COMPACT SPACES WITH A P-BASE

ALAN DOW AND ZIQIN FENG

Abstract. In the paper, we investigate (scattered) compact spaces with a P-base for some poset P. More specifically, we prove that any compact space with an ω^ω-base is metrizable and any scattered compact space with an ω^ω-base is countable under the assumption $\omega_1 < \mathfrak{b}$. These give positive solutions to Problems 8.6.9 and 8.7.7 in [1]. Using forcing, we also prove that in a model of $\omega_1 < \mathfrak{b}$, there is a non-first-countable compact space with a P-base for some poset P with calibre ω_1.

1. Introduction

Let P be a partially ordered set. A topological space X is defined to have a neighborhood P-base at $x \in X$ if there exists a neighborhood base $(U_p[x])_{p\in P}$ at x such that $U_p[x] \subset U_{p'}[x]$ for all $p \geq p'$ in P. We say that a topological space has a P-base if it has a neighborhood P-base at each $x \in X$. All topological spaces in this paper are regular.

We will use Tukey order to compare the cofinal complexity of posets. The Tukey order [19] was originally introduced, early in the 20th century, as a tool to understand convergence in general topological spaces, however it was quickly seen to have broad applicability in comparing partial orders. Given two directed sets P and Q, we say Q is a Tukey quotient of P, denoted by $P \geq_T Q$, if there is a map $\phi : P \to Q$ carrying cofinal subsets of P to cofinal subsets of Q. In our context, where P and Q are both Dedekind complete (every bounded subset has the least upper bound), we have $P \geq_T Q$ if and only if there is a map $\phi : P \to Q$ which is order-preserving and such that $\phi(P)$ is cofinal in Q. If P and Q are mutually Tukey quotients, we say that P and Q are Tukey equivalent, denoted by $P =_T Q$. It is straightforward to see that a topological space X has a P-base if and only if $\mathcal{T}_x(X) \leq_T P$ for each $x \in X$, here, $\mathcal{T}_x(X) = \{U : U \text{ is an open neighborhood of } x\}$.

Topological spaces and function spaces with an ω^ω-base were systematically studied in [1]. Lots of work also have been done with the ω^ω-base in topological groups (see [2], [8], [15], and [18]). In this paper we investigate the Tukey reduction of a P-base in some (scattered) compact spaces with P satisfying some Calibre conditions. This paper is organized in the following way.

In Section 3, we show that if P has Calibre ω_1, then any compact space with a P-base is countable tight. Furthermore, we prove that if a compact space with countable tightness has a $\mathcal{K}(M)$-base for some separable metric space M, then it is first-countable. As a corollary, any compact space with an ω^ω-base is first-countable under the assumption $\omega_1 < \mathfrak{b}$. This gives a positive answer to Problem

Date: March 15, 2021.

2010 *Mathematics Subject Classification.* 54D30, 46B50.

Key words and phrases. (Scattered) compact spaces, scattered height, Tukey order, Calibre ω_1, Calibre (ω_1, ω), convergent free sequence, ω^ω-base, countable tightness .

8.7.7 in [1]. In Section 4, we address Problem 8.6.9 in [1] positively by showing that any scattered compact space with an ω^ω-base is countable under the assumption $\omega_1 < \mathfrak{b}$. It is natural to ask whether under the assumption $\omega_1 < \mathfrak{b}$ any compact with a P-base is first-countable if P satisfies some Calibre properties, for example, Calibre ω_1. In Section 5, we prove that in a model of Martin's Axiom in which $\omega_1 < \mathfrak{b}$, there is a non-first-countable compact space with a P-base for some poset P with calibre ω_1.

2. Preliminaries

For any separable metric space M, $\mathcal{K}(M)$ is the collection of compact subsets of M ordered by set-inclusion. Fremlin observed that if a separable metric space M is locally compact, then $\mathcal{K}(M) =_T \omega$. Its unique successor under Tukey order is the class of Polish but not locally compact spaces. For M in this class, $\mathcal{K}(M) =_T \omega^\omega$ where ω^ω is ordered by $f \leq g$ if $f(n) \leq g(n)$ for each $n \in \omega$. In [9], Gartside and Mamataleshvili constructed a $2^{\mathfrak{c}}$-sized antichain in $\mathcal{K}(\mathcal{M}) = \{\mathcal{K}(M) : M \in \mathcal{M}\}$ where $\mathcal{M}$ is the set of separable metric spaces.

Let P be a directed poset, i.e. for any points $p, p' \in P$, there exists a point $q \in P$ such that $p \leq q$ and $p' \leq q$. A subset C of P is *cofinal* in P if for any $p \in P$, there exists a $q \in C$ such that $p \leq q$. Then $\mathrm{cof}(P) = \min\{|C| : C \text{ is cofinal in } P\}$. We also define $\mathrm{add}(P) = \min\{|Q| : Q \text{ is unbounded in } P\}$. For any $f, g \in \omega^\omega$, we say that $f \leq^* g$ if the set $\{n \in \omega : f(n) > g(n)\}$ is finite. Then $\mathfrak{b} = \mathrm{add}(\omega^\omega, \leq^*)$ and $\mathfrak{d} = \mathrm{cof}(\omega^\omega, \leq^*)$. See [5] for more information about small cardinals.

Let $\kappa \geq \mu \geq \lambda$ be cardinals. We say that a poset P has *calibre* (κ, μ, λ) if for every κ-sized subset S of P there is a μ-sized subset S_0 such that every λ-sized subset of S_0 has an upper bound in P. We write calibre (κ, μ, μ) as calibre (κ, μ) and calibre (κ, κ, κ) as calibre κ. It is known that $\mathcal{K}(M)$ has Calibre (ω_1, ω) for any separable metric space M, hence so does ω^ω. Under the assumption $\omega_1 = \mathfrak{b}$, ω_1 is a Tukey quotient of ω^ω. Furthermore, under the assumption $\omega_1 < \mathfrak{b}$, the poset ω^ω has Calibre $(\omega_1, \omega_1, \omega_1)$, i.e. Calibre ω_1. We will use this fact in several places of this paper.

It is clear that if $P \leq_T Q$ and $Q \leq_T R$ then $P \leq_T R$ for any posets P, Q, and R. So we get the following proposition.

Proposition 2.1. *Let P and Q be posets such that $P \leq_T Q$. Then if a space X has a neighborhood P-base at $x \in X$, then X also has a neighborhood Q-base at x. Hence, any space with a P-base also has a Q-base.*

Proposition 2.2. *If X has a P-base, then any subspace of X also has a P-base.*

Proposition 2.3. *Let P be a poset with $\omega_1 \leq_T P$ and $P =_T \omega \times P$. Then the space $\omega_1 + 1$ has a P-base.*

Proof. For each $\alpha < \omega_1$, the space $\omega_1 + 1$ has a countable local base at α. Hence $\mathcal{T}_\alpha(\omega_1 + 1) \leq_T P$ due to the fact that $P =_T \omega \times P$.

Let ϕ be a map from P to ω_1 which carries confinal subsets of P to confinal subsets of ω_1. Then we define a map ψ from P to $\mathcal{T}_{\omega_1}(\omega_1+1)$ by $\psi(p) = (\phi(p), \omega_1]$ for each $p \in P$. Clearly ψ carries confinal subsets of P to confinal subset of $\mathcal{T}_{\omega_1}(\omega_1+1)$. Hence the space ω_1+1 has a neighborhood P-base at ω_1. This finishes the proof. $\square$

As a result of $\mathfrak{b} \leq_T \omega^\omega$, the space $\omega_1 + 1$ has an ω^ω-base under the assumption $\omega_1 = \mathfrak{b}$. Gartside and Mamatelashvili in [10] proved that $\omega^\omega \times \omega_1 \leq_T \mathcal{K}(\mathbb{Q}) \leq_T \omega^\omega \times [\omega_1]^{<\omega}$, here $\mathbb{Q}$ is the space of rationals. Hence, we have the following result.

Corollary 2.4. *The space $\omega_1 + 1$ has a $\mathcal{K}(\mathbb{Q})$-base.*

A generalization of G_δ-diagonals is P-diagonals for some poset P. A collection $\mathcal{C}$ of subsets of a space X is P-directed if $\mathcal{C}$ can be represented as $\{C_p : p \in P\}$ such that $C_p \subseteq C_{p'}$ whenever $p \leq p'$. We say X has a P-diagonal if $X^2 \setminus \Delta$ has a P-directed compact cover, where $\Delta = \{(x,x) : x \in X\}$. The second author showed that any compact space with a $\mathcal{K}(\mathbb{Q})$-diagonal is metrizable in [7] and Sánchez proved that the same result holds for any compact space with a $\mathcal{K}(M)$-diagonal for some separable metric space M in [17]. Here, we include two results about spaces with (or without) P-diagonal giving that P satisfies some Calibre properties.

Proposition 2.5. *Let P be a poset with Calibre (ω_1, ω). The space $\omega_1 + 1$ doesn't have a P-diagonal.*

Proof. Suppose that $\omega_1 + 1$ has a P-diagonal, i.e., a P-ordered compact covering $\{K_p : p \in P\}$ of $(\omega_1 + 1)^2 \setminus \Delta$ Choose α_γ and β_γ in ω_1 for $\gamma \in \omega_1$ such that

$$\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \ldots < \alpha_\gamma < \beta_\gamma < \ldots.$$

Let p_γ in P be such that $(\alpha_\gamma, \beta_\gamma) \in K_{p_\gamma}$ for each $\gamma \in \omega_1$. By Calibre (ω_1, ω), there are p in P and $\gamma_n \in \omega_1, n \in \omega$ such that $\gamma_0 < \gamma_1 < \ldots$ and $K_{p_{\gamma_n}} \subset K_p$ for each $n \in \omega$. Then $(\delta, \delta) \in K_p$, where $\delta = \sup\{\alpha_{\gamma_n} : n \in \omega\}$. This contradiction finishes the proof. $\square$

Proposition 2.6. *Let P be a poset with Calibre ω_1. Any compact space with a P-diagonal has countable tightness.*

Proof. Let $\{K_p : p \in P\}$ be a P-ordered compact covering of $X^2 \setminus \Delta$. Suppose that X has uncountable tightness. Then, X has a free sequence of length ω_1, hence a convergence free sequence of length ω_1 by [12]. Let $\{x_\alpha : \alpha < \omega_1\}$ be such a sequence and x^* the limit point.

Choose α_γ and β_γ in ω_1 for $\gamma \in \omega_1$ such that $\alpha_0 < \beta_0 < \alpha_1 < \beta_1 < \ldots < \alpha_\gamma < \beta_\gamma < \ldots$. For each $\gamma \in \omega_1$, fix $p_\gamma \in P$ such that $(x_{\alpha_\gamma}, x_{\beta_\gamma}) \in K_{p_\gamma}$. Since P has Calibre ω_1, there is an uncountable subset γ_τ of ω_1 with $\tau < \omega_1$ such that p_{γ_τ} is bounded above by $p^* \in P$. Hence, $K_{p_{\gamma_\tau}} \subseteq K_{p^*}$ for each $\tau < \omega_1$, furthermore, $(x_{\alpha_{\gamma_\tau}}, x_{\beta_{\gamma_\tau}}) \in K_{p^*}$. Since $\{x_\alpha : \alpha < \omega_1\}$ is a convergent sequence with limit point x^*, the subsequences $x_{\alpha_{\gamma_\tau}}$ and $x_{\beta_{\gamma_\tau}}$ both converge to x^*, hence, $(x^*, x^*) \in K_{p^*}$ which contradicts with the fact that $K_{p^*} \subset X \setminus \Delta$. This contradiction finishes the proof. $\square$

3. Compact Spaces

In this section, we study compact spaces possessing a P-base with P satisfying some Calibre property. Mainly, we investigate the problem whether each compact Hausdorff space with an ω^ω-base first countable under the assumption $\omega_1 < \mathfrak{b}$. We start with some ZFC result about tightness of the spaces with a P-base.

Theorem 3.1. *Let κ be an uncountable regular cardinal and P be a poset with Calibre κ. If X is a compact Hausdorff space with a P-base, then $t(X) < \kappa$.*

Proof. Assume, for contradiction, that $t(X) \geq \kappa$. Then, X has a free sequence of length κ. Hence, by [12], X has a convergent sequence of length κ. Let $\{x_\alpha : \alpha < \kappa\}$ be such a sequence and x^* be the limit point. Let $S = \{x_\alpha : \alpha < \kappa\} \cup \{x^*\}$. Notice that for any unbounded subset $\{\alpha_\gamma : \gamma < \kappa\}$, x^* is the limit point of $\{x_{\alpha_\gamma} : \gamma < \kappa\}$. Fix a neighborhood base $\{B_p : p \in P\}$ at x_*. It is straightforward to see that $S \setminus B_p$ has size $< \kappa$ for each $p \in P$.

For each $\alpha < \kappa$, choose $p_\alpha \in P$ such that $x_\alpha \notin B_{p_\alpha}$. Let $P' = \{p_\alpha : \alpha \in \kappa\}$. If the cardinality of P' is $< \kappa$, there exists a $p_\alpha \in P'$ such that $S \setminus B_{p_\alpha}$ has size κ which is a contradiction. Hence, P' have cardinality κ. Since P has Calibre κ, there is a κ-sized subset P'' of P' which is bounded above. List P'' as $\{p_{\alpha_\gamma} : \gamma < \kappa\}$ and pick p^* to be the upper bound of P''. Then, $S \setminus B_{p^*} = \{x_{\alpha_\gamma} : \gamma < \kappa\}$ which is a contradiction. This finishes the proof. □

Corollary 3.2. *Let P be a poset with Calibre ω_1. Each compact Hausdorff space with a P-base is countable tight.*

A poset P has Calibre (ω_1, ω) if it has Calibre ω_1. It is showed in [16] that for a separable metric space M the poset $\mathcal{K}(M)$ has Calibre ω_1 if it has Calibre $(\omega_1, \omega_1, \omega)$. Hence it is natural to ask whether a compact space has countable tightness if it has a P-base with P having Calibre (ω_1, ω). The following example shows that the answer is negative. So the result above is 'optimal' in terms of the Calibre complexity of posets having the form $\mathcal{K}(M)$ with M being a separable metric space.

Example 3.3. *There is a poset P with Calibre (ω_1, ω) and a compact space X with a P-base, but $t(X) > \omega$.*

Proof. Let P be $\mathcal{K}(\mathbb{Q})$ which clearly has Calibre (ω_1, ω). From Proposition 2.4, the space $\omega_1 + 1$ has a P-base, but its tightness is uncountable. □

Again, since ω^ω has Calibre ω_1 under the assumption $\omega_1 < \mathfrak{b}$, we obtain the following result about spaces with an ω^ω-base.

Corollary 3.4. *Assume that $\omega_1 < \mathfrak{b}$. Each compact Hausdorff space with an ω^ω-base is countable tight.*

It is folklore that any GO-space with countable tightness is first countable. Hence, applying Corollary 3.4 to compact GO-spaces, we obtain the following results.

Corollary 3.5. *Let P be a poset with Calibre ω_1. Each compact GO-space has a P-base if and only if it is first countable.*

The following example shows that the result above doesn't hold for general GO-spaces.

Example 3.6. *There is a poset P with Calibre ω_1 such that there exists a GO-space with a P-base and uncountable tightness.*

Proof. Consider the set $\omega_2 + 1$ in the ordinal order. Let $\mathcal{T}$ be the topology on $\omega_2 + 1$ such that every point except ω_2 is isolated and a base at ω_2 is $\{(\alpha, \omega_2] : \alpha < \omega_2\}$. So the space $(\omega_2 + 1, \mathcal{T})$ is a non-first-countable GO-space and clearly has a neighborhood ω_2-base at ω_2. It is straightforward to verify that the poset ω_2 has Calibre ω_1 since every ω_1-sized subset is bounded above. □

The result below was proved in [1] through a different approach. We obtain it here as a result of ω^ω having Calibre ω_1 under the assumption $\omega_1 < \mathfrak{b}$.

Corollary 3.7. *Assume that $\omega_1 < \mathfrak{b}$. Each compact GO-space has an ω^ω-base if and only if it is first countable.*

It is natural to ask as in [1] (Problem 8.7.7) whether the same result holds for any compact space. Next, we'll give a positive answer to this problem by showing that any compact space with a P-base is first countable if $P = \mathcal{K}(M)$ for some separable metric space M has Calibre ω_1.

First, we show that any compact space with countable tightness is first countable if it has a P-base and $P = \mathcal{K}(M)$ for some separable metric space. We use the ideas and techniques from [4].

Theorem 3.8. *Let $P = \mathcal{K}(M)$ for some separable metric space M. If X is a compact space with countable tightness and has a P-base, then X is first-countable.*

Proof. Fix $x \in X$ and an open P-base $\{U_p[x] : p \in P\}$ at x. For each $p \in P$, let $K_p = X \setminus U_p[x]$. Then, $\{K_p : p \in P\}$ is a P-directed compact cover of $X \setminus \{x\}$.

For any separable metric space M, the space $P = \mathcal{K}(M)$ with the Hausdorff metric d^H is also separable, hence second countable. Also if $\{p_n : n \in \omega\}$ is a sequence converging to p in P, then $p^* = p \cup (\bigcup\{p_n : n \in \omega\})$ is compact, hence it is an element in P with $p_n \subset p^*$ and $p \subset p^*$.

Fix a countable base $\{B_n : n \in \omega\}$ of P. For each $n \in \omega$, define $L(B_n) = \bigcup\{K_p : p \in B_n\}$. And for each $p \in P$, define $C(p) = \bigcap\{L(B_n) : p \in B_n\}$. For each $p \in P$, we pick a decreasing local base $\{B^p_{n_i} : i \in \omega\} \subseteq \{B_n : n \in \omega\}$ at p such that for each $i \in \omega$ there is a positive number ϵ_i satisfying that $B^p_{n_i} \supset D_{d^H}(p, \epsilon_i) \supset \overline{B^p_{n_{i+1}}}$ where $D_{d^H}(p, \epsilon_i)$ is the open ball centered at p with radius ϵ_i. Define $C'(p) = \bigcap\{L(B^p_{n_i}) : i \in \omega\}$. It is straightforward to verify that $C'(p) = C(p)$.

First, we claim that x is not in the closure of $C(p)$ for all $p \in P$. Fix $p \in P$. By the countable tightness of X, it suffices to show that if x is not in the closure of any countable subset of $C(p)$. Let $\{y_i : i \in \omega\}$ be a countable subset of $C(p)$. For each $i \in \omega$, choose $q_i \in B^p_{n_i}$ with $y_i \in K_{q_i}$. Clearly $\{q_i : i \in \omega\}$ is a sequence converging to p, hence $p^* = p \cup (\bigcup\{q_i : i \in \omega\})$ is an element in P with $\{y_i : i \in \omega\} \subset K_{p^*}$ which implies that x is not in the closure of $\{y_i : i \in \omega\}$.

Then, we claim that for each $p \in P$, there is an $i \in \omega$ such that x is not in the closure of $L(B^p_{n_i})$. Fix $p \in P$. Choose any open set U such that $\overline{C(p)} \subset U$ and $x \notin \overline{U}$. It suffices to prove that there is an i so that $L(B^p_{n_i}) \subset U$. Suppose not. Choose $y_i \in L(B^p_{n_i}) \setminus U$ for each $i \in \omega$. Then for each $i \in \omega$, choose $q_i \in B^p_{n_i}$ so that $y_i \in K_{q_i}$. Define $p^*_i = p \cup (\bigcup\{q_j : j \geq i\})$. Hence $\{y_j : j \geq i\} \subset K_{p^*_i}$. By the property of the Hausdorff metric d^H, it is straightforward to verify that $d^H(p^*_{i+1}, p) \leq \epsilon_i$, hence $p^*_{i+1} \in B^p_{n_i}$ which implies that $K_{p^*_{i+1}} \subset L(B^p_{n_i})$ for each $i \in \omega$. Therefore, $\bigcap\{K_{p^*_i} : i \in \omega\} \subseteq C(p)$. Then, all the limit points of $\{y_i : i \in \omega\}$ are in $C(p)$ which contradicts with $C(p) \subset U$ and the choices of $\{y_i : i \in \omega\}$.

Finally, we prove that the family $\mathcal{L} = \{\overline{L(B_n)} : x \notin \overline{L(B_n)}\}$ is a cover of $X \setminus \{x\}$, furthermore, $\{B : B = X \setminus S$ for some $S \in \mathcal{L}\}$ is a local base at x. For each $p \in P$, there is an $i \in \omega$ such that that x is not in the closure of $L(B^p_{n_i})$. Hence $\overline{L(B^p_{n_i})} \in \mathcal{L}$. Since $K_p \subset L(B^p_{n_i})$, this completes the proof. $\square$

Theorem 3.9. *Let $P = \mathcal{K}(M)$ for some separable metric space M such that P has Calibre ω_1. Any compact space X with a P-base is first-countable.*

Proof. By Corollary 3.4, X has countable tightness since P has Calibre ω_1. Then by Theorem 3.8, X is first-countable. □

Then using the fact that ω^ω has Calibre ω_1 under the assumption $\omega_1 < \mathfrak{b}$, we get a positive answer to Problem 8.7.7 in [1].

Corollary 3.10. *Assume $\omega_1 < \mathfrak{b}$. A compact space has an ω^ω-base if and only if it is first countable.*

4. Scattered Compact Spaces

We recall that a topological space X is scattered if each non-empty subspace of X has an isolated point. The complexity of a scattered space can be determined by the scattered height.

For any subspace A of a space X, let A' be the set of all non-isolated points of A. It is straightforward to see that A' is a closed subset of A. Let $X^{(0)} = X$ and define $X^{(\alpha)} = \bigcap_{\beta<\alpha}(X^{(\beta)})'$ for each $\alpha > 0$. Then a space X is scattered if $X(\alpha) = \emptyset$ for some ordinal α. If X is scattered, there exists a unique ordinal $h(x)$ such that $x \in X^{(h(x))} \setminus X^{(h(x)+1)}$ for each $x \in X$. The ordinal $h(X) = \sup\{h(x) : x \in X\}$ is called the scattered height of X and is denoted by $h(X)$. It is known that any compact scattered space is zero-dimensional. Also, it is straightforward to show that for any compact scattered space X, $X^{(h(x))}$ is a non-empty finite subset.

Theorem 4.1. *Let P be a poset with Calibre ω_1 and X a scattered compact space with a P-base. Then X is countable.*

Proof. If $h(X) = 0$, then X is countable because it is compact.

Assume $h(X) = \alpha$ and any compact scattered space with a P-base is countable if it has a scattered height $< \alpha$. Since X is compact, $X^{(\alpha)}$ is a nonempty finite subset of X. List $X^{(\alpha)} = \{x_1, \ldots, x_n\}$. For each $i \in \{1, \ldots, n\}$, take a closed and open neighborhood U_i of x_i with $U_i \cap X(\alpha) = \{x_i\}$. Then $X \setminus \bigcup\{U_i : i = 1, \ldots, n\}$ is a scattered compact space with scattered height $< \alpha$, hence it is countable by the assumption. So it is sufficient to show that U_i is countable for each $i = 1, \ldots, n$.

Fix $i \in \{1, \ldots, n\}$. Consider the subspace $Y = U_i \cap X$. By proposition 2.2, Y has a neighborhood P-base $\{B_p : p \in P\}$ at $\{x_i\}$. For each $p \in P$, $Y \setminus B_p$ is a compact subspace with scattered height $< \alpha$, hence is countable by the inductive assumption.

Assume that Y is uncountable. Take an uncountable subset $\{y_\alpha : \alpha < \omega_1\}$ of $Y \setminus \{x_i\}$. For each $\alpha < \omega_1$, we choose $p_\alpha \in P$ such that $y_\alpha \notin B_{p_\alpha}$.

If $\{p_\alpha : \alpha < \omega_1\}$ is countable, there is a $p^* \in \{p_\alpha : \alpha < \omega_1\}$ such that there is an uncountable subset D of $\{y_\alpha : \alpha < \omega_1\}$ such that $D \subset Y \setminus B_{p^*}$ which is a contradiction.

If $\{p_\alpha : \alpha < \omega_1\}$ is uncountable, then it has an uncountable subset P' which is bounded above using the Calibre ω_1 property of P. List $P' = \{p_{\alpha_\gamma} : \gamma < \omega_1\}$. Let p^* be an upper bound of P'. Then we have that $y_{\alpha_\gamma} \notin B_{p^*}$ for each $\gamma < \omega_1$. This also contradicts with the fact that $Y \setminus B_{p^*}$ is countable. This finishes the proof. □

Using the same approach we obtain the following example.

Example 4.2. *The one point Lindelöfication of uncountably many points doesn't have a P-base if P has Calibre ω_1, hence, under the assumption $\omega_1 < \mathfrak{b}$, it doesn't have an ω^ω-base.*

The example above uses the fact that under the assumption $\omega_1 < \mathfrak{b}$, the poset ω^ω has Calibre ω_1. Furthermore, using Theorem 4.1, we obtain the following result which answers Problem 8.6.9 in [1] positively. This also gives a partial positive answer to Problem 8.6.8 in the same paper.

Corollary 4.3. *Assume $\omega_1 < \mathfrak{b}$. Any scattered compact space with an ω^ω-base is countable, hence metrizable.*

It was proved in [1] that any compact spaced with an ω^ω-base and finite scattered height is countable, hence metrizable. Next, we show that the same result holds for any compact space with a P-base and finite scattered height if P has Calibre (ω_1, ω).

Theorem 4.4. *Let P be a poset with Calibre (ω_1, ω) and X a compact Hausdorff space with a P-base and finite scattered height. Then X is countable, hence metrizable.*

Proof. Fix a natural number $n > 0$. Assume that any compact Hausdorff space with scattered height $< n$ is countable. Let X be a compact Hausdorff space with scattered height n. We'll show that X is countable. As discussed in the proof of Theorem 4.1, we could assume that $X^{(n)}$ is a singleton, denoted by x, without loss of generality. Suppose, for contradiction, that X is uncountable. Define m to be the greatest natural number such that $X^{(m)}$ is uncountable and $X^{(m+1)}$ is countable. Then there are two cases: 1. $m = n - 1$; 2. $m < n - 1$. We will obtain contradictions in both cases.

First assume that $m = n - 1$. Then we fix a neighborhood P-base $\{B_p : p \in P\}$ at x. For each $p \in P$, $X^{(m)} \setminus B_p$ is finite as X is compact and B_p is open. Pick an uncountable subset $\{x_\alpha : \alpha < \omega_1\}$ of $X^{(m)}$ with $x_\alpha \neq x$ for all $\alpha < \omega_1$. For each $\alpha < \omega_1$, there is a $p_\alpha \in P$ such that $x_\alpha \notin B_{p_\alpha}$. If $\{p_\alpha : \alpha < \omega_1\}$ is countable, then there exists $p^* \in \{p_\alpha : \alpha < \omega_1\}$ such that $X^{(m)} \setminus B_{p^*}$ is uncountable which is a contradiction. If $\{p_\alpha : \alpha < \omega_1\}$ is uncountable, we can find a countable bounded subset $\{p_{\alpha_n} : n \in \omega\}$ of $\{p_\alpha : \alpha < \omega_1\}$ using the Calibre (ω_1, ω) property of P. Let the upper bound of $\{p_{\alpha_n} : n \in \omega\}$ be p^*. Then, $x_{\alpha_n} \notin B_{p^*}$ for each $n \in \omega$. This is a contradiction.

Now we assume that $m < n - 1$. Then $X^{(m+1)} \setminus \{x\}$ is countable which can be listed as $\{x_\ell : \ell \in \omega\}$. For each ℓ, pick a closed and open neighborhood U_ℓ of x_ℓ. Then for each $\ell < \omega$, U_ℓ is a compact subspace with scattered height $< n$, hence is countable. Therefore, $X^{(m)} \setminus \bigcup\{U_\ell : \ell \in \omega\}$ is uncountable. Pick an uncountable subset $S = \{x_\alpha : \alpha < \omega_1\}$ of $X^{(m)} \setminus (\{x\} \cup (\bigcup\{U_\ell : \ell \in \omega\}))$. Fix a neighborhood P base $\{B_p : p \in P\}$ at x. For each $p \in P$, $S \setminus B_p$ is finite. Similarly as in the proof of case 1, we could obtain a contradiction using the Calibre (ω_1, ω) property of P. □

The result above doesn't hold for compact space with uncountable scattered height since the space $\omega_1 + 1$ has a $\mathcal{K}(\mathbb{Q})$-base. However, we don't know the answer to the following problem.

Question 4.5. *Assume that $\omega_1 = \mathfrak{b}$. Let P be a poset with Calibre (ω_1, ω) and X be any compact Hausdorff space with a P-base and countable scattered height. Is X countable?*

5. Calibre ω_1 and non-first-countable compact space

We prove that there is a model of Martin's Axiom in which there is a compact space that has a P-base for a poset P with Calibre ω_1. This space will be the space constructed by Juhasz, Koszmider, and Soukup in the paper [11]. This article [11] shows there is a forcing notion that forces the existence of a first-countable initially ω_1-compact locally compact space of cardinality ω_2 whose one-point compactification has countable tightness. We must prove that there is a poset P as above. We must also show that extra properties of the space ensure that we can perform a further forcing to obtain a model of Martin's Axiom and that the desired properties of a space naturally generated from the original space possesses these same properties in the final model. The reader may be interested to note that in this way we produce a model of Martin's Axiom and $\mathfrak{c} = \omega_2$ in which there is a compact space of countable tightness that is not sequential. This is interesting because Balogh proved in [3] that the forcing axiom, PFA, implies that compact spaces of countable tightess are sequential. It was first shown in [6] that the celebrated Moore-Mrowka problem was independent of Martin's Axiom plu $\mathfrak{c} = \omega_2$. The methods in [6] are indeed based on the paper [11] using the notion of T-algebras first formulated in [13]. The example in [11] is itself a space generated by a T-algebra but is not explicitly formulated as such because of its simpler structure.

To do all this, at minimum cost, we must explicitly reference a number of statements and proofs from [11]. The construction is modeled on the following natural property of locally compact scattered topology, τ, with base set an ordinal μ in which initial segments are open. The well-ordering on the underlying set arises canonically from the fact that such spaces are right-separated and scattered. There are functions H with domain μ and a function $i : [\mu]^2 \to [\mu]^{<\aleph_0}$ satisfying that for all $\alpha < \beta < \mu$:

(1) $\alpha \in H(\alpha) \subset \alpha + 1$ and $H(\alpha)$ is a compact open set (i.e. $H(\alpha) \in \tau$),
(2) $i(\alpha, \beta)$ is a finite subset of α,
(3) if $\alpha \notin H(\beta)$, then $H(\alpha) \cap H(\beta) \subset \bigcup\{H(\xi) : \xi \in i(\alpha, \beta)\}$
(4) if $\alpha \in H(\beta)$, then $H(\alpha) \setminus H(\beta) \subset \bigcup\{H(\xi) : \xi \in i(\alpha, \beta)\}$.

Conversely if H and i are functions as in (1)-(4) where (1) is replaced by simply

(1') $\alpha \in H(\alpha) \subset \alpha + 1$ (i.e. no mention of topology)

then using the family $\{H(\alpha) : \alpha \in \mu\}$ as a clopen subbase generates a locally compact scattered topology on μ in which H, i satisfy property (1)-(4).

Statements (3) and (4) are combined into a single statement in [11] by adopting the notation

$$H(\alpha) * H(\beta) = \begin{cases} H(\alpha) \cap H(\beta) & \alpha \notin H(\beta) \\ H(\alpha) \setminus H(\beta) & \alpha \in H(\beta) \end{cases} .$$

As noted in [11] a locally compact scattered space can not have the properties listed above, hence the construction must be generalized. Also it is shown above (and in [1] for $P = \omega^\omega$) that a compact scattered space with a P-base that has Calibre ω_1 will be first countable.

The generalization from [11] will use almost the same terminology and ideas to generate a topology on the base set $\omega_2 \times \mathbb{C}$ where $\mathbb{C} = 2^{\mathbb{N}}$ is the usual Cantor set and, for each $\alpha < \omega_2$, $\{\alpha\} \times \mathbb{C}$ will be homeomorphic to $\mathbb{C}$. For $n \in \mathbb{N} = \omega \setminus \{0\}$ and $\epsilon \in 2$, the notation $[n, \epsilon]$ will denote the clopen subset $\{f \in 2^{\mathbb{N}} : f(n) = \epsilon\}$ in

$\mathbb{C}$. However a critically important aspect of the construction to watch for is that every point of the space will have a local base of neighborhoods that *splits* only one of the sets in $\{\{\alpha\} \times \mathbb{C} : \alpha \in \omega_2\}$. The function H will identify the copies that such a subbasic clopen set meets (and contains all except the *top* one). More precisely, $H(\alpha, 0) \times \mathbb{C}$ be a subbasic clopen set, and for $n > 0$, the set $H(\alpha, n) \subset H(\alpha, 0)$ will be used to construct the subbasic clopen set that intersects $\{\alpha\} \times \mathbb{C}$ as the set $\{\alpha\} \times [n, 1]$. Naturally, $H(\alpha, 0) \setminus H(\alpha, n)$ will generate the subbasic clopen set corresponding to $\{\alpha\} \times [n, 0]$. The function i is similarly generalized to be a function from $[\omega_2]^2 \times \omega$ into $[\omega_2]^{<\aleph_0}$. Here is the definition of a suitable pair of functions from [11, Definition 2.4].

Definition 5.1. *A pair $H : \omega_2 \times \omega \to \mathcal{P}(\omega_2)$ and $i : [\omega_2]^2 \times \omega \to [\omega_2]^{<\aleph_0}$ is ω_2-suitable if the following conditions hold for all $\alpha < \beta < \omega_2$ and $n \in \mathbb{N}$:*

(1) $\alpha \in H(\alpha, n) \subset H(\alpha, 0) \subset \alpha + 1$,
(2) $i(\alpha, \beta, n) \in [\alpha]^{<\aleph_0}$,
(3) $H(\alpha, 0) * H(\beta, n) \subset \bigcup\{H(\xi, 0) : \xi \in i(\alpha, \beta, n)\}$.

Also, given an ω_2-suitable pair H, i, define the following sets for $\alpha \in \omega_2$, $F \in [\omega_2]^{<\aleph_0}$ and $n \in \mathbb{N}$:

(4) $U(\alpha) = U(\alpha, \mathbb{C}) = H(\alpha, 0) \times \mathbb{C}$,
(5) $U(\alpha, [n, 1]) = (\{\alpha\} \times [n, 1]) \ \cup \ ((H(\alpha, n) \setminus \{\alpha\}) \times \mathbb{C})$,
(6) $U(\alpha, [n, 0]) = U(\alpha, \mathbb{C}) \setminus U(\alpha, [n, 1])$,
(7) $U[F] = \bigcup\{U(\xi) : \xi \in F\}$.

Next we rephrase [11, Lemma 2.5]:

Proposition 5.2. *If H, i is an ω_2-suitable pair then the subbase*

$$\{U(\alpha, \mathbb{C}) : \alpha \in \omega_2\} \cup \{U(\alpha, [n, \epsilon]) : \alpha \in \omega_2, n \in \mathbb{N}, \epsilon \in 2\}$$

generates a locally compact Hausdorff topology τ_H on $\omega_2 \times \mathbb{C}$ satisfying that for all $\alpha \in \omega_2$, $n \in \mathbb{N}$, and $r \in \mathbb{C}$,

(1) *$U(\alpha, \mathbb{C})$, $U(\alpha, [n, 1])$ are compact,*
(2) *the collection of finite intersections of members of the family*

$$\{U(\alpha, [n, r(n)]) \setminus U[F] : n \in \mathbb{N}, F \in [\alpha]^{<\aleph_0}\}$$

is a local base at (α, r)

Next, the authors of [11] have to work very hard to produce an ω_2-suitable pair so that τ_H is first-countable and initially ω_1-compact. The first step is to work in a model in which there is a special function $f : [\omega_2]^2 \mapsto [\omega_2]^{\leq\aleph_0}$ called a strong Δ-function. Since we will not need any properties of this function we omit the definition, but henceforth assume that f is such a function. We record additional minor modifications of results from [11, 4.1,4.2].

Proposition 5.3. *There is a ccc poset P_f consisting of quadruples $p = (a_p, h_p, i_p, n_p)$ that are finite approximations of an ω_2-suitable pair where*

(1) *$a_p \in [\omega_2]^{<\aleph_0}$, $n_p \in \omega$*
(2) *$h_p : [a_p]^2 \times n_p \mapsto \mathcal{P}(a_p)$,*
(3) *$i_p : [a_p]^2 \times n_p \mapsto [a_p]^{<\aleph_0}$,*

and, for each P_f-generic filter G, the relations

$$H = \bigcup\{h_p : p \in G\} \quad and \quad i = \bigcup\{i_p : p \in G\}$$

are functions that form an ω_2-suitable pair. In particular, if $p \in G$, $\alpha \in h_p(\beta)$, and $i_p(\alpha,\beta,0)=\emptyset$, then (in $V[G]$) $U(\alpha,\mathbb{C}) \subset U(\beta,\mathbb{C})$.

The space $(\omega_2 \times \mathbb{C}, \tau_H)$ is shown to have these additional properties [11, 4.2]:

Proposition 5.4. *If G is P_f-generic and H, i are defined as in Proposition 5.3, then the following hold in $V[G]$:*

1. *$X_H = (\omega_2 \times \mathbb{C}, \tau_H)$ is locally compact 0-dimensional of cardinality $\mathfrak{c} = 2^{\aleph_1} = \aleph_2$,*
2. *X_H is first-countable,*
3. *for every $A \in [X_H]^{\omega_1}$, there is a $\lambda < \omega_2$ such that $A \cap U(\lambda,\mathbb{C})$ is uncountable,*
4. *for every countable $A \subset X_H$, either $\overline{Y}$ is compact or there is an $\alpha < \omega_2$ such that $(\omega_2 \setminus \alpha) \times \mathbb{C}$ is contained in $\overline{Y}$.*

Consequently X_H is a locally compact, 0-dimensional, normal, first-countable, initially ω_1-compact but non-compact space.

Finally, we need the following strengthening of [11, Lemma 7.1] but which is actually proven.

Proposition 5.5. *If $p = (a_p, h_p, i_p, n_p) \in P_f$ and $a_p \subset \lambda \in \omega_2$, then there is a $q < p$ in P_f such, that*

1. *$a_q = a_p \cup \{\lambda\}$ and $n_q = n_p$,*
2. *$a_p \subset h_q(\lambda, 0)$,*
3. *$i_q(\alpha,\lambda,j) = \emptyset$ for all $\alpha \in a_p$ and $j < n_q$.*

We note that for p, q as in Lemma 5.5, if q is in the generic filter G, then $U(\alpha,\mathbb{C})$ is a subset of $U(\lambda,\mathbb{C})$ for all $\alpha \in a_p$. One consequence of this is that the family $\{U(\alpha,\mathbb{C}) : \alpha \in \omega_2\}$ is finitely upwards directed. Equivalently, the family of complements of these sets in the one-point compactification of X_H is a neighborhood base for the point at infinity.

Now we strengthen [11, Lemma 7.2] which will be used to prove that the one-point compactification of X_H has Calibre ω_1. Some of our proofs will require forcing arguments and we refer the reader to [14] for more details. However some remarks may be sufficient to assist many readers. The forcing extension, $V[G_Q]$ by a Q-generic filter G_Q for a poset Q is equal to the valuation, $\mathrm{val}_{G_Q}(\dot{A})$ for the collection of all Q-names $\dot{A}$ that are sets from V. The notation $q \Vdash x \in \dot{A}$ can be read as the assertion that $x \in \mathrm{val}_{G_Q}(\dot{A})$ for any generic filter with $q \in G_Q$. The forcing theorem ([14, VII 3.6]) ensures, for example, that if $\dot{A}$ is a Q-name of a subset of a ground model set B, then b is an element of $\mathrm{val}_{G_Q}(\dot{A})$ exactly when there is an element $q \in G_Q$ such that $q \Vdash b \in \mathrm{val}_{G_Q}(\dot{A})$. Additionally, the set of $q \in Q$ that satisfy that $q \Vdash x \in \dot{A}$ is a set in the ground model, as is the set of x for which there exists a q with $q \Vdash x \in \dot{A}$. This justifies the first line of the next proof.

Lemma 5.6. *In $V[G]$, for each uncountable $A \subset \omega_2$, there is a $\lambda < \omega_2$ such that $U(\alpha,\mathbb{C}) \subset U(\lambda,\mathbb{C})$ for uncountably many $\alpha \in A$.*

Proof. Let $\dot{A}$ be a P_f-name for a subset of ω_2. Fix any condition $p \in G$ and assume that p forces that $\dot{A}$ has cardinality $\aleph_1$. We prove that there is a $q < p$ and a $\lambda \in a_q$ satisfying that if $q \in G$ then there are uncountably many $\alpha \in \mathrm{val}_G(\dot{A})$ such that

$U(\alpha,\mathbb{C}) \subset U(\lambda,\mathbb{C})$. It is a standard fact of forcing that this would then establish the Lemma (i.e. that there is then necessarily such a $q \in G$).

Let I denote the set of $\alpha \in \omega_2$ satisfying that there is some $p_\alpha < p$ (which we choose) forcing that $\alpha \in \dot{A}$. Since p forces that $\dot{A}$ is a subset of I it follows that I has cardinality at least ω_1. Since P_f is ccc, it also follows that I has cardinality equal to ω_1 but it suffices for this argument to choose any $\lambda \in \omega_2$ such that $I \cap \lambda$ is uncountable. For each $\alpha \in I$, choose $q_\alpha < p_\alpha$ so that $a_{q_\alpha} = a_p \cup \{\lambda\}$ and the properties of the pair p_α, q_α are as stated in Proposition 5.5.

Just as in the proof of [11, Lemma 7.2], the fact that P_f is ccc ensures that there is some $q < p$ such that so long as $q \in G$, the set $\{\alpha \in I \cap \lambda : q_\alpha \in G\}$ is uncountable. As remarked after Proposition 5.5, it follows, in $V[G]$, that $U(\alpha,\mathbb{C}) \subset U(\lambda,\mathbb{C})$ for all $\alpha \in \{\alpha \in I \cap \lambda : q_\alpha \in G\}$. □

Theorem 5.7. *If G is a P_f-generic filter, then in $V[G]$, the one-point compactification of the space X_H has a P-base for a poset with Calibre ω_1.*

Proof. The poset P consists of the family $\{U(\alpha,\mathbb{C}) : \alpha \in \omega_2\}$ ordered by inclusion. To complete the proof we have to note that $\omega <_T P$. For this it is enough to prove that there is a countable subset of P that has no upper bound. It is relatively easy to prove that X_H is separable (indeed, that $\omega \times \mathbb{C}$ is dense) but oddly enough this is not stated in [11] and we can more easily simply note that X_H is not σ-compact because by Proposition 5.4 it is countably compact and non-compact. □

An important feature of the construction of X_H from the ω_2-suitable pair H, i is that even in a forcing extension by a ccc poset Q (in fact by any poset that preserves that ω_1 and ω_2 are cardinals), the new interpretation of the space obtained using H, i (i.e. the base set $\omega_2 \times \mathbb{C}$ may change because there can be new elements of $\mathbb{C}$) is still locally compact and 0-dimensional. This is similar to the fact that local compactness of scattered spaces is preserved by any forcing (a result by Kunen). The other properties of X_H, such as first-countability and initial ω_1-compactness, as well as properties of its one-point extension are not immediate and will depend on what subsets of ω_2 have been added.

An unexpected feature of the ω_2-suitable pair is that, in fact, the first countability of X_H is preserved by any forcing.

Lemma 5.8. *For each poset Q in $V[G]$ and Q-generic filter G_Q, the space X_H is first-countable in $V[G][G_Q]$.*

Proof. Of course we will use the fact that, in $V[G]$, X_H is first-countable (as stated in Proposition 5.4). Fix any $\alpha \in \omega_2$ and recall from Proposition 5.2, that the collection of all finite intersections of the family

$$\{U(\alpha,[n,r(n)]) \setminus U[F] : n \in \mathbb{N},\ F \in [\alpha]^{<\aleph_0}\}$$

is a local base at $(\alpha, r) \in \{\alpha\} \times \mathbb{C}$ (in any model). In $V[G]$, for each $r \in \mathbb{C}$, let $Z(\alpha,r) = \bigcap_{n\in\mathbb{N}} U(\alpha,[n,r(n)])$ and let $K(\alpha,r) = \{\xi < \alpha : \{\xi\} \times \mathbb{C} \subset Z(\alpha,r)\}$. Let us recall that $\xi \in K(\alpha,r)$ if and only if $Z(\alpha,r) \cap (\{\xi\} \times \mathbb{C})$ is not empty. Similarly, by the definition of $U(\alpha,[n,r(n)])$ given in Definition 5.1, $K(\alpha,r) = \bigcap\{H(\alpha,[n,r(n)]) : n \in \mathbb{N}\}$. Since, for all $n \in \mathbb{N}$, $\{H(\alpha,[n,0]), H(\alpha,[n,1])\}$ is a partition of $H(\alpha,0)$, it follows that $\{K(\alpha,r) : r \in \mathbb{C}\}$ is also a partition of $H(\alpha,0)$. Since X_H is first-countable (in $V[G]$), for each $r \in \mathbb{C}$, there is a countable $F_r \subset K(\alpha,r)$ such that $K(\alpha,r) \times \mathbb{C} \subset \bigcup\{U(\xi,0) : \xi \in F_r\}$.

Now we are ready to show that, in $V[G][G_Q]$, each point of $\{\alpha\}\times\mathbb{C}$ is a G_δ-point in X_H. For each $r\in\mathbb{C}$, we again define the G_δ-set $Z(\alpha,r)$ and $K(\alpha,r)\subset H(\alpha,0)$ as we did in $V[G]$ but as *calculated* in the new model $V[G][G_Q]$. It is immediate that $Z(\alpha,r)\cap(\{\alpha\}\times\mathbb{C})$ is equal to (α,r). Since there are no changes to the values of $H(\alpha,[n,\epsilon])$ for $(n,\epsilon)\in\mathbb{N}\times 2$, the value of $K(\alpha,r)$ for each $r\in\mathbb{C}\cap V[G]$ is unchanged and the family $\{K(\alpha,r):r\in\mathbb{C}\}$ is a partition of $H(\alpha,0)$. It clearly remains the case that, for $r\in\mathbb{C}\cap V[G]$, $K(\alpha,r)\times\mathbb{C}$ is a subset of $\bigcup\{U(\xi,0):\xi\in F_r\}$. This implies that (α,r) is a G_δ-point for each $r\in\mathbb{C}\cap V[G]$. Now consider a point $s\in\mathbb{C}$ that is not an element of $V[G]$. But now we have that $K(\alpha,s)$ is empty since $H(\alpha,0)$ is covered by the family $\{K(\alpha,r):r\in\mathbb{C}\cap V[G]\}$. This implies that Z_s is equal to the singleton set $\{(\alpha,s)\}$. □

Next we prove that we can extend the model $V[G]$ to obtain a model in which Martin's Axiom holds (and $\mathfrak{c}=\omega_2$). We do so using the following result from [14, VI 7.1, VIII 6.3] (i.e. the standard method to construct a model of Martin's Axiom).

Proposition 5.9. *In the model $V[G]$, there is an increasing chain $\{Q_\xi:\xi\le\omega_2\}$ of partially ordered sets satisfying for each $\xi<\omega_2$*

1. *Q_ξ is a ccc poset of cardinality at most $\aleph_1$,*
2. *each maximal antichain of Q_ξ is a maximal antichain of Q_{ω_2},*
3. *if G_2 is a Q_{ω_2}-generic filter, then in the model $V[G][G_2]$*
   - (a) *Martin's Axiom holds and $\mathfrak{c}=\omega_2$*
   - (b) *for each $A\subset\omega_2\times\mathbb{C}$ of cardinality less than ω_2, there is a $\xi<\omega_2$ such that A is in the model $V[G][G_2\cap Q_\xi]$.*

For the remainder of this section let $\{Q_\xi:\xi\le\omega_2\}$ be the poset as in this Proposition and let G_2 be a Q_{ω_2}-generic filter. The model $V[G][G_2\cap Q_\xi]$ is actually equal to the valuation by G_2 of all Q_ξ-names that are in $V[G]$.

First we prove that the poset of P (consisting of the family $\{U(\alpha,\mathbb{C}):\alpha\in\omega_2\}$ ordered by inclusion) still has Calibre ω_1 in the forcing extension of $V[G]$ by Q_{ω_2}. In fact, by Proposition 5.9, it suffices to prove that any ccc poset Q of cardinality at most $\aleph_1$ preserves that P has Calibre ω_1.

Lemma 5.10. *If G_Q is Q-generic over $V[G]$ for a ccc poset, then P has Calibre ω_1 in the model $V[G][G_Q]$.*

Proof. Let $\dot{A}$ be a Q-name of a subset of ω_2 and let q be any element of Q. Let I be the set of $\alpha\in\omega_2$ such that there exists some $q_\alpha<q$ such that $q_\alpha\Vdash\alpha\in\dot{A}$. For each $\alpha\in I$ choose such a $q_\alpha<q$. Fix any $\lambda<\omega_2$ so that $I_\lambda=\{\alpha\in I:U(\alpha,\mathbb{C})\subset U(\lambda,\mathbb{C})\}$ is uncountable. Choose $\bar{q}<q$ so that for all Q-generic G_Q with $\bar{q}\in G_Q$, the set $\{\alpha\in I_\lambda:q_\alpha\in G_Q\}$ is uncountable. Since $\alpha\in\mathrm{val}_{G_Q}(\dot{A})$ for all $\alpha\in I$ with $q_\alpha\in G_Q$, this completes the proof that P retains the Calibre ω_1 property. □

It follows from the results so far that, in the model $V[G][G_2]$, the one-point compactification of X_H has a P-base and that P has Calibre ω_1. Also, $\{U(\alpha,0):\alpha\in\omega_2\}$ is an open cover of X_H that has no countable subcover, so the one-point compactification is not first-countable. This completes the proof of the desired properties, but it is of independent interest to prove this next result because of the connection to the Moore-Mrowka problem.

Theorem 5.11. *In the model $V[G][G_2]$ there is an ω_2-suitable pair H, i and a poset P of Calibre ω_1 such that each of the following hold:*

(1) *Martin's Axiom and $\mathfrak{c} = \omega_2$,*
(2) *the space X_H is locally compact, 0-dimensional, first-countable, and not compact,*
(3) *the one-point compactification of X_H has a P-base*
(4) *the space X_H is initially ω_1-compact and normal,*
(5) *the one-point compactification of X_H is compact, has countable tightness, and is not sequential.*

Proof. We have already established items (1), (2), and (3). Item (3) implies that the one-point compactification of X_H has countable tightness. Item (5) is an immediate consequence of items (1)-(4). So it remains to prove item (4). This will require a forcing proof over the model $V[G]$. Before we begin, let us notice that:

Fact 1. In $V[G]$, if S is an unbounded subset of ω_2, then the closure of $S \times \mathbb{C}$ will contain $(\omega_2 \setminus \alpha) \times \mathbb{C}$ for some $\alpha \in \omega_2$.

This follows from the property in item (4) because of the facts that $S \times \mathbb{C}$ does not have compact closure and that the one-point compactification of X_H has countable tightness.

Recall, from Proposition 5.4, that, in $V[G]$, the closure in X_H of each countable subset of X_H is either compact or contains $(\omega_2 \setminus \alpha) \times \mathbb{C}$ for some $\alpha \in \omega_2$. We will prove that this statement remains true in $V[G][G_2]$. Before doing so we note that item (4) is a consequence of this claim. It is immediate from (4) that X_H is countably compact. The fact that then X_H is initially ω_1-compact follows from the fact that a compact P-space has no converging ω_1-sequences. The fact that X_H is normal is noted in [11, §8] and is similar to the proof that an Ostaszewski space is normal. Indeed, it follows from item (4) that for any two disjoint closed subsets of X_H at least one of them is compact.

Let $\dot{A}$ be a Q_{ω_2}-name of a countable subset of X_H. Assume there is a $q \in G_2$ such that q forces that the closure of $\dot{A}$ is not compact. Note that q forces that for all finite $F \subset \omega_2$, $\dot{A} \setminus U[F]$ is not empty. Also, that the closure of $\dot{A}$ is forced to miss $\{\alpha\} \times \mathbb{C}$ if and only if $\dot{A}$ is forced to miss $U(\alpha, 0) \setminus U[F]$ for some finite $F \subset \alpha$.

By Proposition 5.9, there is a $\xi < \omega_2$ and a Q_ξ-name $\dot{B}$ satisfying that $\mathrm{val}_{G_2 \cap Q_\xi}(\dot{B})$ is equal to $\mathrm{val}_{G_2}(\dot{A})$. By possibly choosing a larger value of ξ, we may assume that $q \in Q_\xi$. We first note that it suffices to work with $\dot{B}$ and the poset Q_ξ.

Fact 2. For each $\lambda \in \omega_2$, $k \in \mathbb{N}$, and $t : \{1, \ldots, k\} \mapsto 2$, and finite $F \subset \lambda$, the following are equivalent:

(1) $\mathrm{val}_{G_2}(\dot{A})$ misses $\bigcap\{U(\lambda, [n, t(n)]) : 1 \leq n \leq k\} \setminus U[F]$,
(2) $\mathrm{val}_{G_2 \cap Q_\xi}(\dot{B})$ misses $\bigcap\{U(\lambda, [n, t(n)]) : 1 \leq n \leq k\} \setminus U[F]$.

We must prove that the closure of $\mathrm{val}_{G_2 \cap Q_\xi}(\dot{B})$ contains $\{\lambda\} \times \mathbb{C}$ for a co-initial set of $\lambda \in \omega_2$. This means that we are interested in the set of $\lambda \in \omega_2$ such that $\{\lambda\} \times \mathbb{C}$ is not contained in the closure of $\dot{B}$. For any such λ, there must be a $q \geq q_\lambda \in Q_\xi$, an integer k_λ and a function $t_\lambda : \{1, \ldots, k_\lambda\} \mapsto 2$, and a finite $F_\lambda \subset \lambda$ such that q_ξ forces that $\dot{B}$ is disjoint from $\bigcap\{U(\lambda, [n, t_\lambda(n)]) : 1 \leq n \leq k_\lambda\} \setminus U[F_\lambda]$. Let S denote the set of all λ such that such a sequence $\langle q_\lambda, k_\lambda, t_\lambda, F_\lambda \rangle$ exists.

If $\alpha \notin S$, then q forces that $\{\alpha\} \times \mathbb{C}$ is contained in the closure of $\dot{B}$. Of course this also means that q forces that the closure of $\dot{B}$ contains the closure of $(\omega_2 \setminus S) \times \mathbb{C}$. In this case, Fact 1 implies that there is an $\alpha \in \omega_2$ such that the closure of $(\omega_2 \setminus S) \times \mathbb{C}$, and therefore of $\mathrm{val}_{G_2 \cap Q_\xi}(\dot{B})$, will contain $(\omega_2 \setminus \alpha) \times \mathbb{C}$ as required. Therefore we conclude that if S is unbounded, then $\omega_2 \setminus S$ is bounded.

We assume that S is unbounded and obtain a contradiction. Since Q_ξ has cardinality $\aleph_1$, it follows from the pressing down lemma that there is a stationary subset S_1 of S, consisting of limits with cofinality ω_1, and a tuple $\langle \bar{q}, k, t, F\rangle$ such that, for all $\lambda \in S_1$

(1) $\bar{q} = q_\lambda$, $k = k_\lambda$, $t = t_\lambda$, and
(2) $F = F_\lambda$.

Let W be the union of the family $\{\bigcap\{U(\lambda, [n, t(n)]) : 1 \leq n \leq k\} \setminus U[F] : \lambda \in S_1\}$. Since W is open and the property of item (4) holds in $V[G]$, it follows that $X_H \setminus W$ is compact. Choose any finite $F_1 \subset \omega_2$ so that $X_H \subset W \cup U[F_1]$. It now follows that $\bar{q}$ forces that $\dot{B}$ is contained in $U[F \cup F_1]$, which is a contradiction. $\square$

Acknowledgement. The authors would like to express their gratitude to the referee for all his/her valuable comments and suggestions which lead to the improvements of the paper.

References

[1] Taras Banakh, *Topological spaces with an ω^ω-base*, Dissertationes Math. **538** (2019), 141, DOI 10.4064/dm762-4-2018. MR3942223

[2] Taras Banakh and Arkady Leiderman, *ω^ω-dominated function spaces and ω^ω-bases in free objects of topological algebra*, Topology and Appl. **241** (2018), 203–241.

[3] Zoltán T. Balogh, *On compact Hausdorff spaces of countable tightness*, Proc. Amer. Math. Soc. **105** (1989), no. 3, 755–764, DOI 10.2307/2046929. MR930252

[4] B. Cascales, J. Orihuela, and V. V. Tkachuk, *Domination by second countable spaces and Lindelöf Σ-property*, Topology Appl. **158** (2011), no. 2, 204–214, DOI 10.1016/j.topol.2010.10.014. MR2739891

[5] E.K. van Douwen, *The Integers and Topology*, Handbook of Set-Theoretic Topology, North-Holland, Amsterdam (1984), 111–167.

[6] Alan Dow, *Generalized side-conditions and Moore-Mrówka*, Topology Appl. **197** (2016), 75–101, DOI 10.1016/j.topol.2015.10.016. MR3426909

[7] Ziqin Feng, *Spaces with a $\mathbb{Q}$-diagonal*, Fund. Math. **245** (2019), no. 3, 305–320, DOI 10.4064/fm615-8-2018.

[8] Saak Gabriyelyan, Jerzy Kąkol, and A.G. Leiderman, *On topological groups with a small base and metrizability*, Fund. Math. **229** (2015), no. 2, 129–158, DOI 10.4064/fm229-2-3.

[9] Paul Gartside and Ana Mamatelashvili, *The Tukey order on compact subsets of separable metric spaces*, J. Symb. Log. **81** (2016), no. 1, 18–200, DOI 10.1017/jsl.2015.49. MR3471135

[10] ———, *Tukey order, calibres and the rationals*, Annals of Pure and Applied Logic **172** (2021), no. 1, 102873.

[11] István Juhász, Piotr Koszmider, and Lajos Soukup, *A first countable, initially ω_1-compact but non-compact space*, Topology Appl. **156** (2009), no. 10, 1863–1879, DOI 10.1016/j.topol.2009.04.004. MR2519221

[12] István Juhász and Z. Szentmiklóssy, *Convergent free sequences in compact spaces*, Proc. Amer. Math. Soc. **116** (1992), no. 4, 1153–1160, DOI 10.2307/2159502. MR1137223

[13] Piotr Koszmider, *Forcing minimal extensions of Boolean algebras*, Trans. Amer. Math. Soc. **351** (1999), no. 8, 3073–3117, DOI 10.1090/S0002-9947-99-02145-5. MR1467471

[14] Kenneth Kunen, *Set theory*, Studies in Logic and the Foundations of Mathematics, vol. 102, North-Holland Publishing Co., Amsterdam-New York, 1980. An introduction to independence proofs. MR597342

[15] A.G. Leiderman, V. Pestov, and A.H. Tomita, *On topological groups admitting a base at the identity indexed by ω^ω*, Fund. Math. **238** (2017), 79–100.

[16] Ana Mamatelashvili, *Tukey order on sets of compact subsets of topological spaces*, ProQuest LLC, An Arbor, MI, 2014. Thesis (Ph.D.)–University of Pittsburgh.

[17] David Guerrero Sánchez, *Spaces with an M-diagonal*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. Racsam **114** (2020), no. 1, 16–24, DOI 10.1007/S13398-019-00745-x. MR4039696

[18] Rongxin Shen and Ziqin Feng, *On ω^ω-bases and ω^ω-weak bases*, Houston J. Math. **46** (2020), no. 2, 507–518.

[19] John W. Tukey, *Convergence and Uniformity in Topology*, Annals of Mathematics Studies, no. 2, Princeton University Press, Princeton, N. J., 1940. MR0002515

UNC Charlotte
Email address: `adow@uncc.edu`

Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849
Email address: `zzf0006@auburn.edu`