ω_1-sequence $\mathcal{A}$ has the ScP, then $\mathcal{A}$ induces a countably compact topology on ω_1.* needScP

Proof. Let X be an infinite countable subset of ω_1 and suppose that $\widehat{a}_F \cap X$ is finite for all non-empty finite $F \subset \omega_1$. For each $\gamma > \sup(X)$, let X_γ be the finite set $X \cap \widehat{a}_\gamma$. By the pressing down lemma, we may choose a stationary $S \subset \omega_1$ and a single finite set $Y \subset X$ so that $Y = X_\gamma$ for all $\gamma \in S$. We now have a contradiction since it follows that S is not an almost cover. □

For the remainder of the section assume that $\mathcal{A} = \{a_\alpha : \alpha \in \omega_1\}$ is a coherent sequence with ScP.

subset **Lemma 13.** *If $S \subset \omega_1$ is a cover, then for each stationary S', there is a finite $F \subset S$ such that $S' \cup F$ is a cover.*

Proof. Choose finite F_1 such that $S' \cup F_1$ is a cover. For each $\alpha \in F_1$, $\widehat{a}_\alpha$ is a compact space and $\{\widehat{a}_\gamma : \gamma \in S\}$ is an open cover. Therefore, there is a finite $F \subset S$ such that $\bigcup_{\alpha \in F_1} \widehat{a}_\alpha$ is contained in $\widehat{a}_F$. This implies that $S' \cup F$ is a cover. □

bigsets **Lemma 14.** *If $S \subset \omega_1$ is a cover, then there is a finite $F \subset \omega_1$ such that for all $\alpha \in \omega_1 \setminus \widehat{a}_F$, the set of $\gamma \in S$ such that $\alpha \in \widehat{a}_\gamma$ is uncountable.*

Proof. For each limit ordinal $\gamma \in \omega_1$, choose the minimal $\alpha_\gamma \in S$ such that $\gamma \in \widehat{a}_{\alpha_\gamma}$. For each limit γ, choose finite $H_\gamma \subset \gamma$ so that $a_\gamma \setminus a_{\alpha_\gamma} \subset \widehat{a}_{H_\gamma}$. This can be restated as $a_\gamma \subset a_{\alpha_\gamma} \cup \widehat{a}_{H_\gamma}$. By the pressing down lemma, we may choose a stationary $S \subset \omega_1$ so that there is a finite $H \subset \omega_1$ so that $H_\gamma = H$ for all $\gamma \in S$. By Lemma 13, choose, for each $\gamma \in S$, $F_\gamma \subset S$ so that $F_\gamma \cup (S \setminus \gamma)$ is a cover. By another application of the pressing down lemma, there is a stationary $S' \subset S$ and a finite F so that $F = F_\gamma \cap \gamma$ for all $\gamma \in S'$.

We show that $F \cup H$ is as required. Fix any $\alpha \in \omega_1 \setminus \widehat{a}_F$ and choose any $\alpha < \gamma \in S'$. We prove there is a $\delta \in S \setminus \gamma$ with $\alpha \in \widehat{a}_\gamma$. Since $F \cup [(S' \cup S) \setminus \gamma]$ is a cover, $\alpha \in \widehat{a}_\beta$ for some $\beta \in (S' \cup (F_\gamma \setminus \gamma)) \subset (S' \cup S) \setminus \gamma$. If $\beta \in S$, then let $\delta = \beta$. If $\beta \in S' \setminus S$, then $\alpha \in \widehat{a}_{\alpha_\beta} \cup \widehat{a}_H$. Since $\alpha \notin \widehat{a}_H$, then $\delta = \alpha_\beta$ and we indeed have that $\alpha \in \widehat{a}_\delta$ for some $\delta \in S \setminus \gamma$. □

This next result is our first mention of forcing. We refer the reader to [14] for standard background and notation. We will use the notation of placing a dot over a letter, such as $\dot{Q}$, to indicate that we are refering to a name. Also, following [14], we use $\check{x}$ as the notation for the canonical name of the set x from the ground model (with the poset being clear from the context). For a poset P and a set X we use the term nice P-name for a subset of X to mean a name of the form $\dot{Y} = \bigcup\{\{\check{x}\} \times A_x : x \in X\}$ where, for each $x \in X$, A_x is a (possibly empty) antichain of P. If G is a P-generic filter, then $\mathrm{val}_G(\dot{Y})$ denotes the valuation or interpretation of the name $\dot{Y}$ and is defined to be the set $\{x \in X : G \cap A_x \neq \varnothing\}$. Every subset of X in the forcing extension does equal the valuation of a nice name, so it is often sufficient to only consider nice names.

lemma7 **Lemma 15** ([6]). *For a limit ordinal λ, if $\langle P_\xi, \dot{Q}_\xi : \xi \in \lambda\rangle$ is a finite support ccc iteration such that, for all $\xi < \lambda$, P_ξ forces that $\mathcal{A}$ has the ScP, then P_λ forces that $\mathcal{A}$ has the ScP (hence Cohen forcing preserves that $\mathcal{A}$ has the ScP).*

Proof. Let $\dot{S}$ be a P_λ-name and let $p_0 \in P_\lambda$ force that $\dot{S} \in NS^+_{\omega_1}$. We prove that p_0 does not force that $\dot{S}$ is a witness to the failure of the ScP. Let S_1 denote the set of limit ordinals $\xi \in \omega_1$ such that there is a $p_\xi < p_0$ with $p_\xi \Vdash \xi \in \dot{S}$. Since $p_0 \Vdash \dot{S} \subset S_1$ it follows that S_1 is stationary. For each $\xi \in S_1$, let H_ξ denote the support of p_ξ in the iteration sequence. By the standard Δ-system arguments, there is a finite H and a stationary set $S_2 \subset S_1$ such that $H_\xi \cap H_\eta = H$ for all $\xi < \eta$ in S_2.

Fix any $\mu < \lambda$ such that $H \subset \mu$ and $p_0 \in P_\mu$. Let $\dot{S}_3$ be the nice P_μ-name $\{(\check{\xi}, p_\xi \restriction \mu) : \xi \in S_2\}$. We prove the well-known fact that there is P_μ-generic filter G_μ satisfying that $\mathrm{val}_{G_\mu}(\dot{S}_3)$ is stationary. Since P_μ is ccc, it is a standard fact (see [14, VII H1]) that if $\dot{C}$ is a P_μ-name of a cub subset of ω_1, then there is cub C_1 such that $1 \Vdash C_1 \subset \dot{C}$. Let $\dot{C}$ be a P_μ-name of a cub satisfying that 1 forces that if $\dot{S}_3 \in NS_{\omega_1}$, then $\dot{S}_3 \cap \dot{C}$ is empty. Choose a cub C_1 as above, and choose any $\xi \in C_1 \cap S_1$. Since $p_\xi \restriction \mu \Vdash \xi \in \dot{C} \cap \dot{S}_3$, it follows that $p_\xi \restriction \mu$ forces that $\dot{S}_3$ is stationary.

Now let G_μ be a P_μ-generic filter with $p_0 \in G_\mu$ and satisfying that $S_4 = \mathrm{val}_{G_\mu}(\dot{S}_3)$ is stationary. We finish the proof by working in $V[G_\mu]$ (where $\mathcal{A}$ has the ScP). Choose any finite $F_0 \subset \omega_1$ so that $S_4 \cup F_0$ is a cover, and then choose finite $F_0 \subset F \subset \omega_1$ as in Lemma 14. Thus, for each $\alpha \in \omega_1 \setminus \widehat{a}_F$, the set of $\gamma \in S_4$ with $\alpha \in \widehat{a}_\gamma$ is uncountable. We prove that if $q \in P_\lambda$ and $q \restriction \mu \in G_\mu$, then q forces that $F \cup \dot{S}$ is a cover. Let $q \in P_\lambda$ with $q \restriction \mu \in G_\mu$ and fix any $\alpha \in \omega_1 \setminus \widehat{a}_F$. Choose any $\xi \in S_4$ satisfying that $H_\xi \setminus H$ is disjoint from the support of q and that $\alpha \in \widehat{a}_\xi$. Since $p_\xi \restriction \mu$ and $q \restriction \mu$ are in G_μ, it follows that p_ξ and q are compatible. Since $p_\xi \Vdash \xi \in \dot{S}$, it follows that q does not force that α is not covered by $F \cup \dot{S}$. □

In this next definition we keep the notation simpler by assuming that $\mathcal{D}$ will be clear from the context.

Definition 16. *Let $\dot{S}$ be an $\mathbb{L}(\mathcal{D})$-name for a filter $\mathcal{D}$ on ω. For $t \in \omega^{<\omega}$, say that $t \Vdash_w \gamma \in \dot{S}$ if there is some $T \in \mathbb{L}(\mathcal{D})$ such that $T \Vdash \gamma \in \dot{S}$ and t is the stem of T. Also, for a name $\dot{S}$ and $t \in T$, let $S_t = \{\gamma \in \omega_1 : t \Vdash_w \gamma \in \dot{S}\}$.*

Lemma 17. *If $\mathcal{D}$ is a filter with countable character, then $\mathbb{L}(\mathcal{D})$ preserves that $\mathcal{A}$ has the ScP.*

Proof. Let $\dot{S}$ be a name of a stationary set. For each $t \in \omega^{<\omega}$, let $S_t = \{\gamma : t \Vdash_w \gamma \in \dot{S}\}$. We must prove that there is a $T \in \mathbb{L}(\mathcal{D})$ and an $F \in [\omega_1]^{<\aleph_0}$ such that $T \Vdash \dot{S} \cup F$ is a cover. Note that if $T \Vdash \dot{S} \notin NS_{\omega_1}$, then $S_t \notin NS_{\omega_1}$ for all $t \in T$. Fix any $t \in \omega^{<\omega}$ such that S_t is stationary Fix any finite $F \subset \omega_1$ such that $S_t \cup F$ is a cover. Let $\{D_n : n \in \omega\}$ be a descending base for $\mathcal{D}$. For each $\gamma \in S_t$, choose $T_\gamma <_0 (\omega^{<\omega})_t$ forcing that $\gamma \in \dot{S}$. For each $\gamma \in S_t$, choose $n_\gamma \in \omega$ so that $D_{n_\gamma} \supset \{j : t^\frown j \in T_\gamma\}$. Choose n so that $S_t^n = \{\gamma \in S_t : n_\gamma \leq n\}$ is stationary. Note that $S_t^n \subset S_{t^\frown j}$ for all $j \in D_n$. By Lemma 13, we may choose a finite $F' \subset S_t$ so that $F \cup S_t^n \cup F'$ is a cover. Since $F' \subset S_t$ we can increase n and arrange that $F' \subset S_t^n$. It now follows that $S_{t^\frown j} \cup F$ is a cover for all $j \in D_n$. Continuing this construction we can produce $T_1 <_0 (\omega^{<\omega})_t$ such that, for all $t \in T_1$, $S_t \cup F$ is a cover. It then follows easily that $\dot{S} \cup F$ is a cover. □

Lemma 18. *Assume that $\mathcal{D}$ is an ultrafilter and that $\dot{S}$ is an $\mathbb{L}(\mathcal{D})$-name of a stationary subset of ω_1. If, for some $T \in \mathbb{L}(\mathcal{D})$, S_t is a cover for all $t \in T$, then* isacover
$T \Vdash \dot{S}$ is a cover.

Proof. Fix any α and $T' < T$. Let t be the stem of T'. Choose any $\gamma \in S_t$ so that $\alpha \in \widehat{a}_\gamma$ and choose $T_\gamma <_0 T$ so that $T_\gamma \Vdash \gamma \in \dot{S}$. Then $T' \cap T_\gamma$ forces that $\gamma \in \dot{S}$. □

Proposition 19. *For an $\mathbb{L}(\mathcal{D})$-name $\dot{S}$, if $S_t \in NS^+_{\omega_1}$ for all $t \in T \in \mathbb{L}(\mathcal{D})$, then $T \Vdash \dot{S} \in NS^+_{\omega_1}$.*

Proof. Since $\mathbb{L}(\mathcal{D})$ is ccc (σ-centered) it suffices show that if C is a cub and $T' < T$, then $T' \not\Vdash \dot{S} \cap C$ is not empty. Of course this is immediate from the fact that $S_t \cap C$ is not empty where $t = \mathsf{stem}(T')$. □

allstat **Lemma 20.** *If $\mathcal{D}$ is a filter and $T \Vdash \dot{S} \in NS^+_{\omega_1}$ then there is $T' <_0 T$ such that $S_t \in NS^+_{\omega_1}$ for all $t \in T'$.*

Proof. Fix any $t \in T$ such that S_t is stationary. Fix any countable elementary submodel M such that $T, \dot{S}$ are in M and such that $M \cap \omega_1 = \delta \in S_t$. Choose $T' <_0 T_t$ such that $T' \Vdash \delta \in \dot{S}$. Evidently, $\delta \in S_{t'}$ for all $t' \in T'$. Since $\delta \in S_{t'} \in M$ for all $t' \in T'$, it follows that each such $S_{t'}$ is in $NS^+_{\omega_1}$. □

thirteen **Lemma 21.** *If $\mathbb{L}(\mathcal{D})$ forces that $\mathcal{A}$ does not have the ScP, then there is a name $\dot{S}$ of a stationary set and a $T \in \mathbb{L}(\mathcal{D})$ and $\{\{\beta_t\} \cup F_t : t \in T\} \subset [\omega_1]^{<\aleph_0}$ such that*

1. *S_t is stationary for all $t \in T$,*
2. *$S_t \cup F_t \cup F_{\mathsf{stem}(T)}$ is a cover,*
3. *for each $\mathsf{stem}(T) \neq t \in T$, $\beta_t \in \widehat{a}_{F_t} \setminus \widehat{a}_{F_{\mathsf{stem}(T)}}$,*
4. *for each $t, t^\frown j \in T$, $F_{t^\frown j} \subset S_t$,*
5. *for each $T' < T$, there is a $t \in T'$ such that the set*
$$\bigcup_{t^\frown j \in T'} \{\beta_{t^\frown j}\} \setminus \left(\widehat{a}_{F_t} \cup \bigcup \{\widehat{a}_\gamma : \gamma \in S_{t^\frown j}\}\right)$$
is infinite.

Proof. Fix T forcing that $\dot{S}$ is stationary and that $\dot{S}$ is not an almost cover. Let M be a countable elementary submodel so that there is a $t_0 \in T$ with $M \cap \omega_1 \in S_{t_0}$. We may assume that $t_0 = \mathsf{stem}(T)$ and, as in Lemma 20, that S_t is stationary for all $t \in T$. In fact, we can assume that $\bigcap\{S_{t \restriction \ell} : |t_0| \leq \ell \leq |t|\}$ is stationary for all $t \in T$. Let $S^-_{t_0} = S_{t_0}$ and, for each $t_0 \subset t \in T$, let S^-_t denote the stationary set $\bigcap\{S_{t\restriction \ell} : |t_0| \leq \ell \leq |t|\}$. We show that we may assume that $T_t \Vdash \dot{S} \subset S^-_t$ for all $t \in T$. Consider the name $\dot{S}_1$ where
$$\dot{S}_1 = \{(\gamma, \tilde{T}) : \tilde{T} \leq T,\ \tilde{T} \Vdash \gamma \in \dot{S}, \gamma \in S^-_{\mathsf{stem}(\tilde{T})}\} \ .$$
It is easily checked that, for all $t \in T$, $t \Vdash_w \gamma \in \dot{S}_1$ if and only if $\gamma \in S^-_t$. Of course it follows, by Lemma 20, that T forces that $\dot{S}_1$ is a stationary subset of $\dot{S}$ (and so is not an almost cover), that $\dot{S}_1 \in M$, and that if we replace $\dot{S}$ by $\dot{S}_1$, then we have that $S^-_t = S_t$ for all $t \in T$.

Choose any finite F_{t_0} so that $S_{t_0} \cup F_{t_0}$ is a cover. By Lemma 18 and by possibly replacing t_0 by an extension in T, we may assume that $F_{t_0} \cup S_t$ is not a cover for all $t_0 \subsetneq t \in T$. For all $t_0 \subsetneq t^\frown j \in T$, choose a finite $F_{t^\frown j} \subset S_t$ so that $S_{t^\frown j} \cup F_{t^\frown j} \cup F_{t_0}$ is a cover (by Lemma 13). We now choose $\beta_{t^\frown j}$ using the fact that $S_{t^\frown j} \cup F_{t_0}$ is not a cover. If $S_{t^\frown j} \cup F_t \cup F_{t_0}$ is a cover, then choose $\beta_{t^\frown j} \in \widehat{a}_{F_t} \setminus \widehat{a}_{F_{t_0}}$. Otherwise, let $\beta_{t^\frown j}$ be the minimal element of $\widehat{a}_{F_{t^\frown j}} \setminus \bigcup\{\widehat{a}_\gamma : \gamma \in S_{t^\frown j} \cup F_t \cup F_{t_0}\}$. Note that $\beta_{t^\frown j} \notin \widehat{a}_{F_{\mathsf{stem}(T)}}$.

Choose any $T' < T$ and suppose that

$$H_t = \bigcup_{t^\frown j \in T'} \{\beta_{t^\frown j}\} \setminus \left(\widehat{a}_{F_t} \cup \bigcup\{\widehat{a}_\gamma : \gamma \in S_{t^\frown j}\}\right)$$

is finite for all $t \in T'$. We proceed to a contradiction. Since each $F_{t^\frown j} \subset S_t$, there is a finite set $I_t \subset S_t$ such that $H_t \subset \widehat{a}_{I_t}$. Since $I_t \subset S_t$, there is a $\tilde{T} <_0 T'_t$ forcing that $I_t \subset \dot{S}$. Notice then that $I_t \subset S_{t^\frown j}$ for all $t^\frown j \in \tilde{T}$. It then follows that $S_{t^\frown j} \cup F_t \cup F_{t_0}$ is a cover for all $t^\frown j \in \tilde{T}$. But now it follows, by a fusion, that there is $T_1 <_0 T'$ satisfying that $F_{t_0} \cup S_t$ is a cover for all $t \in T_1$. By Lemma 18, we have that such a T_1 forces that $F_{t_0} \cup \dot{S}$ is a cover, contradicting that T forces that $\dot{S}$ is not an almost cover. □

Theorem 22 ($\diamondsuit$)**.** *There is an ultrafilter $\mathcal{D}$ so that $\mathbb{L}(\mathcal{D})$ preserves that $\mathcal{A}$ has the ScP.* justone

Proof. Let $\{Y_\xi : \xi \in \omega_1\}$ be any enumeration of $\mathcal{P}(\omega)$. Using standard coding techniques, we may assume we have a sequence $\langle \mathcal{M}_\delta : \omega \leq \delta \in \omega_1\rangle$ with the property that, for each $\omega \leq \delta < \omega_1$

$$\mathcal{M}_\delta = \langle\langle D(\delta,\alpha) : \alpha < \delta\rangle, \langle S(\delta,t), F(\delta,t), \beta(\delta,t) : t \in \omega^{<\omega}\rangle$$

where each $D(\delta,\alpha) \subset \omega$ and each $S(\delta,t) \cup F(\delta,t) \cup \{\beta(\delta,t)\} \subset \delta$, and satisfying that for each ultrafilter $\mathcal{D}$ on ω and each sequence $\{S_t, F_t, \{\beta_t\} : t \in \omega^{<\omega}\}$ of subsets of ω_1, there is a stationary set of δ satisfying that

1. $\mathcal{D} \supset \{D(\delta,\alpha) : \alpha < \delta\}$,
2. $(S_t \cap \delta, F_t \cap \delta, \beta_t) = (S(\delta,t), F(\delta,t), \beta(\delta,t))$ for all $t \in \omega^{<\omega}$.

We define a mod finite decreasing sequence $\langle D_\xi : \xi \in \omega_1\rangle$ by induction on $\xi \in \omega_1$. For $n < \omega$, $D_n = \omega \setminus n$ and choose any infinite D_ω so that, for each $n < \omega$, D_ω is mod finite contained in one of $\{Y_n, \omega \setminus Y_n\}$. We similarly require that for each limit δ and $n \in \omega$, $Y_{\delta+\omega}$ is mod finite contained in one of $\{Y_{\delta+n}, \omega \setminus Y_{\delta+n}\}$. These inductive assumptions, if successfully completed, ensure that the filter generated by $\langle D_\xi : \xi \in \omega_1\rangle$ is an ultrafilter.

Let C be any cub satisfying that for each $\delta \in C$ there is a countable elementary submodel M of $H(\omega_2)$ satisfying that $M \cap \omega_1 = \delta$. Note that each $\delta \in C$ is a limit of limit ordinals.

Our induction will proceed along limit ordinals in that for each limit ordinal δ we will define the sequence $\langle D_\beta : \delta \leq \beta < \delta+\omega\rangle$. If $\delta \notin C$ and we have defined the mod finite descending sequence $\langle D_\xi : \xi < \delta\rangle$, then choose an infinite pseudointersection D_δ so that D_δ is also mod finite contained in one of $\{Y_\xi, \omega \setminus Y_\xi\}$ for all $\xi < \delta$. For each $\delta < \beta < \delta + \omega$, we simply set $D_\beta = D_\delta$.

Now assume that $\delta \in C$. Choose, if possible, a countable elementary submodel $M \prec H(\omega_2)$ so that

1. $M \cap \omega_1 = \delta$ and $\{Y_\xi : \xi \in \omega_1\} \in M$,
2. there is an ultrafilter $\mathcal{D} \in M$ such that $\mathcal{D} \cap M = \langle D_\beta : \beta \in \delta\rangle$
3. there is an $\mathbb{L}(\mathcal{D})$-name $\dot{S}$ in M and a $T \in \mathbb{L}(\mathcal{D}) \cap M$ forcing that $\mathcal{A}$ does not have the ScP,
4. there is a family $\{S_t, F_t, \beta_t : t \in T\} \in M$ as in Lemma 21 satisfying that
   (a) $(S(\delta,t), F(\delta,t), \beta(\delta,t)) = (\varnothing, \varnothing, 0)$ for all t such that $t \notin T$ or $t \subsetneq \mathsf{stem}(T)$,
   (b) for all $t \in T$, $(S_t \cap \delta, F_t, \beta_t) = (S(\delta,t), F(\delta,t), \beta(\delta,t))$.

For definiteness, we may assume we have a well-ordering of $H(\aleph_2)$ and that M_δ is chosen to be the minimal (in this well-ordering) such countable elementary submodel. Let D_δ be any infinite pseudointersection of $\langle D_\xi : \xi < \delta\rangle$ and let $\{t_n : n \in \omega\}$ be an enumeration of $\{t : \mathsf{stem}(T) \subseteq t \in T\}$. By induction on $n \in \omega$ choose an infinite $D_{\delta+n+1} \subset D_{\delta+n}$ and an $\alpha_{t_n} \in \omega_1$ so that if $\bigcup_{j\in D_{\delta+n+1}}\{\beta(\delta, t_n^\frown j)\} \setminus \left(\widehat{a}_{F(\delta,t_n)} \cup \bigcup\{\widehat{a}_\gamma : \gamma \in S(\delta, t_n^\frown j)\}\right)$ is infinite then it converges to some α_{t_n}. This completes the construction of the ultrafilter $\mathcal{D}$.

Now assume that $T \in \mathbb{L}(\mathcal{D})$. We prove that T does not force that $\mathcal{A}$ fails the ScP. We consider an $\mathbb{L}(\mathcal{D})$-name, $\dot{S}$ such that T forces that $\dot{S}$ is a stationary subset of ω_1 and such that $\{\{\beta_t\} \cup F_t : t \in T\} \subset [\omega_1]^{<\aleph_0}$ satisfy properties (1)-(4) in the statement of Lemma 21. We prove that property (5) fails. For any $t \in \omega^{<\omega} \setminus T$ and $t \subsetneq \mathsf{stem}(T)$, set $\beta_t = 0$ and $S_t = F_t = \varnothing$. We may choose $\delta \in \omega_1$ so that $\mathcal{D} \supset \{D(\delta, \alpha) : \alpha \in \delta\}$ and $(S_t \cap \delta, F_t \cap \delta, \beta_t) = (S(\delta, t), F(\delta, t), \beta(\delta, t))$ for all $t \in \omega^{<\omega}$. Let $T_1 \in \mathbb{L}(\mathcal{D})$ be any condition satisfying that $T_1 \subset T$ and that for all $t \in T_1$, the set $\{j \in \omega : t^\frown j\}$ is a subset of $D_{\delta+\omega}$. For each $t \in T_1$, let α_t denote the ordinal chosen for t in the stages δ to $\delta + \omega$ of the construction. Let L be the set of $t \in T_1$ satisfying that there is a $\gamma_t \in S_t$ with $\alpha_t \in \widehat{a}_{\gamma_t}$. We recall that for $t \in L$, $t \Vdash_w \gamma_t \in \dot{S}$. By induction on levels, we can now perform a simple fusion to choose $T' \leq T_1$ so that for all $t \in T' \cap L$, $(T')_t \Vdash \gamma_t \in \dot{S}$. It then follows that, for all $t \in L \cap T'$ and $t^\frown j \in T'$, $\gamma_t \in S_{t^\frown j}$. Now, choose any $\mathsf{stem}(T') \subseteq t \in T'$. Let $J = \{j : t^\frown j \in T' \text{ and } \beta(\delta, t^\frown j) \notin \left(\widehat{a}_{F(\delta,t)} \cup \bigcup\{\widehat{a}_\gamma : \gamma \in S(\delta, t^\frown j)\}\right)\}$. Property (5) will fail if we prove that $\{\beta_{t^\frown j} : j \in J\}$ is finite. So we assume that it is infinite and obtain a contradiction. Since $S_t \cup F_t \cup F_{\mathsf{stem}(T)}$ is a cover, we may choose $\gamma \in S_t \cup F_t \cup F_{\mathsf{stem}(T)}$ so that $\alpha_t \in \widehat{a}_\gamma$. Since all but finitely many members of the set $\{\beta_{t^\frown j} : j \in J\}$ are in $\widehat{a}_{\alpha_t} \cap \widehat{a}_\gamma$ and, by (3) of Lemma 21 and the definition of J, $\widehat{a}_{F_t} \cup \widehat{a}_{F_{\mathsf{stem}(T)}}$ is disjoint from $\{\beta_{t^\frown j} : j \in D_{\delta+n+1}\}$, so it follows that $\gamma \in S_t$ and therefore that $t \in L$. However this is the contradiction we seek since we can assume that γ was chosen to be γ_t, and with $\gamma_t \in S_{t^\frown j}$ for all $t^\frown j \in T'$, $\{\beta_{t^\frown j} : j \in J\}$ is disjoint from $\widehat{a}_{\gamma_t}$. □

3. Basic properties of $\mathbb{T}$-algebras

In the previous section we explored how to produce and preserve the ScP property for a single proper coherent sequence. In this section we use the concept of $\mathbb{T}$-algebras to do so for a large system of proper coherent sequences. These were introduced in [13] and utilized in [6, 7] for the purposes of constructing Efimov spaces.

Say that $\mathcal{A}_\Gamma$ has the ScP if $\lambda \leq \omega_1$ and $\mathcal{A}_{\Gamma,x}$ has the ScP for all $x \in 2^{\omega_1}$.

Let us note that the set $X(\Gamma)$ may grow, e.g. $\dot{X}(\Gamma)$ would be a name of all branches, if we enlarge the model. It may also be useful to note that for $x \in X(\Gamma)$ (or $x \in \check{X}(\Gamma)$) in the ground model, $(\lambda_x{+}1, \tau(\mathcal{A}_{\Gamma,x}))$ will be unchanged and so $\tau(\mathcal{A}_\Gamma)$ will induce the subspace topology on the elements of $\check{X}(\Gamma)$. Before proving this next result we recall S. Koppelberg's result [11] that adding a Cohen real ensures that every Stone space of a ground model infinite Boolean algebra will have a non-trivial converging sequence. It is also well-known (as shown in [15, Theorem 9]) that a σ-centered forcing will not add any new uncountable branches to Γ and so $\lambda_x < \omega_1$ for all new $x \in X(\Gamma)$.

Lemma 23. *If $\mathcal{A}_\Gamma$ is a $\mathbb{T}$-algebra with the ScP and if G is $Fn(\omega,2)$ generic, then in $V[G]$, $(X(\Gamma),\tau(\mathcal{A}_\Gamma))$ is sequentially compact.* seqcpt

Proof. Let $\dot K$ be the $Fn(\omega,2)$ name of the closed set of limit points of an infinite discrete subset of $\dot X(\Gamma)$. Clearly the infinite set has a converging subsequence if $\dot K$ has any points of countable character. So we may assume that $\dot K$ is an uncountable subset of $\dot X(\Gamma)\setminus \dot X(\Gamma,\aleph_0)$, which is contained in $\check X(\Gamma)$. By recursion on $\alpha\in\omega_1$, choose a nice $Fn(\omega,2)$ name $\dot x_\alpha$ for an element of $\dot K\setminus\{\dot x_\beta:\beta<\alpha\}$. Fix any $p\in G$ satisfying that $K_p=\{\alpha\in\omega_1:(\exists x_\alpha)\ p\Vdash \dot x_\alpha = x_\alpha\}$ is uncountable. By Koppelberg's result, the closure of $K_p\subset \dot K$ in $\dot X(\Gamma)$ will contain an infinite converging sequence. Since $\mathcal{A}_\Gamma$ retains the ScP, it follows from Proposition 10 that this sequence must converge to a new point x. However this is a contradiction since we assumed $\dot K$ is contained in $\check X(\Gamma)$. □

4. Preserving the ScP for $\mathbb{T}$-algebras

Theorem 24 ($\diamondsuit$)**.** *If $\mathcal{A}_\Gamma$ is a $\mathbb{T}$-algebra with the ScP, then in the forcing extension by $Fn(\omega_1,2)$, there is an ultrafilter $\mathcal{D}$ satisfying that $\mathbb{L}(\mathcal{D})$ preserves that $\mathcal{A}_\Gamma$ has the ScP.* Dpreserve

Proof. Using a standard coding technique, we can arrange that there is a $\diamondsuit$-sequence with the form $\vec{\mathcal{M}}=\{\mathcal{M}_\delta(i):i<5,\delta\in\omega_1\}$ where,

(1) $\mathcal{M}_\delta(0)\in 2^\delta$,
(2) $\mathcal{M}_\delta(1)=\{\dot D(\delta,\alpha):\alpha<\delta\}$ is a sequence of $Fn(\delta,2)$-names of subsets of ω,
(3) $\mathcal{M}_\delta(2)=\{\dot S(\delta,t):t\in\omega^{<\omega}\}$ and $\mathcal{M}_\delta(3)=\{\dot F(\delta,t):t\in\omega^{<\omega}\}$ are sequences of $Fn(\delta,2)$-names of subsets of δ,
(4) $\mathcal{M}_\delta(4)=\{\dot\beta(\delta,t):t\in\omega^{<\omega}\}$ is a sequence of $Fn(\delta,2)$-names of elements of δ.

satisfying that for all $x\in 2^{\omega_1}$, sequences $\{\dot D_\alpha:\alpha\in\omega_1\}$ of nice $Fn(\omega_1,2)$ names of subsets of ω, all sequences $\{\dot S_t,\dot F_t:t\in\omega^{<\omega}\}$ of pairs of nice $Fn(\omega_1,2)$-names of subsets ω_1, and $\{\dot\beta_t:t\in\omega^{<\omega}\}$ of nice $Fn(\omega_1,2)$-names of ordinals in ω_1,

there is a stationary set of δ such that

(1) $\mathcal{M}_\delta(0)=x\restriction\delta$,
(2) $\dot D(\delta,\alpha)=\dot D_\alpha$ for $\alpha<\delta$,
(3) $\dot S(\delta,t)=\dot S_t$ for $t\in\omega^{<\omega}$,
(4) $\dot F(\delta,t)=\dot F_t$ for $t\in\omega^{<\omega}$,
(5) $\dot\beta(\delta,t)=\dot\beta_t$ for $t\in\omega^{<\omega}$.

We construct a sequence $\langle\dot D_\alpha:\alpha\in\omega_1\rangle$ of nice $Fn(\omega_1,2)$-names of subsets of ω with the intention that $\dot{\mathcal{D}}$ (the filter generated by this sequence) is forced to be an ultrafilter. Also, that it is forced that $\mathbb{L}(\mathcal{D})$ forces that $\mathcal{A}_\Gamma$ has the ScP.

One simple inductive assumption on $\delta<\omega_1$ is that the sequence $\langle\dot D_\alpha:\alpha<\delta\rangle$ is that 1 forces that finite intersections are infinite. We may also assume that each $\dot D_\alpha$ is a nice $Fn(\alpha+\omega,2)$-name. For $n\in\omega$, set $\dot D_n$ to be the canonical name for $\omega\setminus n$. We will emulate the proof of Theorem 22. Again, let $C\subset\omega_1$ be a cub satisfying that for each $\delta\in C$, there is a countable elementary submodel M of $H(\omega_2)$ satisfying that $M\cap\omega_1=\delta$. Let $\{\dot Y_\xi:\xi\in\omega_1\}$ be an enumeration of all nice $Fn(\omega_1,2)$ names of subsets of ω enumerated in such a way that $\dot Y_\xi$ is an $Fn(\xi,2)$-name. We may again inductively assume that for limit δ, 1 forces, for each

$n\in\omega$, that $\dot D_{\delta+\omega}$ is mod finite contained in one of $\{\dot Y_{\delta+n}, \omega\setminus \dot Y_{\delta+n}\}$. Again, if δ is a limit not in C, then $\dot D_\delta$ is any nice $Fn(\delta,2)$-name of an infinite pseudointersection of the sequence $\langle \dot D_\beta : \beta<\delta\rangle$, and for $\delta<\beta<\delta+\omega$, $\dot D_\beta = \dot D_\delta$.

Our main task is accomplished for *critical* values $\delta\in C$. Choose, if possible, a countable elementary submodel $M\prec H(\omega_2)$ and an $x\in X(\Gamma)\cap 2^{\omega_1}$, again choose the least such pair by some well-ordering, so that

(1) $M\cap\omega_1=\delta$, $\{\dot Y_\xi : \xi\in\omega_1\}\in M$, and $M_\delta(0) = x\restriction\delta$,
(2) there is an nice $Fn(\omega_1,2)$ name of a ultrafilter $\dot{\mathcal D}\in M$ such that 1 forces that $\dot{\mathcal D}\cap M\supset \langle \dot D(\delta,\beta):\beta\in\delta\rangle$
(by elementarity, $\dot{\mathcal D}$ will be unique up to forcing equivalence)
(3) there are a nice $Fn(\omega_1,2)$-names $\dot T$ in M forced to be in $\mathbb L(\dot{\mathcal D})$ and $\dot{\mathcal S}$ in M such that 1 forces that $\dot{\mathcal S}$ is a nice $\mathbb L(\dot{\mathcal D})$-name $\dot S$ that is forced by $\dot T$ to witness that $\mathcal A_{\Gamma,x}$ does not have the ScP,
(4) there is a family $\{\dot S_t, \dot F_t, \dot\beta_t : t\in\dot T\}\in M$ that is forced to be as in Lemma 21 satisfying that, it is forced for all $t\in\omega^{<\omega}$
   (a) $(\dot S(\delta,t), \dot F(\delta,t), \dot\beta(\delta,t)) = (\varnothing,\varnothing,0)$ if $t\notin\dot T$ or $t\subsetneq \mathsf{stem}(\dot T)$,
   (b) $(\dot S_t\cap\delta, \dot F_t, \dot\beta_t) = (\dot S(\delta,t), \dot F(\delta,t), \dot\beta(\delta,t))$ if $t\in\dot T$,
(5) and $\dot S_t = \{\gamma : t\Vdash_w \gamma\in\dot S\}$ for $\mathsf{stem}(\dot T)\subset t\in\dot T$.

Let G_δ be a generic filter for $Fn(\delta,2)$. We complete the construction by working in the forcing extension $V[G_\delta]$. For each $\beta<\delta$, fix any $x_\beta\in\check X(\Gamma)$ such that $\varphi_x(x_\beta)=\beta$. Let D_δ be any infinite pseudointersection of the sequence $\langle \mathrm{val}_G(\dot D_\beta):\beta<\delta\rangle$. Let $\{t_n:n\in\omega\}$ be an enumeration of $\{t\in\omega^{<\omega} : \mathsf{stem}(\mathrm{val}_G(\dot T))\le t\in\mathrm{val}_G(\dot T)\}$ with $t_0=\mathsf{stem}(\mathrm{val}_G(\dot T))$. By removing a finite set, we may assume that $\{t_0^\frown j : j\in D_\delta\}$ is a subset of $\mathrm{val}_G(\dot T)$.

By induction on n, we utilize Lemma 23 to choose an infinite $D_{\delta+n+1}$ contained in $\{j\in D_{\delta+n} : (t_{n+1})^\frown j\in T\}$ so that, if

$$\bigcup_{j\in D_{\delta+n}}\{\beta(\delta,t_n^\frown j)\}\setminus\left(\widehat a_{F(\delta,t_n)}\cup\bigcup\{\widehat a_\gamma:\gamma\in S(\delta,t_n^\frown j)\}\right)$$

is infinite, then $\{x_\beta : \beta\in B_n\}$ converges in $X(\Gamma)$ (to some point of countable character) where

$$B_n=\bigcup_{j\in D_{\delta+n+1}}\{\beta(\delta,t_n^\frown j)\}\setminus\left(\widehat a_{F(\delta,t_n)}\cup\bigcup\{\widehat a_\gamma:\gamma\in S(\delta,t_n^\frown j)\}\right).$$

To finish the construction, simply choose a sequence $\langle \dot D_{\delta+n}:n\in\omega\rangle$ of nice $Fn(\delta,2)$-names that is forced to satisfy the properties of the above sequence in $V[G_\delta]$.

It is time to verify that this works. We have constructed the nice name $\dot{\mathcal D}=\langle \dot D_\alpha:\alpha\in\omega_1\rangle$ and we must verify that 1 forces that $\mathbb L(\dot{\mathcal D})$ preserves that $\mathcal A_\Gamma$ has the ScP. Let G be a generic filter for $Fn(\omega_1,2)$ and, in $V[G]$, let $\dot S$ be a nice $\mathbb L(\mathrm{val}_G(\dot{\mathcal D}))$ name of a stationary subset of ω_1 and let $x\in\check X(\Gamma)$. Assume that there is some $T\in\mathbb L(\mathrm{val}_G(\dot{\mathcal D}))$ that forces that $\dot S$ is not an almost cover for $\mathcal A_{\Gamma,x}$. We may thus assume that T has the property that there is a sequence $\{\{\beta_t\}\cup F_t, S_t : t\in T\}\subset[\omega_1]^{<\aleph_0}$ as in Lemma 21. Let $\dot T$ be a nice $Fn(\omega_1,2)$ name for T and similarly let $\{\dot\beta_t,\dot F_t,\dot S_t : t\in\dot T\}$ be nice $Fn(\omega_1,2)$ names for the objects as chosen by Lemma 21. Choose any $p\in G$ that forces the above mentioned properties. Modify if necessary, the above mentioned names so that if $q\perp p$ is in

$Fn(\omega_1,2)$, q forces that $\dot\beta_t = 0$ and $\dot F_t, \dot S_t$ are empty. Similarly, if $q < p$ forces that $t \notin \dot T$ or if $t \subsetneq \mathsf{stem}(\dot T)$, then q forces that $\dot\beta_t = 0$ and $\dot F_t, \dot S_t$ are empty.

Fix any continuous $\in$-chain $\{M^x_\alpha : \alpha \in \omega_1\}$ of countable elementary submodels of $H(\aleph_2)$ such that $\{\vec{\mathcal{M}}, x, p, \{\dot Y_\xi : \xi \in \omega_1\}, \dot T, \{\{\dot\beta_t\}, \dot F_t, \dot S_t : t \in \dot T\}\}$ is an element of M^x_0. Let C_x be a cub such that $M^x_\gamma \cap \omega_1 = \gamma$ for all $\gamma \in C_x$. Choose $\delta \in C_x \cap C$ so that M^x_δ and these chosen names satisfy the requirements of a critical value of C.

Let $G_\delta = G \cap Fn(\delta, 2)$ and let, for $n \in \omega$,

$$B_n = \bigcup_{j \in D_{\delta+n+1}} \{\beta(\delta, t_n^\frown j)\} \setminus \Big(\widehat{a}_{F(\delta,t_n)} \cup \bigcup\{\widehat{a}_\gamma : \gamma \in S(\delta, t_n^\frown j)\}\Big)$$

be as defined in the construction. Let I be the set of n such that B_n is infinite, and, for each $n \in I$ let $y_n \in X(\Gamma)$ be the point such that $\{x_\beta : \beta \in B_n\}$ converges to y_n. Of course $y_n \neq x$ for all n since no sequence converges to x.

We finish our work in $V[G]$. Let $T_\delta <_0 T = \mathrm{val}_G(\dot T)$ be any extension satisfying that for all $t \in T_\delta$, the set $\{j : t^\frown j \in T_\delta\} \subset \mathrm{val}_G(D_{\delta+\omega})$. Fix any n so that $t = t_n \in T_\delta$ and let $\alpha = \varphi_x(y_n)$. Since $S_t \cup F_t \cup F_{\mathsf{stem}(T)}$ is a cover, there is a $\gamma_t \in S_t \cup F_t \cup F_{\mathsf{stem}(T)}$ such that $\alpha \in \widehat{a}^x_\gamma$ (and $a^x_{\gamma_t} = a_{x\restriction\gamma_t+1}$). Since $\{x_\beta : \beta \in B_n\}$ converges to y_n, $\{\beta \in B_n : \varphi_x(x_\beta) = \beta \notin \widehat{a}^x_\gamma\}$ is finite. By property (3) and (5) of Lemma 21, B_n is disjoint from $\widehat{a}^x_{F_{\mathsf{stem}(T)}} \cup \widehat{a}^x_{F_t}$, hence $\gamma_t \in S_t$. By definition of S_t, $t \Vdash_w \gamma_t \in \dot S$. It now follows, by a simple fusion, that there is a $T' <_0 T_\delta$ in $\mathbb{L}(\mathcal{D})$ satisfying that, for all $t = t_n \in T'$ and $n \in I$, $T' \Vdash \gamma_t \in \dot S$. Furthermore, for all $n \in I$ and all $t_n^\frown j \in T'$, $\gamma_{t_n} \in S_{t_n^\frown j}$. It now follows that the set

$$\bigcup_{t_n^\frown j \in T'} \{\beta(\delta, t_n^\frown j)\} \setminus \Big(\widehat{a}_{F(\delta,t_n)} \cup \bigcup\{\widehat{a}_\gamma : \gamma \in S(\delta, t_n^\frown j)\}\Big)$$

from (5) of Lemma 21 is a subset of the finite set $B_n \setminus \widehat{a}^x_{\gamma_{t_n}}$. This is our desired contradiction. □

5. THE FINAL MODEL

We construct some $\mathbb{T}$-algebras. For $\sigma \in 2^{<\omega}$, we let $[\sigma] = \{\tau \in 2^{<\omega} : \sigma \subseteq \tau\}$. Our first example is a flexible method of constructing a $\mathbb{T}$-algebra on $\Gamma = 2^{<\omega}$. Let $\{\sigma_k : k \in \omega\}$ be the standard lexicographic enumeration of $2^{<\omega}$.

Proposition 25. *Let π be any permuation on ω and let $L \subset \omega$ have the property that $[\sigma] \cap \{\sigma_k : k \in \pi(L)\}$ is not empty for all $\sigma \in 2^{<\omega}$. For $\sigma \in 2^{<\omega}$, set $a_{\sigma^\frown 0} = \{k \in \pi(L) : \sigma_k \in [\sigma^\frown 1]\}$ and $a_{\sigma^\frown 1} = \omega \setminus a_{\sigma^\frown 0}$. As required, let $a_\varnothing = \varnothing$.* basic
Then $\{a_\sigma : \sigma \in 2^{<\omega}\}$ is a $\mathbb{T}$-algebra.

Proof. For each $\sigma \in 2^{<\omega}$, let $[\sigma]_L$ denote the set $\{k \in \pi(L) : \sigma \subseteq \sigma_k\}$. Therefore $a_{\sigma^\frown 0} = [\sigma^\frown 1]_L$ and it follows that $a_\sigma \cap [\sigma]_L$ is empty for all $\sigma \in 2^{<\omega}$. This ensures that $a_\sigma \setminus \bigcup\{a_{\sigma\restriction j} : j < |\sigma|\}$ is infinite for all $\varnothing \neq \sigma \in 2^{<\omega}$. It is also now evident that $a_{\sigma^\frown 0}$ is disjoint from $a_{\sigma\restriction j}$ for all $j \leq |\sigma|$. This implies that $a_{\sigma^\frown 1}$ contains $a_{\sigma\restriction j}$ for all $j \leq |\sigma|$. This completes the proof. □

We can lift this simple construction to extend any $\mathbb{T}$-algebra $\mathcal{A}_\Gamma$ so long as there are $x \in X(\Gamma)$ with $\lambda_x < \omega_1$. Fix a sequence $\{e_\lambda : \omega \leq \lambda \in \omega_1\}$ where, for each $\lambda \in \omega_1$, e_λ is a bijection from ω to λ. For convenience, Γ will always denote some non-empty subtree of $2^{<\omega_1}$ that is twinned and has no maximal elements. For definiteness, we assume for the remainder of this section, that for every $\mathbb{T}$-algebra $\mathcal{A}_\Gamma$, the family $\{a_\sigma : \sigma \in 2^{<\omega}\}$ is obtained as in Proposition 25 when π is the identity map and $L = \omega$.

nextstep **Definition 26.** *Let $\mathcal{A}_\Gamma$ be a $\mathbb{T}$-algebra and let $x \in X(\Gamma, \aleph_0)$ and let π be a permutation on ω. We define:*

(1) $\Gamma^x = \Gamma \cup \{x^\frown\sigma : \sigma \in 2^{<\omega}\}$,
(2) *recursively for $n \in \omega$, let* $c^x_n = a^x_{e_{\lambda_x}(n)} \setminus \bigcup_{k<n} c^x_k$,
(3) $L_x = \{n : c^x_n \neq \emptyset\}$,
(4) $a^{\pi,x}_x = \emptyset$ *and* $a^{\pi,x}_{\sigma^\frown 0} = \bigcup\{c^x_k : \sigma_{\pi(k)} \in [\sigma^\frown 1]\}$ *for* $\sigma \in 2^{<\omega}$,
(5) $\mathcal{A}_\Gamma[\pi, x]$ *is the collection* $\mathcal{A}_\gamma \cup \{a^{\pi,x}_\sigma : \sigma \in 2^{<\omega}\}$,
(6) *for all* $Y \subset X(\Gamma, \aleph_0)$, $\Gamma^Y = \bigcup\{\Gamma^y : y \in Y\}$, *and* $\mathcal{A}_\Gamma[\pi, Y] = \bigcup\{\mathcal{A}_\Gamma[\pi, y] : y \in Y\}$.

One could also define $\mathcal{A}_\Gamma[\langle \pi_y, y : y \in Y\rangle]$ to equal $\bigcup\{\mathcal{A}_\Gamma[\pi_y, y] : y \in Y\}$ but we will not need this.

Lemma 27. *The collection $\mathcal{A}_\Gamma[\pi, x]$ is a $\mathbb{T}$-algebra on Γ^x so long as $[\sigma] \cap \{\sigma_k : k \in \pi(L_x)\}$ is not empty for all $\sigma \in 2^{<\omega}$.*

We skip the proof of this next result since the only non-immediate property in the definition of being a $\mathbb{T}$-algebra depends only on the behavior of each $\mathcal{A}_{\Gamma,x}$.

Proposition 28. *Let $Y \subset X(\Gamma, \aleph_0)$ and let π be a permutation on ω. $\mathcal{A}_\Gamma[\pi, Y]$ is a $\mathbb{T}$-algebra on Γ^Y so long as $\mathcal{A}_\Gamma[\pi, y]$ is a $\mathbb{T}$-algebra for all $y \in Y$.*

Let Q_{perm} denote the poset consisting of functions $\psi \in \omega^{<\omega}$ that are 1-to-1. Q_{perm} is ordered by extension and is forcing isomorphic to $Fn(\omega, 2)$. If G is the generic for Q_{perm}, then $\pi_G = \bigcup G$ is a permutation on ω.

Lemma 29. *Let $\mathcal{A}_\Gamma$ be a $\mathbb{T}$-algebra and let $x \in X(\Gamma, \aleph_0)$ and let $\{x_n : n \in \omega\} \subset X(\Gamma)$ converge to x in $\tau(\mathcal{A}_\Gamma)$. If G is a Q_{perm} generic filter and $\pi = \bigcup G$, then $\mathcal{A}_\Gamma[\pi, x]$ is a $\mathbb{T}$-algebra on Γ^x and $\{x_n : n \in \omega\}$ does not converge in $\tau(\mathcal{A}_\Gamma[\pi, x])$.*

Here is where Cohen genericity will help create a canonical $\mathbb{T}$-algebra that has the ScP.

canonical **Lemma 30.** *Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \mu\rangle$ be a finite support iteration of σ-centered posets. Assume that E is a set of limit ordinals that is cofinal in μ and that $\dot{Q}_\alpha$ is the canonical P_α name for Q_{perm} for all $\alpha \in E$. Let $\mathcal{A}_\Gamma$ be a $\mathbb{T}$-algebra and let $G \subset P_\mu$ be a generic filter. For each $\alpha < \mu$, let $G_\alpha = G \cap P_\alpha$ and let π_α be the permutation obtained from $G_{\alpha+1} \cap \mathrm{val}_{G_\alpha}(\dot{Q}_\alpha)$.*
In $V[G]$, we recursively define $\langle \Gamma_\alpha, \Gamma_{\alpha+1} : \alpha \in E\rangle$ and $\langle \mathcal{A}_{\Gamma_\alpha}, \mathcal{A}_{\Gamma_{\alpha+1}} : \alpha \in E\rangle$:

(1) $\Gamma_\alpha = \Gamma \cup \{\Gamma_{\beta+1} : \beta \in E \cap \alpha\}$ *and* $\mathcal{A}_{\Gamma_\alpha} = \mathcal{A}_\Gamma \cup \bigcup\{\mathcal{A}_{\Gamma_{\beta+1}} : \beta \in E \cap \alpha\}$,
(2) $\Gamma_{\alpha+1} = \Gamma^{Y_\alpha}_\alpha$ *where* $Y_\alpha = X(\Gamma_\alpha, \aleph_0) \cap V[G_\alpha]$,
(3) $\mathcal{A}_{\Gamma_{\alpha+1}}$ *is defined as* $\mathcal{A}_{\Gamma_\alpha}[\pi_\alpha, Y_\alpha]$.

Then, in $V[G]$, $\mathcal{A}_{\Gamma_\mu}$ is a $\mathbb{T}$-algebra, and $\mathcal{A}_{\Gamma_\mu}$ has the ScP so long as each $\mathcal{A}_{\Gamma_\alpha}$ has the ScP.

Proof. We leave the routine verification that $\mathcal{A}_{\Gamma_\mu}$ is a $\mathbb{T}$-algebra as an exercise. Assume that $\mathcal{A}_{\Gamma_\alpha}$ has the ScP for all $\alpha \in E$. Fix any $x \in X(\Gamma_\mu)$ with $\lambda_x = \omega_1$. We must prove that $\mathcal{A}_{\Gamma_\mu,x}$ has the ScP. It is immediate that if $x \in X(\Gamma_\alpha)$ for some $\alpha \in E$, then $\mathcal{A}_{\Gamma_\mu,x} = \mathcal{A}_{\Gamma_\alpha,x}$ does have the ScP. Therefore we may assume that

$x \notin \bigcup_{\alpha\in E} X(\Gamma_\alpha)$. Since $\Gamma_\mu = \bigcup_{\alpha\in E}\Gamma_\alpha$, we may choose $\gamma_\alpha \in \omega_1$, for each $\alpha \in E$, so that that $x \restriction \gamma_\alpha \in X(\Gamma_\alpha)$. Note that $\langle \gamma_\alpha : \alpha \in E\rangle$ is monotone increasing and cofinal in ω_1. Therefore the proof is complete if μ does not have cofinality ω_1.

Now we return to V and argue by forcing. For each $\alpha \in E$, let $\dot\Gamma_\alpha$ be a nice name for Γ_α. Fix a nice P_μ-name for $\dot x$ and $p \in P_\mu$ forcing that $\dot x \in 2^{\omega_1}$ and, for each $\alpha \in E$, let $\dot\gamma_\alpha$ be a nice P_μ name so that $p \Vdash \dot x \restriction \dot\gamma_\alpha \in X(\dot\Gamma_\alpha)$. Since P_μ is ccc and $\dot\gamma_\alpha$ is a nice name, there is a minimal $\beta_\alpha < \mu$ such that $\dot x \restriction \dot\gamma_\alpha$ is a P_{β_α} name. This implies that 1 forces that $\dot x \restriction \dot\gamma_\alpha \in \dot\Gamma_{\beta_\alpha+1}$. Fix a cofinal sequence $\{\alpha_\xi : \xi \in \omega_1\} \subset E$ so that α_η and β_{α_η} are less than α_ξ for all $\eta < \xi$. For each limit $\delta \in \omega_1$, let $\mu_\delta \le \alpha_\delta$ be the supremum of $\{\alpha_\eta : \eta < \delta\}$. Note also that for each limit δ, $\dot x \restriction \dot\gamma_{\alpha_\delta}$ is a P_{μ_δ}-name since it is forced to equal the union of $\{\dot x \restriction \dot\gamma_{\alpha_\eta} : \eta < \delta\}$.

Now also let $\dot S$ be a nice P_μ-name for a stationary subset of ω_1. There is a stationary set S_1 such that for $\delta \in S_1$, there is a $p_\delta < p$ forcing that $\delta \in \dot S$ and a γ_δ such that $p_\delta \restriction \mu_\delta \Vdash \dot\gamma_{\alpha_\delta} = \gamma_\delta$. We may also assume that p_δ decides the value of $\dot x(\gamma_\delta{+}1)$. By the pressing down lemma, we can assume that there is some $\zeta < \mu$ such that $supp(p_\delta)\cap\mu_\delta \subset \zeta$ for all $\delta \in S_1$. Note that $p \in P_\zeta$ and, as is well-known, we can choose a P_ζ-generic G_ζ with $p \in G_\zeta$ so that, in $V[G_\zeta]$, $S = \{\delta \in S_1 : p_\delta \restriction \zeta \in G_\zeta\}$ is stationary. We continue our work in $V[G_\zeta]$. By passing to a stationary subset, we can assume that for all $\delta < \gamma$ in S, $\mathrm{dom}(p_\delta) \subset \mu_\gamma$. A final reduction on S is to assume that there is a single $\psi \in Q_{perm}$ and function $\rho \in \omega_1^m$ so that, for all $\delta \in S$, $p_\delta(\alpha_\delta) = \psi$ and $\rho = e_{\lambda_{\gamma_\delta}} \restriction \mathrm{dom}(\psi)$.

Let $F = \{\rho(k) : k \in \mathrm{dom}(\psi)\}$. We are ready to prove that there is some $q \in G_\zeta$ forcing that $F \cup \dot S$ is a cover. Choose $\bar p \in P_\mu$ such that $\bar p \restriction \zeta \in G_\zeta$ and any $\xi \in \omega_1$. We may assume that $\bar p \Vdash \xi \notin \widehat{a}^{\,\dot x}_F$. Fix any $\delta \in S$ so that $\xi < \gamma_\delta$ and $\bar p \in P_{\mu_\delta}$. Now jump first to $V[G_{\mu_\delta}]$ with $\bar p \in G_{\alpha_\delta}$. Recall that $p_\delta \restriction \mu_\delta \in G_\zeta \subset G_{\mu_\delta}$, so we may assume that $\bar p < p_\delta$. Now pass further to $V[G_{\alpha_\delta}]$ so that $p_\delta \restriction \alpha_\delta \in G_{\alpha_\delta}$. We now prove there is an extension q of p_δ (with $q \restriction \mu_\delta < \bar p$) that forces that $\xi \in \widehat{a}^{\,\dot x}_\delta$. For simpler notation we prove just the case when p_δ forces that $\dot x(\gamma_\delta + 1) = 0$. Let π denote $\dot\pi_{\alpha_\delta}$ and note that p_δ forces (only) that $\psi \subset \pi$.

Let y denote $\mathrm{val}_{G_{\alpha_\delta}}(\dot x \restriction \gamma_\delta)$. Let $\langle a_\eta : \eta < \gamma_\delta\rangle$ denote the sequence $\langle \mathrm{val}_{G_{\alpha_\delta}}(a^y_\eta) : \eta < \gamma_\delta\rangle$. These are infinite subsets of ω and are also in $V[G_{\mu_\delta}]$. Also let λ denote λ_{γ_δ}. Set $c^y_n = a^y_{e_\lambda(n)} \setminus \bigcup_{\ell<n} c^y_\ell$ for all $n \in \omega$ (as in Definition 26). Choose j minimal so that $\xi \in \widehat{a}^{\,y}_{e_\lambda(j)}$. Since F is the initial segment $\{e_\lambda(n) : n \in \mathrm{dom}(\psi)\}$, it follows that $\widehat{a}_F = \bigcup_{k\in\mathrm{dom}(\psi)} c^y_k$ and so j is not in the domain of ψ. Recall that $a^{\dot x}_\delta = a^{\pi,x_\delta}_{\langle 0\rangle}$ is defined to equal $\bigcup\{c^y_k : \sigma_{\pi(k)} \in [\langle 1\rangle]\}$. Choose any $q < p_\delta$ with $q \restriction \alpha_\delta \in G_{\alpha_\delta}$ so that $q(\alpha_\delta) = \tilde\psi \supset \psi = p_\delta(\alpha_\delta)$ is an element of Q_{perm} satisfying that $\sigma_{\tilde\psi(j)} \in [\langle 1\rangle]$. It follows from the minimality of j that $\xi \in \widehat{a}^{\,y}_{e_\lambda(j)} \setminus \bigcup\{\widehat{a}^{\,y}_{e_\lambda(\ell)} : \ell < j\}$. Since q forces that $c^y_j = a^{\,y}_{e_\lambda(j)} \setminus \bigcup\{a^{\,y}_{e_\lambda(\ell)} : \ell < j\}$ is a subset of $a^{\dot x}_\delta$, it should be clear that q forces that $\widehat{a}^{\,\dot x}_\delta$ contains $\widehat{a}^{\,y}_{e_\lambda(j)} \setminus \bigcup\{\widehat{a}^{\,y}_{e_\lambda(\ell)} : \ell < j\}$, and therefore ξ as required. □

Now the main theorem.

Theorem 31. *It is consistent that there is an Efimov space with character $\aleph_1$ while the splitting number is $\aleph_2$. In particular, it is consistent with $\mathfrak{s} = \aleph_2$ to have a $\mathbb{T}$-algebra on $2^{<\omega_1}$ whose Stone space is an Efimov space.* maintheorem

Proof. We assume that $\diamondsuit$ and $2^{\aleph_1} = \aleph_2$ holds and we produce a ccc poset to prove the consistency of the statement. Let S^2_1 denote the ordinals in ω_2 that have uncountable cofinality. Let $(S^2_1)'$ denotes the limit points of S^2_1 in ω_2 and let $E = S^2_1 \cap (S^2_1)'$ (i.e. E is the limit points of uncountable cofinality). Fix a well-ordering $\sqsubset$ of $H(\aleph_2)$. Let Γ_ω be $2^{<\omega}$ and $\mathcal{A}_{\Gamma_\omega}$ be as defined immediately following Proposition 25.

We define a finite support iteration $\langle P_\alpha, \dot{Q}_\alpha : \alpha \in \omega_2\}$ of σ-centered posets by a recursion on α. We also let $\dot\Gamma_\alpha, \dot\Gamma_{\alpha+1}, \dot{\mathcal{A}}_{\Gamma_\alpha}, \dot{\mathcal{A}}_{\Gamma_{\alpha+1}}$ be recursively defined as in Lemma 30,

(1) for $\alpha \notin S^2_1$, $\dot{Q}_\alpha$ is the P_α name for $Fn(\omega, 2)$,
(2) for $\alpha \in E$, $\dot\Gamma_\alpha$ is a P_α name for $2^{<\omega} \cup \bigcup_{\beta \in E\cap\alpha} \dot\Gamma_{\beta+1}$,
(3) for $\alpha \in E$, $\dot{Y}_\alpha$ is the P_α name for $X(\dot\Gamma_\alpha, \aleph_0)$ and $\dot\Gamma_{\alpha+1} = \dot\Gamma_\alpha^{\dot{Y}_\alpha}$ (as in Definition 26,
(4) for all $\alpha \in E$, $\dot{Q}_\alpha$ is the P_α name for Q_{Perm} and $\dot\pi_\alpha$ is the $P_{\alpha+1}$ name for the permutation added by $\dot{Q}_\alpha$,
(5) for $\alpha \in E$, $\dot{\mathcal{A}}_{\Gamma_\alpha}$ is the P_α name for $\mathcal{A}_{\Gamma_\omega} \cup \bigcup\{\dot{\mathcal{A}}_{\Gamma_{\beta+1}} : \beta \in E \cap \alpha\}$ and $\dot{\mathcal{A}}_{\Gamma_{\alpha+1}}$ is the $P_{\alpha+1}$ name for $\dot{\mathcal{A}}_{\Gamma_\alpha}[\pi_\alpha, \dot{Y}_\alpha]$, and finally,
(6) for $\alpha \in S^2_1 \setminus (S^2_1)'$ and $\mu_\alpha = \sup(E \cap \alpha)$, $\dot{\mathcal{D}}_\alpha$ is the $\sqsubset$-least P_α-name of an ultrafilter on ω such that $P_\alpha * \mathbb{L}(\dot{\mathcal{D}}_\alpha)$ forces that $\dot{\mathcal{A}}_{\Gamma_{\mu_\alpha}}$ has the ScP (if one exists), and $\dot{Q}_\alpha$ is the P_α name for $\mathbb{L}(\dot{\mathcal{D}}_\alpha)$. If no such $\dot{\mathcal{D}}_\alpha$ exists, then $\dot{Q}_\alpha$ is the P_α name for $Fn(\omega, 2)$.

Claim 1. For each $\mu \in (S^2_1)' \cup \{\omega_2\}$, P_μ forces that $\dot{\mathcal{A}}_{\dot\Gamma_\mu}$ is a $\mathbb{T}$-algebra with the ScP.

Proof of Claim: We prove the Claim by induction on μ. If $E \cap \mu$ is cofinal in μ, then this follows from the inductive assumption and Lemma 30. Otherwise, let $\alpha = \sup(E \cap \mu)$ and we break into two cases. In the first case $\alpha \in E$ and $\dot{\mathcal{A}}_{\dot\Gamma_\mu}$ is just the $P_{\alpha+1}$ name for $\dot{\mathcal{A}}_{\dot\Gamma_{\alpha+1}}$. Since $\dot{\mathcal{A}}_{\dot\Gamma_\alpha}$ is assumed to have the ScP in the forcing extension by P_α, and since $\dot\Gamma_{\alpha+1} \setminus \dot\Gamma_\alpha$ is forced to be a subset of $2^{<\omega_1}$, it follows that $\dot{\mathcal{A}}_{\dot\Gamma_\mu}$ is forced to have the ScP. Now assume that $\alpha \notin E$. It then follows that $\alpha \in S^2_1 \setminus (S^2_1)'$. In this case, $\dot{\mathcal{A}}_{\Gamma_\mu}$ is forced to equal $\dot{\mathcal{A}}_{\Gamma_\alpha}$ (since $E \cap [\alpha, \mu)$ is empty) and so we must verify that P_μ preserves that $\dot{\mathcal{A}}_{\Gamma_\alpha}$ has the ScP. By the inductive assumption, P_α forces that $\dot{\mathcal{A}}_{\Gamma_\alpha}$ has the ScP and $Fn(\omega, 2)$ always preserves the ScP (Lemma 15). Therefore, by the choice of $\dot{Q}_\alpha$ in clause (6), $P_{\alpha+1}$ forces that $\dot{\mathcal{A}}_{\Gamma_\alpha}$ has the ScP. For all $\alpha < \beta < \mu$, $\dot{Q}_\beta$ is chosen to be $Fn(\omega, 2)$, and so, again by Lemma 15, P_μ forces that $\dot{\mathcal{A}}_{\dot\Gamma_\alpha} = \dot{\mathcal{A}}_{\dot\Gamma_\mu}$ has the ScP. □

Claim 2. P_{ω_2} forces that $\mathfrak{s} = \aleph_2$.

Proof of Claim: Since P_{ω_2} is ccc, it suffices to prove that for all $\mu \in S^2_1 \setminus (S^2_1)'$, P_μ forces that $\dot{Q}_\mu$ adds an unsplit real. Fix any $\mu \in S^2_1 \setminus (S^2_1)'$ and let $\alpha = \sup(E \cap \mu) < \mu$. It is well-known that the generic real added by $\mathbb{L}(\mathcal{D})$ is unsplit providing $\mathcal{D}$ is an ultrafilter on ω. By Claim 1, P_μ forces that $\dot{\mathcal{A}}_{\dot\Gamma_{\alpha+1}}$ has the ScP. Let $G_{\alpha+1}$ be any $P_{\alpha+1}$-generic filter and let $\mathcal{A}_{\Gamma_{\alpha+1}}$ (which has the ScP) be the valuation by $G_{\alpha+1}$ of $\dot{\mathcal{A}}_{\dot\Gamma_{\alpha+1}}$. Since $\diamondsuit$ is assumed to hold in the ground model and $P_{\alpha+1}$ is a ccc poset of cardinality $\aleph_1$, $\diamondsuit$ also holds in $V[G_{\alpha+1}]$ ([14]VII.H.8). By Theorem 24, there is a $Fn(\omega_1, 2)$ name, $\dot{\mathcal{D}}$ satisfying that $Fn(\omega_1, 2) * \mathbb{L}(\dot{\mathcal{D}})$ preserves that $\mathcal{A}_{\Gamma_{\alpha+1}}$ has the ScP. Since the forcing extension by P_μ is equal to the forcing extension of

$V[G_{\alpha+1}]$ by $Fn(\omega_1, 2)$, it follows that there is a $\dot{D}_\mu$ as required in clause (6) of the construction so that P_μ forces that $\dot{Q}_\mu$ is $\mathbb{L}(\mathcal{D})$ for an ultrafilter $\mathcal{D}$. □

This completes the proof of the Theorem. □

6. Splitting number and Scarborough-Stone problem

The Scarborough-Stone problem asks if each product of sequentially compact spaces is countably compact. A space is sequentially compact if every infinite sequence has a converging subsequence. Just as with the Efimov space question, this problem is not known to be independent of ZFC. With this, the original formulation of the problem, it is not known if an affirmative answer is consistent with ZFC. By the results in [7], we have the following corollary to Theorem 31.

Corollary 32. *It is consistent that there is a family of first-countable sequentially compact spaces of cardinality less than $\mathfrak{s}$ whose product is not countably compact.* smallsize

We give a brief sketch of the proof.

Proof. Let $\Gamma = 2^{<\omega_1}$ and let $\mathcal{A}_\Gamma = \{a_\sigma : \sigma \in 2^{<\omega_1}\}$ be the $\mathbb{T}$-algebra constructed in Theorem 31 in the model in which $\mathfrak{s} = \aleph_2$. For each $x \in 2^{\omega_1}$, let $\mathcal{A}_{\Gamma,x}$ be the proper coherent sequence as defined in Definition 7 and let Y_x denote the topology on ω_1 as a subspace of of $\omega_1 + 1 = \lambda_x + 1$ as defined in Proposition 6. Also let φ_x be the continuous mapping from $X(\Gamma)$ onto $Y_x \cup \{\omega_1\}$ as defined in Lemma 9. It is evident from Proposition 6 that Y_x is first-countable. We prove that Y_x is sequentially compact. Fix any infinite sequence $\{y_n : n \in \omega\} \subset Y_x$ and choose any sequence $\{x_n : n \in \omega\} \subset X(\Gamma)$ satisfying that $\varphi_x(x_n) = y_n$ for all $n \in \omega$. Since $X(\Gamma)$ is compact and has no converging sequences, there is some limit $z \in X(\Gamma)$ of $\{x_n : n \in \omega\}$ with $z \neq x$. Let $y = \varphi(z) \in Y_x$ and note that y is a limit of $\{y_n : n \in \omega\}$. Since y has countable character in Y_x, $\{y_n : n \in \omega\}$ has a subsequence converging to y. Now we prove that the product of the family $\{Y_x : x \in 2^{\omega_1}\}$ is not countably compact. Choose any $r \in 2^\omega$ and consider the product of the subfamily $\{Y_x : r \subset x \in 2^{\omega_1}\}$. For each $n \in \omega$, let $\overline{n}$ denote the function in the product $\Pi\{Y_x : r \subset x \in 2^{\omega_1}\}$ that has value n in every coordinate (recall that Y_x has base set ω_1). Let $\vec{y}$ be any point in $\Pi\{Y_x : r \subset x \in 2^{\omega_1}\}$ and we show that $\vec{y}$ is not a limit point of $\{\overline{n} : n \in \omega\}$. Since each $n \in \omega$ is isolated in each Y_x, we may assume that $\vec{y}(x) \geq \omega$ for all $r \subset x \in 2^{\omega_1}$. For each $\sigma \in 2^{<\omega_1}$, let $X_\sigma = \{x \in 2^{\omega_1} : \sigma \subset x\}$. For each $x \in X_r$, let $\sigma_x = x \restriction (y(x)+1)$ and note that $\widehat{a}_{\sigma_x}$ is a neighborhood of $y(x)$ in Y_x. Let W_x be the product neighborhood $\pi_x^{-1}[\widehat{a}_{\sigma_x}]$ in $\Pi\{Y_x : x \in X_r\}$. By Definition 7 (4), a_{σ_x} and $a_{\sigma_x^\dagger}$ are disjoint and by the coherence property, so are $\widehat{a}_{\sigma_x}$ and $\widehat{a}_{\sigma_x^\dagger}$. If there are $x, x_1 \in X_r$ such that $\sigma_{x_1} = \sigma_x^\dagger$, then $W_x \cap W_{x_1} \cap \{\overline{n} : n \in \omega\}$ is equal to $\{\overline{n} : n \in \widehat{a}_{\sigma_x} \cap \widehat{a}_{\sigma_x^\dagger}\}$, and so is empty. Therefore we consider the case that for all $r \subset \sigma \in 2^{<\omega_1}$, one of $\{\sigma^\frown 0, \sigma^\frown 1\}$ is not in the set $\{\sigma_x : x \in X_r\}$. Beginning with $\sigma_0 = r$, we can now recursively construct a strictly increasing sequence $\{\sigma_\xi : \xi \in \omega_1\} \subset 2^{<\omega_1}$ satisfying that for each $\xi \in \omega_1$ and each $x \in X_{\sigma_\xi}$, $\sigma_\xi \subsetneq \sigma_x$. Of course $x = \bigcup\{\sigma_\xi : \xi \in \omega_1\}$ contradicts the assumption that $y(x) < \omega_1$. □

In the other direction, it is a natural question to ask if a negative answer to the Scarborough-Stone problem follows from $\mathfrak{s} = \aleph_1$. Many partial results are known, see, for example, Vaughan's survey article [17]. It is shown in [9] that the assumption $\diamondsuit(\mathfrak{s})$, a strengthening of $\mathfrak{s} = \aleph_1$, suffices. In the spirit of Corollary 32, we pose the following problems.

(1) Does $\mathfrak{s} = \aleph_1$ imply a negative answer to the Scarborough-Stone problem?
(2) Does $\mathfrak{s} = \aleph_1$ imply there is a family of sequentially compact spaces, each of cardinality at most $\aleph_1$, whose product is not countably compact?
(3) Does $\mathfrak{s} = \aleph_1$ imply there is a $\mathbb{T}$-algebra on $2^{<\omega_1}$ whose Stone space has no converging sequences?

Two well-known strengthenings of the assumption $\mathfrak{s} = \aleph_1$ may be relevant to these questions. A splitting family $\mathcal{S}$ is $\aleph_0$-splitting if for every countable family $\{b_n : n \in \omega\}$ of infinite subsets of $\mathbb{N}$, there is a single member of $\mathcal{S}$ that splits each of them. It is not known if an $\aleph_0$-splitting family of cardinality $\mathfrak{s}$ necessarily exists. A splitting family $\mathcal{S} = \{s_\alpha : \alpha \in \mathfrak{s}\}$ is tail-splitting if for each infinite $b \subset \mathbb{N}$, the set of members of $\mathcal{S}$ that split b contains a final segment of $\mathcal{S}$. It is known to be consistent that there is no tail-splitting sequence of cardinality $\mathfrak{s}$. We formulate a still stronger condition that is sufficient to obtain the conclusions of problems (1)-(3). This condition will hold in a forcing extension by uncountably many Random reals, or by a finite support iteration with cofinality equal to ω_1. Recall that $H(\omega_1)$ is equal to the set of sets whose transitive closure is countable. Say that $M \subset H(\omega_1)$ is a model if it, i.e. $(M, \in)$, is a model of all the axioms of ZF with the exception of the power set axiom. Of course $H(\omega_1)$ itself is a model (see [14, IV]).

Proposition 33. *Assume that $H(\omega_1)$ can written as an increasing chain $\{M_\xi : \xi \in \omega_1\}$ of models in such a way that for each ξ, there is a subset of ω that splits every member of $M_\xi \cap [\omega]^\omega$, then there is a $\mathbb{T}$-algebra on $2^{<\omega_1}$ whose Stone space has no converging sequences.*

The proof is a minor variant of similar proofs in [7, 13].

Proof. For each $\alpha \in \omega_1$, let $\Gamma_\alpha = 2^{<\alpha}$. Also, for each $\omega \leq \alpha \in \omega_1$, let $e_\alpha : \omega \to \alpha$ be a bijection onto the successor ordinals in α. Let $\{a_\sigma : \sigma \in \Gamma_\omega\}$ be the $\mathbb{T}$-algebra as defined in Definition 26. For each $x \in 2^\omega$, fix a $\xi_x \geq \omega$ so that e_ω and $\{a_{x\restriction n} : n \in \omega\}$ are in M_{ξ_x}. By induction on $\omega \leq \alpha < \omega_1$ we construct a $\mathbb{T}$-algebra $\mathcal{A}_{\Gamma_\alpha} = \{a^\alpha_\sigma : \sigma \in \Gamma_\alpha\}$ and choose ordinals $\{\xi_\sigma : \sigma \in \Gamma_\alpha\}$ so that the following induction hypotheses hold for all $\sigma \in \Gamma_\alpha$:

(1) if $\beta < \alpha$ and $\sigma \in \Gamma_\beta$, then $a^\beta_\sigma = a^\alpha_\sigma$,
(2) $\mathrm{dom}(\sigma) < \xi_\sigma$, $e_{\mathrm{dom}(\sigma)} \in M_{\xi_\sigma}$ and $\{a^\alpha_{\sigma\restriction\beta} : \beta \in \mathrm{dom}(\sigma)\} \in M_{\xi_\sigma}$,
(3) if $\sigma^\frown 0 \in \Gamma_\alpha$, then $\{n : c_\sigma(n) \subset a^\alpha_{\sigma^\frown 0}\}$ splits every infinite $b \subset \omega$ in M_{ξ_σ} where, for each $n \in \omega$, $c_\sigma(n)$ denotes the set $a^\alpha_{\sigma\restriction e_{\mathrm{dom}(\sigma)}(n)} \setminus \bigcup\{a^\alpha_{\sigma\restriction e_{\mathrm{dom}(\sigma)}(m)} : m < n\}$.

The inductive construction is routine and can be omitted. We note that properties (2) and (3) ensure that each of $\{a^\alpha_{\sigma\restriction\beta} : \beta \in \mathrm{dom}(\sigma)\} \cup \{a^\alpha_{\sigma^\frown 0}\}$ and $\{a^\alpha_{\sigma\restriction\beta} : \beta \in \mathrm{dom}(\sigma)\} \cup \{a^\alpha_{\sigma^\frown 1}\}$ are proper coherent sequences. We finish by proving that $X(\Gamma_{\omega_1})$ has no converging sequences. Let $\{x_n : n \in \omega\}$ be an infinite subset of 2^{ω_1} and we show that the sequence does not converge to $x \in 2^{\omega_1}$. Let φ_x be the mapping as in Lemma 9, and let, for $n \in \omega$, $y_n = \varphi_x(x_n)$. Following Definition 7, let $a^x_\alpha = a_{x\restriction\alpha+1}$ for all $\alpha \in \omega_1$. It suffices to find a $\beta < \omega_1$ so that $\{n : y_n \in \widehat{a^x_\beta}\}$ is infinite. If $\{y_n : n \in \omega\}$ is finite, then this is immediate, so assume that it is infinite. For each

$k \in \omega$, let $\beta_k+1 = e_{\mathrm{dom}(\sigma)}(k)$, and for each $n \in \omega$, choose the minimal $k_n \in \omega$ so that $y_n \in \widehat{a^x_{\beta_{k_n}}}$.

Now let L be the infinite set $\{x \restriction y_n+1 : n \in \omega\}$ and choose $\sigma \in \{x \restriction \beta : \beta < \omega_1\}$ large enough so that $\{y_n : n \in \omega\} \subset \mathrm{dom}(\sigma)$ and each of L and $\{k_n : n \in \omega\}$ are elements of M_{ξ_σ}. Let $\mathrm{dom}(\sigma) = \alpha$ and let $\{c_\sigma(k) : k \in \omega\}$ be as in condition (3). Then there is an infinite set b_σ chosen so that $a^{\alpha+1}_{\sigma^\frown 0} = \bigcup\{c_\sigma(k) : k \in b_\sigma\}$. Since $a^{\alpha+1}_{\sigma^\frown 1} = \omega \setminus a^{\alpha+1}_{\sigma^\frown 0}$, we also have that $a^{\alpha+1}_{\sigma^\frown 1} = \bigcup\{c_\sigma(k) : k \in \omega \setminus b_\sigma\}$. Since a^x_α is one of $a^{\alpha+1}_{\sigma^\frown 0}, a^{\alpha+1}_{\sigma^\frown 1}$, we have that $b^x_\alpha = \{k \in \omega : c_\sigma(k) \subset a^x_\alpha\}$ splits $\{k_n : n \in \omega\}$. We finish by checking that $y_n \in \widehat{a^{\alpha+1}_\alpha}$ for the infinitely many n such that $k_n \in b^x_\alpha$. Fix any n with $k_n \in b^x_\alpha$ and recall that $y_n \in \widehat{a^\alpha_{\beta_{k_n}}} \setminus \bigcup_{m<k_n} \widehat{a^\alpha_{\beta_m}}$. Since $c_\sigma(k_n) = a^\alpha_{\beta_{k_n}} \setminus \bigcup_{m<n} a^\alpha_{\beta_m}$, it follows that $y_n \in \widehat{a^x_\alpha}$ since $\widehat{a^x_\alpha}$ contains $\widehat{a^\alpha_{\beta_{k_n}}} \setminus \bigcup_{m<k_n} \widehat{a^\alpha_{\beta_m}}$. □

References

[1] Andreas Blass, *Selective ultrafilters and homogeneity*, Ann. Pure Appl. Logic **38** (1988), no. 3, 215–255, DOI 10.1016/0168-0072(88)90027-9. MR942525

[2] Jörg Brendle and Saharon Shelah, *Ultrafilters on ω—their ideals and their cardinal characteristics*, Trans. Amer. Math. Soc. **351** (1999), no. 7, 2643–2674, DOI 10.1090/S0002-9947-99-02257-6. MR1686797

[3] W. Brian and A. Dow, *Small cardinals and small Efimov spaces* (2019). submitted.

[4] Alan Dow, *Efimov spaces and the splitting number*, Topology Proc. **29** (2005), no. 1, 105–113. Spring Topology and Dynamical Systems Conference. MR2182920

[5] A. Dow, *Compact C-closed spaces need not be sequential*, Acta Math. Hungar. **153** (2017), no. 1, 1–15, DOI 10.1007/s10474-017-0739-x. MR3713559

[6] Alan Dow and Roberto Pichardo-Mendoza, *Efimov's problem and Boolean algebras*, Topology Appl. **160** (2013), no. 17, 2207–2231, DOI 10.1016/j.topol.2013.09.006. MR3116583

[7] Alan Dow and Saharon Shelah, *An Efimov space from Martin's axiom*, Houston J. Math. **39** (2013), no. 4, 1423–1435. MR3164725

[8] ———, *On the cofinality of the splitting number*, Indag. Math. (N.S.) **29** (2018), no. 1, 382–395, DOI 10.1016/j.indag.2017.01.010. MR3739621

[9] Miguel Ángel Gaspar-Arreola, Fernando Hernández-Hernández, and Michael Hrušák, *Scattered spaces from weak diamonds*, Israel J. Math. **225** (2018), no. 1, 427–449, DOI 10.1007/s11856-018-1669-1. MR3805653

[10] Jaime I. Ihoda and Saharon Shelah, *Δ^1_2-sets of reals*, Ann. Pure Appl. Logic **42** (1989), no. 3, 207–223, DOI 10.1016/0168-0072(89)90016-X. MR998607

[11] Sabine Koppelberg, *Boolean algebras as unions of chains of subalgebras*, Algebra Universalis **7** (1977), no. 2, 195–203, DOI 10.1007/BF02485429. MR0434914

[12] ———, *Minimally generated Boolean algebras*, Order **5** (1989), no. 4, 393–406, DOI 10.1007/BF00353658. MR1010388

[13] Piotr Koszmider, *Forcing minimal extensions of Boolean algebras*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3073–3117, DOI 10.1090/S0002-9947-99-02145-5.

[14] Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs. MR597342

[15] Kenneth Kunen and Franklin D. Tall, *Between Martin's axiom and Souslin's hypothesis*, Fund. Math. **102** (1979), no. 3, 173–181, DOI 10.4064/fm-102-3-173-181. MR532951

[16] Richard Laver, *On the consistency of Borel's conjecture*, Acta Math. **137** (1976), no. 3-4, 151–169. MR0422027

[17] Jerry Vaughan, *Open problems in topology. II* (Elliott Pearl, ed.), Elsevier B. V., Amsterdam, 2007. MR2367385