SOME SPECIAL REGULAR OPEN SUBSETS OF \( \omega^* \)

Abstract. We prove the Main Lemma and Theorem 4.11 of [3] (i.e. Bezhanishvili and Harding) in ZFC.

For any family \( A \subset [\omega]^{\aleph_0} \), \( A^+ \) denotes the set of \( b \in [\omega]^{\aleph_0} \) that are almost disjoint from each \( a \in A \). \( A^+ \) denotes the set of \( X \subset \omega \) that are not in the ideal generated by \( A \cup A^- \). In particular, if \( A \) is an adf (almost disjoint family), then \( A^+ \) denotes the set of \( X \subset \omega \) that meet infinitely many members of \( A \) in an infinite set. If \( U \) is an open subset of \( \omega^* \) and \( A_U \) is the ideal of those infinite \( a \subset \omega \) satisfying that \( a^* \subset U \), then \( X \) being in \( A^+ \) is equivalent to \( X^* \) meeting the boundary of \( U \).

Lemma 1. For any \( A \subset [\omega]^{\aleph_0} \), \( (A^+)^+ \subset A^+ \).

Proof. Since \( A \subset (A^+)^+ \), if \( X \) is not in the ideal generated by \( A^+ \cup (A^+)^+ \), then \( X \) is not in the ideal generated by \( A^+ \cup A = A \cup A^+ \). This proves the Lemma. \( \square \)

Definition 2. A family \( A \subset [\omega]^{\aleph_0} \) is completely separable if for all \( X \in A^+ \), there is an \( a \in A \) such that \( a^* \subset X \).

Proposition 3 ([2]). There is an infinite completely separable adf.

Lemma 4. For any \( m \in \omega \), there are \( B_i \) (\( i \leq m \)) such that for all \( i \neq j \leq m \),

1. \( B_i \) is an infinite completely separable adf,
2. \( B_i \subset B_j^+ \),
3. \( B_i^+ = B_j^+ \) for \( i, j \leq m \).

Proof. Let \( A \) be a completely separable adf as in Proposition 3 and let \( \{a_\alpha : \alpha \in \mathfrak{c} \} \) be an enumeration of \( A \). It is shown in [2, 4.9] that each infinite completely separable adf has cardinality \( \mathfrak{c} \) and that \( \{a \in A : a \subset^* X \} \) has cardinality \( \mathfrak{c} \) for all \( X \in A^+ \). Let \( \{X_\xi : \xi \in \mathfrak{c} \} \) be an enumeration of \( A^+ \) so that each \( X \in A^+ \) is listed infinitely many times. By induction on \( \xi \in \mathfrak{c} \), choose \( H_\xi \in [\mathfrak{c} \setminus \bigcup_{\eta < \xi} H_\eta]^{m+1} \) so that \( a_\alpha \subset^* X_\xi \) for each \( \alpha \in H_\xi \). Choose pairwise disjoint subsets of \( \mathfrak{c} \), \( \{J_i : i < m \} \), so that \( |J_i \cap H_\xi| = 1 \) for all \( i < m \) and \( \xi < \mathfrak{c} \). For \( i < m \), set \( B_i = \{a_\alpha : \alpha \in J_i \} \) and let \( B_m = \{a_\alpha : \alpha \in \mathfrak{c} \setminus \bigcup_{i < m} J_i \} \). Clearly each \( X \in A^+ \) contains mod finite infinitely many elements of \( B_i \) for each \( i \leq m \). It thus follows that each of \( \{B_i : i < m \} \) is completely separable and that \( B_i^+ = A^+ \) for each \( i \leq m \). Since \( A \) is an adf and the family \( \{J_i : i < m \} \) are pairwise disjoint, we also have that \( B_i \subset B_j^+ \) for \( i \neq j \leq m \). \( \square \)

Definition 5. \( B \prec^+ A \) if

1. for each \( b \in B \), there is an \( a \in A \) with \( b \subset^* a \) (or \( B \prec A \)),
2. for each \( X \in A^+ \), there is an \( a \in A \) with \( X \cap a \in B^+ \).

Lemma 6. For each completely separable adf \( A \) and each \( m < \omega \), there is a family \( \{B_i : i \leq m \} \) such that, for each \( i \neq j \leq m \),

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(1) $B_i \prec^+ A$ and is an infinite completely separable adf,
(2) $B_i \subset B_j^+$,
(3) $B_i^+ = B_j^+$ and $A \subset B_i^+$,
(4) $B_m^+ = \left((B_0 \cup \cdots \cup B_{m-1})^+\right)^+$.

Proof. For each $a \in A$, choose $\{B_i(a) : i < m\}$ as in Lemma 4 so that $B_i(a) \subset [a]^{<\omega}$ for each $i \leq m$. Set $B = \bigcup\{B_i(a) : a \in A\}$. We verify each item.

(1) It is clear that $B_i \prec A$. Similarly, if $X \in A^+$, then there is an $a \in A$ such that $a \subset X$, hence it follows that $X \in B_i^+$.
(2) It is obvious that $B_i \cap B_j$ is empty.
(3) Suppose that $X \in B_i^+$. If there is an $a \in A$ such that $X \cap a \in B_i(a)^+$, then $X \in B_j^+$. Otherwise $X \in A^+$, and so there is an $a \in A$ such that $a \subset X$. Of course this ensures that $X \in B_j(a)^+$.
(4) It is immediate from (1) - (3) that $B_m \subset (B_0 \cup \cdots \cup B_{m-1})^+$ and this implies that $B_m^+ \subset \left((B_0 \cup \cdots \cup B_{m-1})^+\right)^+$. Now assume that $X \notin B_m^+$. By (3), there is a $B$ in the ideal generated by $\bigcup_{i \leq m} B_i$ such that $X \setminus B$ is in $B_i^+$ for each $i \leq m$. Therefore, $X \setminus B$ is in $(B_0 \cup \cdots \cup B_{m-1})^+$ and so $X$ is in the ideal generated by $(B_0 \cup \cdots \cup B_{m-1}) \cup (B_0 \cup \cdots \cup B_{m-1})^+$. Equivalently, $X$ is not in $(B_0 \cup \cdots \cup B_{m-1})^+$.

By Lemma 1, this completes the proof of (4).

\[\Box\]

Definition 7. If $A$ is an adf, let $\{B_i : i \leq m\} \prec_m A$ denote the relations as in Lemma 6.

Using an easy inductive process and Lemma 6 we have the following.

Corollary 8. Let $0 < m, n \in \omega$ and let $T$ be the maximum subtree of $(m+1)^{<\omega}$ satisfying that $t \in T$ is maximal if $t(k) = m$ for some (unique) $k \in \text{dom}(t)$ (of course $k$ is the maximum element of $\text{dom}(t)$). Then there is a sequence $\{B_i : t \in T\}$ satisfying

(1) $B_0 = \{\omega\}$,
(2) if $t \in T$ is not maximal, then $\{B_i : i \leq m\} \prec_m B_i$.

Following [3], for a finite tree $T \subset \omega^{<\omega}$ the topology $\tau_T$ is defined by simply saying that a set $U \subset T$ is open if for each $t \in U$, $t^+ = \{s : t \subseteq s \in T\}$ is a subset of $U$. Thus each maximal node is isolated and the closure of any node equals the set of all nodes below it.

Lemma 9. If $t$ is a maximal node of a finite tree $T$ and if $T^{-1}$ is the subtree of $T$ obtained by removing $t$, then $f : (T, \tau_T) \rightarrow (T^{-1}, \tau_{T^{-1}})$ is open and continuous (and onto) if $f(s) = s$ for $s \in T^{-1}$ and $f(t) = x$ is any maximal node of $T^{-1}$ that is above the immediate predecessor of $t$.

Proof. We first prove that $f$ is continuous. Let $U \in \tau_{T^{-1}}$ and consider any $s \in f^{-1}(U)$. We must show that $s \preceq f^{-1}(U)$. Note that $U \subset f^{-1}(U)$. Since each of $t$ and $x$ are maximal, we may assume that $s \notin \{t, x\}$. If $s$ is not below the immediate predecessor of $t$, then $s \preceq$ (in $T$) is contained in $U$ and therefore in $f^{-1}(U)$. If $s$ is below the immediate predecessor of $t$, then both $x$ and each point of $s \preceq$ (in $T^{-1}$)
are in $U$. This implies that $s^t$ (in $T$) is contained in $U$ and completes the proof that $f$ is continuous.

Now assume that $U$ is an open subset of $T$. It is immediate that $U \cap T^{-t}$ is an open subset of $T^{-t}$. Since each of $U$ and $U \cup \{x\}$ are open in $T^{-t}$, it follows that $f(U)$ is open in $T^{-t}$. \hfill \square

**Proposition 10.** If $f : X \to Y$ and $g : Y \to Z$ are open, continuous, and onto maps, then $g \circ f : X \to Z$ is also open, continuous, and onto.

**Corollary 11.** There is an open, continuous, and onto mapping from the tree topology $(T, \tau_T)$ of Corollary 8 to $m^{<\omega}$ with the subspace topology.

**Theorem 12.** For each $m, n \in \omega$, there is an open, continuous, and onto mapping from $\omega^* \mbox{ to } m^{<\omega}$ such that the preimage of every point is locally compact and has no isolated points.

**Proof.** By Lemma 9 and Proposition 10 it suffices to prove the Theorem for values of $m > 0$. Similarly, it suffices to show that for each $T$ as in Lemma 8, there is an open, continuous mapping $f$ from $\omega^*$ onto $T$ also with the stated property on point pre-images. Let $\{B_t : t \in T\}$ be the family as stated in Lemma 8. For each maximal $t \in m^{<\omega}$, let $U_t = \bigcup \{b^t : b \in B_t\}$ and set $f(U_t) = t$. For each non-maximal $t \in T$, let

$$U_{t-m} = \bigcup \{b^t : b \in (B_{t-0} \cup \cdots \cup B_{t-(m-1)})^+\} \quad \text{and} \quad f(U_{t-m}) = t^--m.$$  

Define $U_0 = \omega^*$ and for non-maximal $\emptyset \neq t \in T$, $U_t = \bigcup \{b^t : b \in B_t\}$. We set $f(U_t \setminus \bigcup \{U_s : t \subset s \in T\}) = t$. For each $b \in B_t$, $f^{-1}(t) \cap b$ is closed so it follows that $f^{-1}(t)$ is locally compact.

**Claim 1.** For each $t \in T$, $t^* = f(U_t)$ and

$$t^* \in f(X^*) \iff X \in B^m_{t^*} \quad \text{if } t \text{ is non-maximal and } X \subset \omega.$$  

The statement of the claim clearly holds for each maximal $t \in T$. We prove the claim by reverse induction on $\text{dom}(t)$. We note that, by definition, $f(U_t) \subset t^*$ for all $t \in T$. Fix any non-maximal $t \in T$. To show that $f(U_t) = t^*$, it suffices, proceeding by induction, to show that $t \in f(U_t)$. Choose any $b \in B_t$ and note that $b \in B^m_{t+i}$ for each $i \leq m$. It follows that $b^t \cap U_{t-0}$ is non-compact and disjoint from $U_{t-i}$ for all $0 < i \leq m$. Since $b^t \cap U_{t-0} \cap U_{t-i}$ is a non-empty subset of $U_t \setminus \bigcup_{i \leq m} U_{t-i}$, it is mapped to. Since $B_{t-0}$ is completely separable, it therefore follows that $t \in f(X^*)$ for each $X \in B^m_{t^*}$. Now assume that $x \in X^*$ (for some $X \subset \omega$) and that $f(x) = t$. Choose the unique $b \in B_t$ so that $x \in b^t$. Since $x \notin U_{t-m}$, we have that $X \cap b$ is not in $(\bigcup_{i \leq m} B_{t-i})^+$. Additionally, $X \cap b$ is not in the ideal generated by $\bigcup_{i \leq m} B_{t-i}$ since $x \notin \bigcup_{i \leq m} U_{t-i}$. Therefore we have, as needed, that $X \in \bigcup_{i < m} B_{t-0}^m = B^m_{t^*}$.

It follows from Claim 1 that $f$ is continuous (i.e. $f^{-1}(t^*)$ is open for each $t \in T$) and onto. We finish by proving that $f$ is open. Choose any infinite $X \subset \omega$ and let $t \in f(X^*)$. We must prove that $t^* \subset f(X^*)$. Again, we can proceed by reverse induction on $\text{dom}(t)$. By Claim 1, $X \in B^m_{t^*}$, and therefore by the assumptions of Corollary 8, $X \in B^m_{t^*}$ for all $i \leq m$. Each $B^m_{t-i}$ is completely separable, hence there are $\{b_i : i < m\} \subset [X]^\omega$ such that $b_i \in B_{t-i}$, for each $i \leq m$. Since $b^m_{m} \subset U_{t-m}$
and \( t^{-m} \) is maximal in \( T \), \( t^{-m} \in f(X^*) \). Fix any \( i < m \), and note that, by (3) of Lemma 6, \( b_i \in B_{i^{-1}}^{j^{-1}} \). Therefore, by Claim 1, \( t^{-i} \in f(X^*) \) and, by the induction hypothesis, \( (t^{-i})^\dagger \subset f(X^*) \).

Finally we prove that \( f^{-1}(t) \) has no isolated points. By Claim 1, it suffices to show that \( X^* \cap f^{-1}(t) \) is not a single point for any \( X \in B_{i^{-1}}^{j^{-1}} \). Choose any infinite \( \{b_n : n \in \omega\} \subset B_{i^{-1}}^{j^{-1}} \) such that \( X \cap b_n \) is infinite for each \( n \). Let \( Y = \bigcup\{b_{2n} : k \leq n \} \). Note that, for each \( n \in \omega \), \( X \cap Y \cap b_{2n} \) is infinite and \( (X \setminus Y) \cap b_{2n+1} \) is infinite. Therefore, \( X \cap Y \) and \( X \setminus Y \) are both in \( B_{i^{-1}}^{j^{-1}} \). Since \( (X \cap Y)^* \) and \( (X \setminus Y)^* \) are disjoint, the proof is complete. \( \square \)

References