ON THE COFINALITY OF THE SPLITTING NUMBER

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Abstract. The splitting number $s$ can be singular. The key method is to construct a forcing poset with finite support matrix iterations of ccc posets introduced by Blass and the second author [Ultrafilters with small generating sets, Israel J. Math., 65, (1989)]

1. Introduction

The cardinal invariants of the continuum discussed in this article are very well known (see [5, van Douwen, p111]) so we just give a brief reminder. They deal with the mod finite ordering of the infinite subsets of the integers. A set $S \subseteq \omega$ is unsplit by a family $\mathcal{Y} \subseteq [\omega]^{\aleph_0}$ if $S$ is mod finite contained in one member of $\{ Y, \omega \setminus Y \}$ for each $Y \in \mathcal{Y}$. The splitting number $s$ is the minimum cardinal of a family $\mathcal{Y}$ for which there is no infinite set unsplit by $\mathcal{Y}$ (equivalently every $S \in [\omega]^{\aleph_0}$ is split by some member of $\mathcal{Y}$). It is mentioned in [2] that it is currently unknown if $s$ can be a singular cardinal.

Proposition 1.1. The cofinality of the splitting number is not countable.

Proof. Assume that $\theta$ is the supremum of $\{ \kappa_n : n \in \omega \}$ and that there is no splitting family of cardinality less than $\theta$. Let $\mathcal{Y} = \{ Y_\alpha : \alpha < \theta \}$ be a family of subsets of $\omega$. Let $S_0 = \omega$ and by induction on $n$, choose an infinite subset $S_{n+1}$ of $S_n$ so that $S_{n+1}$ is not split by the family $\{ Y_\alpha : \alpha < \kappa_n \}$. If $S$ is any pseudointersection of $\{ S_n : n \in \omega \}$, then $S$ is not split by any member of $\mathcal{Y}$. □
One can easily generalize the previous result and proof to show that the cofinality of the splitting number is at least $t$. In this paper we prove the following.

**Theorem 1.2.** If $\kappa$ is any uncountable regular cardinal, then there is a $\lambda > \kappa$ with $\text{cf}(\lambda) = \kappa$ and a ccc forcing $\mathbb{P}$ satisfying that $s = \lambda$ in the forcing extension.

To prove the theorem, we construct $\mathbb{P}$ using matrix iterations.

### 2. A special splitting family

**Definition 2.1.** Let us say that a family $\{x_i : i \in I\} \subset [\omega]^{\omega}$ is $\theta$-Luzin (for an uncountable cardinal $\theta$) if for each $J \in [I]^{\theta}$, $\bigcap\{x_i : i \in J\}$ is finite and $\bigcup\{x_i : i \in J\}$ is cofinite.

Clearly a family is $\theta$-Luzin if every $\theta$-sized subfamily is $\theta$-Luzin. We leave to the reader the easy verification that for a regular uncountable cardinal $\theta$, each $\theta$-Luzin family is a splitting family. A poset being $\theta$-Luzin preserving will have the obvious meaning. For example, any poset of cardinality less than a regular cardinal $\theta$ is $\theta$-Luzin preserving.

**Lemma 2.2.** If $\theta$ is a regular uncountable cardinal then any ccc finite support iteration of $\theta$-Luzin preserving posets is again $\theta$-Luzin preserving.

*Proof.* We prove this by induction on the length of the iteration. Fix any $\theta$-Luzin family $\{x_i : i \in I\}$ and let $\langle \langle \mathbb{P}_\alpha : \alpha \leq \gamma \rangle, \langle \dot{\mathbb{Q}}_\alpha : \alpha < \gamma \rangle \rangle$ be a finite support iteration of ccc posets satisfying that $\mathbb{P}_\alpha$ forces that $\dot{\mathbb{Q}}_\alpha$ is ccc and $\theta$-Luzin preserving, for all $\alpha < \gamma$.

If $\gamma$ is a successor ordinal $\beta + 1$, then for any $\mathbb{P}_\beta$-generic filter $G_\beta$, the family $\{x_i : i \in I\}$ is a $\theta$-Luzin family in $V[G_\beta]$. By the hypothesis on $\dot{\mathbb{Q}}_\beta$, this family remains $\theta$-Luzin after further forcing by $\dot{\mathbb{Q}}_\beta$.

Now we assume that $\alpha$ is a limit. Let $\dot{J}_0$ be any $\mathbb{P}_\gamma$-name of a subset of $I$ and assume that $p \in \mathbb{P}_\gamma$ forces that $|\dot{J}_0| = \theta$. We must produce a $q < p$ that forces that $\dot{J}_0$ is as in the definition of $\theta$-Luzin. There is a set $J_1 \subset I$ of cardinality $\theta$ satisfying that, for each $i \in J_1$, there is a $p_i < p$ with $p_i \forces i \in \dot{J}_0$. The case when the cofinality of $\alpha$ not equal to $\theta$ is almost immediate. There is a $\beta < \alpha$ such that $J_2 = \{i \in J_1 : p_i \in \mathbb{P}_\beta\}$ has cardinality $\theta$. There is a $\mathbb{P}_\beta$-generic filter $G_\beta$ such that $\dot{J}_3 = \{i \in J_2 : p_i \in G_\beta\}$ has cardinality $\theta$. By the induction hypothesis, the family $\{x_i : i \in I\}$ is $\theta$-Luzin in $V[G_\beta]$. By the induction hypothesis, the family $\{x_i : i \in I\}$ is $\theta$-Luzin in $V[G_\beta]$ and we have that $\bigcap\{x_i : i \in J_3\}$ is finite and $\bigcup\{x_i : i \in J_3\}$ is co-finite. Choose any $q < p$ in $G_\beta$ and a name $\dot{J}_3$ for $J_3$ so that $q$ forces this
property for \( \dot{J}_3 \). Since \( q \) forces that \( \dot{J}_3 \subset \dot{J}_0 \), we have that \( q \) forces the same property for \( \dot{J}_0 \).

Finally we assume that \( \alpha \) has cofinality \( \theta \). Naturally we may assume that the collection \( \{ \text{dom}(p_i) : i \in J_1 \} \) forms a \( \Delta \)-system with root contained in some \( \beta < \alpha \). Again, we may choose a \( \mathbb{P}_\beta \)-generic filter \( G_\beta \) satisfying that \( J_2 = \{ i \in J_1 : p_i \upharpoonright \beta \in G_\beta \} \) has cardinality \( \theta \). In \( V[G_\beta] \), let \( \{ J_{2,\xi} : \xi \in \omega_1 \} \) be a partition of \( J_2 \) into pieces of size \( \theta \). For each \( \xi \in \omega_1 \), apply the induction hypothesis in the model \( V[G_\beta] \), and so we have that \( \bigcap \{ x_i : i \in J_{2,\xi} \} \) is finite and \( \bigcup \{ x_i : i \in J_{2,\xi} \} \) is co-finite. For each \( \xi \in \omega_1 \) let \( m_\xi \) be an integer large enough so that \( \bigcap \{ x_i : i \in J_{2,\xi} \} \subset m_\xi \) and \( \bigcup \{ x_i : i \in J_{2,\xi} \} \supset \omega \setminus m_\xi \). Let \( m \) be any integer such that \( m_\xi = m \) for uncountably many \( \xi \). Choose any condition \( \bar{p} \in \mathbb{P}_\alpha \) so that \( \bar{p} \upharpoonright \beta \in G_\beta \). We prove that for each \( n > m \) there is a \( \bar{p}_n < \bar{p} \) so that \( \bar{p}_n \models n \notin \bigcap \{ x_i : i \in I \} \) and \( \bar{p}_n \not\models n \in \bigcup \{ x_i : i \in I \} \). Choose any \( \xi \in \omega_1 \) so that \( m_\xi = m \) and \( \text{dom}(p_i) \cap \text{dom}(\bar{p}) \subset \beta \) for all \( i \in J_{2,\xi} \). Now choose any \( i_0 \in J_{2,\xi} \) so that \( n \notin x_{i_0} \). Next choose a distinct \( \xi' \) with \( m_{\xi'} = m \) so that \( \text{dom}(p_i) \cap (\text{dom}(\bar{p}) \cup \text{dom}(p_{i_0})) \subset \beta \) for all \( i \in J_{2,\xi'} \). Now choose \( i_1 \in J_{2,\xi'} \) so that \( n \in x_{i_1} \). We now have that \( \bar{p} \cup p_{i_0} \cup p_{i_1} \) is a condition that forces \( \{ i_0, i_1 \} \subset I \). □

Next we introduce a \( \sigma \)-centered poset that will render a given family non-splitting.

**Definition 2.3.** For a filter \( \mathcal{D} \) on \( \omega \), we define the Laver style poset \( \mathbb{L}(\mathcal{D}) \) to be the set of trees \( T \subset \omega^{<\omega} \) with the property that \( T \) has a minimal branching node \( \text{stem}(T) \) and for all \( \text{stem}(T) \subseteq t \in T \), the branching set \( \{ k : t \upharpoonright k \in T \} \) is an element of \( \mathcal{D} \). If \( \mathcal{D} \) is a filter base for a filter \( \mathcal{D}' \), then \( \mathbb{L}(\mathcal{D}) \) will also denote \( \mathbb{L}(\mathcal{D}') \).

The name \( \dot{L} = \{ (k, T) : (\exists t) t \upharpoonright k \subset \text{stem}(T) \} \) will be referred to as the canonical name for the real added by \( \mathbb{L}(\mathcal{D}) \).

If \( \mathcal{D} \) is a principal (fixed) ultrafilter on \( \omega \), then \( \mathbb{L}(\mathcal{D}) \) has a minimum element and so is forcing isomorphic to the trivial poset. If \( \mathcal{D} \) is principal but not an ultrafilter, then \( \mathbb{L}(\mathcal{D}) \) is isomorphic to Cohen forcing. If \( \mathcal{D} \) is a free filter, then \( \mathbb{L}(\mathcal{D}) \) adds a dominating real and has similarities to Hechler forcing. As usual, for a filter (or filter base) \( \mathcal{D} \) of subsets of \( \omega \), we use \( \mathcal{D}^+ \) to denote the set of all subsets of \( \omega \) that meet every member of \( \mathcal{D} \).

**Definition 2.4.** If \( E \) is a dense subset of \( \mathbb{L}(\mathcal{D}) \), then a function \( \rho_E \) from \( \omega^{<\omega} \) into \( \omega_1 \) is a rank function for \( E \) if \( \rho_E(t) = 0 \) if and only if \( t = \text{stem}(T) \) for some \( T \in E \), and for all \( t \in \omega^{<\omega} \) and \( 0 < \alpha \in \omega_1 \), \( \rho_E(t) \leq \alpha \) providing the set \( \{ k \in \omega : \rho_E(t \upharpoonright k) < \alpha \} \) is in \( \mathcal{D}^+ \).
When $\mathfrak{D}$ is a free filter, then $\mathbb{L}(\mathfrak{D})$ has cardinality $c$, but nevertheless, if $\mathfrak{D}$ has a base of cardinality less than a regular cardinal $\theta$, $\mathbb{L}(\mathfrak{D})$ is $\theta$-Luzin preserving.

**Lemma 2.5.** If $\mathfrak{D}$ is a free filter on $\omega$ and if $\mathfrak{D}$ has a base of cardinality less than a regular uncountable cardinal $\theta$, then $\mathbb{L}(\mathfrak{D})$ is $\theta$-Luzin preserving.

**Proof.** Let $\{x_i : i \in \theta\}$ be a $\theta$-Luzin family with $\theta$ as in the Lemma. Let $\check{J}$ be a $\mathbb{L}(\mathfrak{D})$-name of a subset of $\theta$. We prove that if $\bigcap \{x_i : i \in \check{J}\}$ is not finite, then $\check{J}$ is bounded in $\theta$. By symmetry, it will also prove that if $\bigcup \{x_i : i \in \check{J}\}$ is not cofinite, then $\check{J}$ is bounded in $\theta$. Let $\check{y}$ be the $\mathbb{L}(\mathfrak{D})$-name of the intersection, and let $T_0$ be any member of $\mathbb{L}(\mathfrak{D})$ that forces that $\check{y}$ is infinite. Let $M$ be any $<\theta$-sized elementary submodel of $H(2^\omega)^+$ such that $T_0, \mathfrak{D}, \check{J}$, and $\{x_i : i \in \theta\}$ are all members of $M$ and such that $M \cap \mathfrak{D}$ contains a base for $\mathfrak{D}$. Let $i_M = \sup(M \cap \theta)$. If $x \in M \cap [\omega]^\omega$, then $I_x = \{i \in \theta : x \in x_i\}$ is an element of $M$ and has cardinality less than $\theta$. Therefore, if $i \in \theta \setminus i_M$, then $x_i$ does not contain any infinite subset of $\omega$ that is an element of $M$. We prove that $x_i$ is forced by $T_0$ to also not contain $\check{y}$. This will prove that $\check{J}$ is bounded by $i_M$. Let $T_1 < T_0$ be any condition in $\mathbb{L}(\mathfrak{D})$ and let $t_1 = \text{stem}(T_1)$. We show that $T_1$ does not force that $x_i \supset \check{y}$. We define the relation $\models_\check{w}$ on $T_0 \times \omega$ to be the set

$$\{(t, n) \in T_0 \times \omega : \text{there is no } T \leq T_0, \text{stem}(T) = t, \text{s.t. } T \models_\check{w} n \notin \check{y}\}.$$ 

For convenience we may write, for $T \leq T_0$, $T \models_\check{w} n \in \check{y}$ providing $(\text{stem}(T), n)$ is in $\models_\check{w}$, and this is equivalent to the relation that $T$ has no stem preserving extension forcing that $n$ is not in $\check{y}$. Let $T_2 \in M$ be any extension of $T_0$ with stem $t_1$. Let $L$ denote the set of $\ell \in \omega$ such that $T_2 \models_\check{w} \ell \in \check{y}$. If $L$ is infinite, then, since $L \in M$, there is an $\ell \in L \setminus x_i$. This implies that $T_1$ does not force $x_i \supset \check{y}$, since $T_2 \models_\check{w} j \in \check{y}$ implies that $T_1$ fails to force that $\ell \notin \check{y}$.

Therefore we may assume that $L$ is finite and let $\ell$ be the maximum of $L$. Define the set $E \subset \mathbb{L}(\mathfrak{D})$ according to $T \in E$ providing that either $t_1 \notin T$ or there is a $j > \ell$ such that $T \models_\check{w} j \in \check{y}$. Again this set $E$ is in $M$ and is easily seen to be a dense subset of $\mathbb{L}(\mathfrak{D})$. By the choice of $\ell$, we note that $\rho_E(t_1) > 0$. If $\rho_E(t_1) > 1$, then the set $\{k \in \omega : 0 < \rho_E(t_1^\omega k) < \rho_E(t_1)\}$ is in $\mathfrak{D}^+$ and so there is a $k_1$ in this set such that $t_1^\omega k_1 \in T_1 \cap T_2$. By a finite induction, we can choose an extension $t_2 \supset t_1$ so that $t_2 \in T_1 \cap T_2$ and $\rho_E(t_2) = 1$. Now, there is a set $D \in \mathfrak{D} \cap M$ contained in $\{k : t_2^\omega k \in T_1 \cap T_2\}$ since $M$ contains a base for $\mathfrak{D}$. Also, $D_E = \{k \in D : \rho_E(t_2^\omega k) = 0\}$ is in $\mathfrak{D}^+$. For each $k \in D_E$, choose the minimal $j_k$ so that $T_2^\omega k \models j_k \in \check{y}$. The set
\{ j_k : k \in D_E \} \text{ is an element of } M. \text{ This set is not finite because if it were then there would be a single } j \text{ such that } \{ k \in D_E : j_k = j \} \in D^+, \text{ which would contradict that } \rho_E(t_2) > 0. \text{ This means that there is a } k \in D_E^+ \text{ with } j_k \notin x_i, \text{ and again we have shown that } T_1 \text{ fails to force that } x_i \text{ contains } \dot{y}. \quad \square

3. Matrix Iterations

The terminology “matrix iterations” is used in [3], see also forthcoming preprint (F1222) from the second author. The paper [3] nicely expands on the method of matrix iterated forcing first introduced in [1].

Let us recall that a poset \((P, <_P)\) is a complete suborder of a poset \((Q, <_Q)\) providing \(P \subset Q, <_P \subset <_Q\), and each maximal antichain of \((P, <_P)\) is also a maximal antichain of \((Q, <_Q)\). Note that it follows that incomparable members of \((P, <_P)\) are still incomparable in \((Q, <_Q)\), i.e. \(p_1 \perp_P p_2 \implies p_1 \perp_Q p_2\). We use the notation \((P, <_P) \bowtie (Q, <_Q)\) to abbreviate the complete suborder relation, and similarly use \(P <_P Q\) if \(<_P\) and \(<_Q\) are clear from the context. An element \(p\) of \(P\) is a reduction of \(q \in Q\) if \(r \not\perp_Q q\) for each \(r <_P p\). If \(P \subset Q, <_P \subset <_Q, \perp_P \subset \perp_Q\), and each element of \(Q\) has a reduction in \(P\), then \(P <_P Q\). The reason is that if \(A \subset P\) is a maximal antichain and \(p \in P\) is a reduction of \(q \in Q\), then there is an \(a \in A\) and an \(r\) less than both \(p\) and \(a\) in \(P\), such that \(r \not\perp_Q q\).

**Definition 3.1.** We will say that an object \(P\) is a matrix iteration if there is an infinite cardinal \(\kappa\) and an ordinal \(\gamma\) (thence a \((\kappa, \gamma)\)-matrix iteration) such that \(P = \langle\langle \mathbb{P}_{i,\alpha} : i \leq \kappa, \alpha \leq \gamma\rangle, \langle \dot{Q}_{i,\alpha}^P : i \leq \kappa, \alpha < \gamma\rangle\rangle\) where, for each \((i, \alpha) \in \kappa^+ \times \gamma\) and each \(j < i\),

1. \(\mathbb{P}_{j,\alpha}\) is a complete suborder of the poset \(\mathbb{P}_{i,\alpha}\) (i.e. \(\mathbb{P}_{j,\alpha} <_P \mathbb{P}_{i,\alpha}\)),
2. \(\dot{Q}_{i,\alpha}^P\) is a \(\mathbb{P}_{i,\alpha}\)-name of a ccc poset, \(\mathbb{P}_{i,A+1}^P\) is equal to \(\mathbb{P}_{i,\alpha}^P \ast \dot{Q}_{i,\alpha}^P\),
3. for limit \(\delta \leq \gamma\), \(\mathbb{P}_{i,\delta}^P\) is equal to the union of the family \(\{\mathbb{P}_{i,\beta}^P : \beta < \delta\}\)
4. \(\mathbb{P}_{j,\alpha}^P\) is the union of the chain \(\{\mathbb{P}_{j,\alpha}^P : j < \kappa\}\).

When the context makes it clear, we omit the superscript \(P\) when discussing a matrix iteration. Throughout the paper, \(\kappa\) will be a fixed uncountable regular cardinal.

**Definition 3.2.** A sequence \(\tilde{\lambda}\) is \(\kappa\)-tall if \(\tilde{\lambda} = \langle \mu_{\xi}, \lambda_{\xi} : \xi < \kappa \rangle\) is a sequence of pairs of regular cardinals satisfying that \(\mu_0 = \omega < \kappa < \lambda_0\) and, for \(0 < \eta < \kappa\), \(\mu_{\eta} < \lambda_{\eta}\) where \(\mu_{\eta} = (2^{\sup\{ \lambda_{\xi} : \xi < \eta \}})^+\).
Also for the remainder of the paper, we fix a \( \kappa \)-tall sequence \( \vec{\lambda} \) and \( \lambda \) will denote the supremum of the set \( \{ \lambda_\xi : \xi \in \kappa \} \). For simpler notation, whenever we discuss a matrix iteration \( \mathbf{P} \) we shall henceforth assume that it is a \( (\kappa, \gamma) \)-matrix iteration for some ordinal \( \gamma \). We may refer to a forcing extension by \( \mathbf{P} \) as an abbreviation for the forcing extension by \( \mathbb{P}_{\kappa,\gamma} \).

For any poset \( P \), any \( P \)-name \( \dot{D} \), and \( P \)-generic filter \( G \), \( \dot{D}[G] \) will denote the valuation of \( \dot{D} \) by \( G \). For any ground model \( x \), \( \check{x} \) denotes the canonical name so that \( \check{x}[G] = x \). When \( x \) is an ordinal (or an integer) we will suppress the accent in \( \check{x} \). A \( \mathcal{P} \)-name \( \dot{A} \) of a subset of \( \omega \) will be said to be nice or canonical if for each integer \( j \in \omega \), there is an antichain \( A_j \) such that \( \dot{A} = \bigcup \{ \{ j \} \times A_j : j \in \omega \} \). We will say that \( \dot{D} \) is a nice \( \mathcal{P} \)-name of a family of subsets of \( \omega \) just to mean that \( \dot{D} \) is a collection of nice \( \mathcal{P} \)-names of subsets of \( \omega \). We will use \( \langle \dot{D} \rangle \mathcal{P} \) if we need to emphasize that we mean the \( \mathcal{P} \)-name.

Following these conventions, the following notation will be helpful.

**Definition 3.3.** For a \( (\kappa, \gamma) \)-matrix \( \mathbf{P} \) and \( i < \kappa \), we let \( \mathbb{B}_{i,\gamma}^\mathbf{P} \) denote the set of all nice \( \mathbb{P}_{i,\gamma}^\mathbf{P} \)-names of subsets of \( \omega \). We note that this then is the nice \( \mathbb{P}_{i,\gamma}^\mathbf{P} \)-name for the power set of \( \omega \). As usual, when possible we suppress the \( \mathbf{P} \) superscript.

For a nice \( \mathcal{P} \)-name \( \dot{D} \) of a filter (or filter base) of subsets of \( \omega \), we let \( (\dot{D})^+ \) denote the set of all nice \( \mathcal{P} \)-names that are forced to meet every member of \( \dot{D} \). It follows that \( (\dot{D})^+ \) is the nice \( \mathcal{P} \)-name for the usual defined notion \( (\dot{D})^+ \) in the forcing extension by \( \mathbf{P} \). We let \( \langle \dot{D} \rangle \) denote the nice \( \mathcal{P} \)-name of the filter generated by \( \dot{D} \). We use the same notational conventions if, for some poset \( \mathbb{P} \), \( \dot{D} \) is a nice \( \mathbb{P} \)-name of a filter (or filter base) of subsets of \( \omega \).

The main idea for controlling the splitting number in the extension by \( \mathbf{P} \) will involve having many of the subposets being \( \theta \)-Luzin preserving for \( \theta \in \{ \lambda_\xi : \xi \in \kappa \} \). Motivated by the fact that posets of the form \( \mathbb{L}(\mathcal{D}) \) (our proposed iterands) are \( \theta \)-Luzin preserving when \( \mathcal{D} \) is sufficiently small we adopt the name \( \vec{\lambda} \)-thin for this next notion.

**Definition 3.4.** For a \( \kappa \)-tall sequence \( \vec{\lambda} \), we will say that a \( (\kappa, \gamma) \)-matrix-iteration \( \mathbf{P} \) is \( \vec{\lambda} \)-thin providing that for each \( \xi < \kappa \) and \( \alpha \leq \gamma \), \( \mathbb{P}_{\xi,\alpha}^\mathbf{P} \) is \( \lambda_\xi \)-Luzin preserving.
Now we combine the notion of $\bar{\lambda}$-thin matrix-iteration with Lemma 2.2. We adopt Kunen’s notation that for a set $I$, $\text{Fn}(I, 2)$ denotes the usual poset for adding Cohen reals (finite partial functions from $I$ into $2$ ordered by superset).

**Lemma 3.5.** Suppose that $P$ is a $\bar{\lambda}\text{-thin } (\kappa, \gamma)$-matrix iteration for some $\kappa$-tall sequence $\bar{\lambda}$. Further suppose that $\dot{Q}_{i,0}$ is the $P_{i,0}$-name of the poset $\text{Fn}(\lambda, 2)$ for each $\xi \in \kappa$, and therefore $P_{i,1}$ is isomorphic to $\text{Fn}(\lambda, 2)$. Let $\dot{y}$ denote the generic function from $\lambda$ onto $2$ added by $P_{i,1}$ and, for $i < \lambda$, let $\dot{x}_i$ be the canonical name of the subset of $\omega$: $\dot{y}(i + n) = 1$. Then the family $\{\dot{x}_i : i < \lambda\}$ is forced by $P$ to be a splitting family.

**Proof.** Let $G_{\kappa, \gamma}$ be a $P_{\kappa, \gamma}$-generic filter. For each $\xi \in \kappa$ and $\alpha \leq \gamma$, let $G_{\xi, \alpha} = G_{\kappa, \gamma} \cap P_{\xi, \alpha}$. Let $\dot{y}$ be any nice $P_{\kappa, \gamma}$-name for a subset of $\omega$. Since $\dot{y}$ is a countable name, we may choose a $\xi < \kappa$ so that $\dot{y}$ is a $P_{\xi, \gamma}$-name. It is easily shown, and very well-known, that the family $\{\dot{x}_i : i < \lambda\}$ is forced by $P_{\xi, 1}$ (i.e. $\text{Fn}(\lambda, 2)$) to be a $\lambda_{\xi}$-Luzin family. By the hypothesis that $P$ is $\bar{\lambda}$-thin, we have, by Lemma 2.2, that $\{\dot{x}_i : i < \lambda\}$ is still $\lambda_{\xi}$-Luzin in $V[G \cap P_{\xi, \gamma}]$. Since $\dot{y}$ is a $P_{\xi, \gamma}$-name, there is an $i < \lambda_{\xi}$ such that $\dot{y}[G_{\xi, \gamma}] \cap \dot{x}_i[G_{\xi, \gamma}]$ and $\dot{y}[G_{\xi, \gamma}] \setminus \dot{x}_i[G_{\xi, \gamma}]$ are infinite. \qed

### 4. The construction of $P$

When constructing a matrix-iteration by recursion, we will need notation and language for extension. We will use, for an ordinal $\gamma$, $P^\gamma$ to indicate that $P^\gamma$ is a $(\kappa, \gamma)$-matrix iteration.

**Definition 4.1.** (1) A matrix iteration $P^\gamma$ is an extension of $P^\delta$ providing $\delta \leq \gamma$, and, for each $\alpha \leq \delta$ and $i < \kappa$, $P_{i, \alpha} = P_{i, \alpha}^\gamma$. We can use $P^\gamma \upharpoonright \delta$ to denote the unique $(\kappa, \delta)$-matrix iteration extended by $P^\gamma$.

(2) If, for each $i < \kappa$, $\dot{Q}_{i, \gamma}$ is a $P_{i, \gamma}$-name of a ccc poset satisfying that, for each $i < j < \kappa$, $\dot{Q}_{i, \gamma} \ast \dot{Q}_{j, \gamma}$ is a complete subposet of $P_{i, \gamma} \ast P_{j, \gamma}$, then we let $P \ast (\dot{Q}_{i, \gamma} : i < \kappa)$ denote the $(\kappa, \gamma + 1)$-matrix $\langle (P_{i, \alpha} : i \leq \kappa, \alpha \leq \gamma + 1), (\dot{Q}_{i, \alpha} : i \leq \kappa, \alpha < \gamma + 1) \rangle$), where $\dot{Q}_{\kappa, \gamma}$ is the $P$-name of the union of $\{\dot{Q}_{i, \gamma} : i < \kappa\}$ and, for $i \leq \kappa$, $\dot{P}_{i, \gamma} = P_{i, \gamma}$, $\dot{P}_{i, \gamma + 1} = P_{i, \gamma} \ast \dot{Q}_{i, \gamma}$, and for $\alpha < \gamma$, $(\dot{P}_{i, \alpha}, \dot{Q}_{i, \alpha}) = (P_{i, \alpha} \upharpoonright \alpha, \dot{Q}_{i, \alpha})$.

The following, from [3, Lemma 3.10], shows that extension at limit steps is canonical.
Lemma 4.2. If $\gamma$ is a limit and if $\{P^\delta : \delta < \gamma\}$ is a sequence of matrix iterations satisfying that for $\beta < \delta < \gamma$, $P^\delta \upharpoonright \beta = P^\beta$, then there is a unique matrix iteration $P^\gamma$ such that $P^\gamma \upharpoonright \delta = P^\delta$ for all $\delta < \gamma$.

Proof. For each $\delta < \gamma$ and $i < \kappa$, we define $P^\gamma_{i,\delta}$ to be $P^\delta_{i,\delta}$ and $Q^\gamma_{i,\delta}$ to be $Q^{P^\gamma_{i,\delta}}$. It follows that $Q^\gamma_{i,\delta}$ is a $P^\delta_{i,\delta}$-name. Since $\gamma$ is a limit, the definition of $P^\gamma_{i,\gamma}$ is required to be $\bigcup\{P^\delta_{i,\delta} : \delta < \gamma\}$ for $i < \kappa$. Similarly, the definition of $P^\gamma_{\kappa,\gamma}$ is required to be $\bigcup\{P^\delta_{i,\gamma} : i < \kappa\}$. Let us note that $P^\gamma_{\kappa,\gamma}$ is also required to be the union of the chain $\bigcup\{P^\delta_{i,\gamma} : \delta < \gamma\}$, and this holds by assumption on the sequence $\{P^\delta : \delta < \gamma\}$.

To prove that $P^\gamma$ is a $(\kappa, \gamma)$-matrix it remains to prove that for $j < i \leq \kappa$, and each $q \in P^\gamma_{i,\gamma}$, there is a reduction $p$ in $P^\gamma_{j,\gamma}$. Since $\gamma$ is a limit, there is an $\alpha < \gamma$ such that $q \in P^\gamma_{i,\alpha}$ and, by assumption, there is a reduction, $p$, of $q$ in $P^\gamma_{i,\alpha}$. By induction on $\beta$ ($\alpha \leq \beta \leq \gamma$) we note that $q \in P^\gamma_{i,\beta}$ and that $p$ is a reduction of $q$ in $P^\gamma_{i,\beta}$. For limit $\beta$ it is trivial, and for successor $\beta$ it follows from condition (1) in the definition of matrix iteration.

We also will need the next result taken from [3, Lemma 13], which they describe as well known, for stepping diagonally in the array of posets.

Lemma 4.3. Let $P, Q$ be partial orders such that $P$ is a complete suborder of $Q$. Let $\dot{A}$ be a $P$-name for a forcing notion and let $\dot{B}$ be a $Q$-name for a forcing notion such that $\Vdash_Q \dot{A} \subseteq \dot{B}$, and every $P$-name of a maximal antichain of $\dot{A}$ is also forced by $Q$ to be a maximal antichain of $\dot{B}$. Then $P * A <_\mathbb{Q} Q * B$.

Let us also note if $\dot{B}$ is equal to $\dot{A}$ in Lemma 4.3, then the hypothesis and the conclusion of the Lemma are immediate. On the other hand, if $\dot{A}$ is the $P$-name of $L(\dot{D})$ for some $P$-name of a filter $\dot{D}$, then the $Q$-name of $L(\dot{D})$ is not necessarily equal to $\dot{A}$.

Lemma 4.4 ([6, 1.9]). Suppose that $P, Q$ are posets with $P <_\mathbb{Q} Q$. Suppose also that $\dot{D}_0$ is a $P$-name of a filter on $\omega$ and $\dot{D}_1$ is a $Q$-name of a filter on $\omega$. If $\Vdash_Q \dot{D}_0 \subseteq \dot{D}_1$ then $P * L(\dot{D}_0)$ is a complete subposet of $Q * L(\dot{D}_1)$ if either of the two equivalent conditions hold:

1. $\Vdash_Q ((\dot{D}_0)^+) \subseteq \dot{D}_1^+$,
2. $\Vdash_Q \dot{D}_1 \cap V^P \subseteq \langle \dot{D}_0 \rangle$ (where $V^P$ is the class of $P$-names).

Proof. Let $\dot{E}$ be any $P$-name of a maximal antichain of $L(\dot{D}_0)$. By Lemma 4.3, it suffices to show that $Q$ forces that every member of
\( \mathbb{L}(\mathcal{D}_i) \) is compatible with some member of \( \mathcal{E} \). Let \( G \) be any \( \mathbb{Q} \)-generic filter and let \( E \) denote the valuation of \( \mathcal{E} \) by \( G \cap \mathbb{P} \). Working in the model \( V[G \cap \mathbb{P}] \), we have the function \( \rho_E \) as in Lemma 2.4. Choose \( \delta \in \omega_1 \) satisfying that \( \rho_E(t) < \delta \) for all \( t \in \omega^{< \omega} \). Now, working in \( V[G] \), we consider any \( T \in \mathbb{L}(\mathcal{D}_1) \) and we find an element of \( E \) that is compatible with \( T \). In fact, by induction on \( \alpha < \delta \), one easily proves that for each \( T \in \mathbb{L}(\mathcal{D}_i) \) with \( \rho_E(\text{stem}(T)) \leq \alpha \), \( T \) is compatible with some member of \( E \).

\begin{definition}
For a \((\kappa, \gamma)\)-matrix-iteration \( \mathbb{P} \), and ordinal \( i_\gamma < \kappa \), we say that an increasing sequence \( \langle \mathcal{D}_i : i < \kappa \rangle \) is a \((\mathbb{P}, \vec{\lambda}(i_\gamma))\)-thin sequence of filter bases, if for each \( i < j < \kappa \)

1. \( \mathcal{D}_i \) is a subset of \( \mathbb{B}_{i, \gamma} \) (hence a nice \( \mathbb{P}_{i, \gamma} \)-name)
2. \( \vDash_{\mathbb{P}_{i, \gamma}} \mathcal{D}_i \) is a filter with a base of cardinality at most \( \mu_{i_\gamma} \),
3. if \( i_\gamma \leq i \), then \( \vDash_{\mathbb{P}_{j, \gamma}} \langle \mathcal{D}_j \rangle \cap \mathbb{B}_{i, \gamma} \subseteq \langle \mathcal{D}_i \rangle \).

Notice that a \((\mathbb{P}, \vec{\lambda}(i_\gamma))\)-thin sequence of filter bases can be (essentially) eventually constant. Thus we will say that a sequence \( \langle \mathcal{D}_i : i < \kappa \rangle \) (for some \( j < \kappa \)) is a \((\mathbb{P}, \vec{\lambda}(i_\gamma))\)-thin sequence of filter bases if the sequence \( \langle \mathcal{D}_i : i < \kappa \rangle \) is a \((\mathbb{P}, \vec{\lambda}(i_j))\)-thin sequence of filter bases where \( \mathcal{D}_i \) is the \( \mathbb{P}_{i, \gamma} \)-name for \( \mathbb{B}_{i, \gamma} \cap \langle \mathcal{D}_j \rangle \) for \( j < i \leq \kappa \). When \( \mathbb{P} \) is clear from the context, we will use \( \vec{\lambda}(i_\gamma) \)-thin as an abbreviation for \((\mathbb{P}, \vec{\lambda}(i_\gamma))\)-thin.

\begin{corollary}
For a \((\kappa, \gamma)\)-matrix-iteration \( \mathbb{P} \), ordinal \( i_\gamma < \kappa \), and a \((\mathbb{P}, \vec{\lambda}(i_\gamma))\)-thin sequence of filter bases \( \langle \mathcal{D}_\xi : i < \kappa \rangle \), \( \mathbb{P} \ast \langle \mathcal{Q}_{i_\gamma} : i \leq \xi \rangle \) is a \( \gamma + 1 \)-extension of \( \mathbb{P} \), where, for each \( i \leq i_\gamma \), \( \mathcal{Q}_{i_\gamma} \) is the trivial poset, and for \( i_\gamma + 1 < i < \kappa \), \( \mathcal{Q}_{i_\gamma} \) is \( \mathbb{L}(\mathcal{D}_i) \).

\end{corollary}

\begin{definition}
Whenever \( \langle \mathcal{D}_i : i < \kappa \rangle \) is a \((\mathbb{P}, \vec{\lambda}(i_\gamma))\)-thin sequence of filter bases, let \( \mathbb{P} \ast \mathbb{L}(\langle \mathcal{D}_i : i_\gamma \leq i < \kappa \rangle) \) denote the \( \gamma + 1 \)-extension described in Corollary 4.6.

This next corollary is immediate.

\end{definition}

\begin{corollary}
If \( \mathbb{P} \) is a \( \vec{\lambda} \)-thin \((\kappa, \gamma)\)-matrix and if \( \langle \mathcal{D}_i : i < \kappa \rangle \) is a \((\mathbb{P}, \vec{\lambda}(i_\gamma))\)-thin sequence of filter bases, then \( \mathbb{P} \ast \mathbb{L}(\langle \mathcal{D}_i : i_\gamma \leq i < \kappa \rangle) \) is a \( \vec{\lambda} \)-thin \((\kappa, \gamma + 1)\)-matrix.

We now describe a first approximation of the scheme, \( \mathcal{K}(\vec{\lambda}) \), of posets that we will be using to produce the model.

\begin{definition}
For an ordinal \( \gamma > 0 \) and a \((\kappa, \gamma)\)-matrix iteration \( \mathbb{P} \), we will say that \( \mathbb{P} \in \mathcal{K}(\vec{\lambda}) \) providing for each \( 0 < \alpha < \gamma \),
(1) for each $i \leq \kappa$, $P_{i,1}$ is $\text{Fn}_(\lambda_i,2)$, and
(2) there is an $i_\alpha = i_\alpha P < \kappa$ and a $(P \upharpoonright \alpha, \tilde{\lambda}(i_\alpha))$-thin sequence
$\langle \tilde{D}_i^\alpha : i < \kappa \rangle$ of filter bases, such that $P \upharpoonright \alpha + 1$ is equal to
$P \upharpoonright \alpha \ast L(\langle \tilde{D}_i^\alpha : i_\alpha \leq i < \kappa \rangle)$.
For each $0 < \alpha < \gamma$, we let $\tilde{D}_\alpha$ denote the $P \upharpoonright \alpha$-name of the union
$\bigcup \{ \tilde{D}_\alpha i : i_\alpha \leq i < \kappa \}$, and we let $L_\alpha$ denote the canonical $P \upharpoonright \alpha + 1$-name
of the subset of $\omega$ added by $L(D_\alpha \kappa)$.

Let us note that each $P \in K(\tilde{\lambda})$ is $\tilde{\lambda}$-thin. Furthermore, by Lemma
3.5, this means that each $P \in K(\tilde{\lambda})$ forces that $s \leq \lambda$. We begin a new
section for the task of proving that there is a $P \in K(\tilde{\lambda})$ that forces that
$s \geq \lambda$.

It will be important to be able to construct $(P, \tilde{\lambda}(i_\gamma))$-thin sequences
of filter bases, and it seems we will need some help.

**Definition 4.10.** For an ordinal $\gamma > 0$ and a $(\kappa, \gamma)$-matrix iteration
$P$ we will say that $P \in H(\tilde{\lambda})$ if $P$ is in $K(\tilde{\lambda})$ and for each $0 < \alpha < \gamma$,
if $i_\alpha = i_\alpha P > 0$ then $\omega_1 \leq \text{cf}(\alpha) \leq \mu_i$ and there is a $\beta_\alpha < \alpha$ such that
(1) for $\beta_\alpha \leq \xi < \alpha$ of countable cofinality, $i_\xi = 0$ and $\tilde{D}_i^\xi$ is a free
filter with a countable base that is strictly descending mod finite,
(2) if $\beta_\alpha \leq \eta < \alpha$, $i_\eta > 0$ and $\xi = \eta + \omega_1 \leq \alpha$, then $\tilde{L}_\eta \in \tilde{D}_i^\xi$, and
$P_{i_\xi, \xi} \models \tilde{D}_i^\alpha$ has a descending mod finite base of cardinality $\omega_1$,
(3) if $\beta_\alpha < \xi \leq \alpha$, $i_\xi > 0$, and $\eta + \omega_1 < \xi$ for all $\eta < \xi$, then
$\{ \tilde{L}_\eta : \beta_\alpha \leq \eta < \alpha, \text{cf}(\eta) \geq \omega_1 \}$ is a base for $\tilde{D}_i^\xi$.

## 5. Producing $\tilde{\lambda}$-thin Filter Sequences

In this section we prove this main lemma.

**Lemma 5.1.** Suppose that $P^\gamma \in H(\tilde{\lambda})$ and that $\mathcal{Y}$ is a set of fewer
than $\lambda$ nice $P^\gamma$-names of subsets of $\omega$, then there is a $\delta < \gamma + \lambda$ and
an extension $P^\delta$ of $P^\gamma$ in $H(\tilde{\lambda})$ that forces that the family $\mathcal{Y}$ is not a
splitting family.

The main theorem follows easily.

**Proof of Theorem 1.2.** Let $\theta$ be any regular cardinal so that $\theta^{< \lambda} = \theta$
(for example, $\theta = (2^{\lambda})^+$). Construct $P^\theta \in H(\tilde{\lambda})$ so that for all $\mathcal{Y} \subset \mathbb{B}_{\kappa, \theta}$
with $|\mathcal{Y}| < \lambda$, there is a $\gamma < \delta < \theta$ so that $\mathcal{Y} \subset \mathbb{B}_{\kappa, \gamma}$ and, by applying
Lemma 5.1, such that $P^\theta \upharpoonright \delta$ forces that $\mathcal{Y}$ is not a splitting family. □

We begin by reducing our job to simply finding a $(P, \tilde{\lambda}(i_\gamma))$-thin sequence.
For the remainder of the paper, we always assume that
when discussing \( p \in \mathbb{P}^\gamma \), that for each \( \xi \in \text{dom}(p) \) there is a \( t^{\rho}_\xi \in \omega^{<\omega} \) such that \( p \mid \xi \vdash t^{\rho}_\xi = \text{stem}(p(\xi)) \).

**Definition 5.2.** For a \((\kappa, \gamma)\)-matrix-iteration \( \mathbb{P}^\gamma \), we say that a subset \( \mathcal{E} \) of \( \mathbb{B}_{\kappa, \gamma} \) is a \((\mathbb{P}^\gamma, \check{\lambda}(i_\gamma))\)-thin filter subbase if, for each \( i_\gamma < \kappa \), \( |\mathcal{E}| \leq \mu_{i_\gamma} \), and the sequence \( \langle \langle \mathcal{E} \cap \mathbb{B}_{i, \gamma} : i < \kappa \rangle \rangle \) is a \((\mathbb{P}^\gamma, \check{\lambda}(i_\gamma))\)-thin sequence of filter bases.

**Lemma 5.3.** For any \( \mathbb{P}^\gamma \in \mathcal{H}(\check{\lambda}) \), and any \((\mathbb{P}^\gamma, \check{\lambda}(i_\gamma))\)-thin filter base \( \mathcal{E} \), there is an \( \alpha \leq \gamma + \mu_{i_\gamma} \) and extensions \( \mathbb{P}^\alpha, \mathbb{P}^{\alpha+1} \) of \( \mathbb{P}^\gamma \) in \( \mathcal{H}(\check{\lambda}) \), such that, \( \mathbb{P}^{\alpha+1} = \mathbb{P}^\alpha \ast \mathbb{L}(\langle \mathbb{D}^\alpha_i : i_\alpha \leq i < \kappa \rangle) \) and \( \mathbb{P}^\alpha \) forces that \( \mathcal{E} \cap \mathbb{B}_{i, \gamma} \) is a subset of \( \check{\alpha}_i \) for all \( i < \kappa \).

**Proof.** The case \( i_\gamma = 0 \) is trivial, so we assume \( i_\gamma > 0 \). There is no loss of generality to assume that \( \mathcal{E} \cap \mathbb{B}_{i_\gamma, \gamma} \) has character \( \mu_{i_\gamma} \). Let \( \{ \dot{E}_\xi : \xi < \mu_{i_\gamma} \} \subset \mathcal{E} \cap \mathbb{B}_{i_\gamma, \gamma} \) enumerate a filter base for \( \langle \mathcal{E} \cap \mathbb{B}_{i_\gamma, \gamma} \rangle \). We can assume that this enumeration satisfies that \( \dot{E}_\xi \setminus \dot{E}_{\xi+1} \) is forced to be infinite for all \( \xi < \mu_{i_\gamma} \). Let \( \mathcal{A} \) be any countably generated free filter on \( \omega \) that is not principal mod finite. By induction on \( \xi < \mu_{i_\gamma} \), we define \( \mathbb{P}^{\gamma+\xi} \) by simply defining \( i_{\gamma+\xi} \) and the sequence \( \langle \dot{D}^\gamma_i : i_{\gamma+\xi} \leq i < \kappa \rangle \). We will also recursively define, for each \( \xi < \mu_{i_\gamma} \), a \( \mathbb{P}^{\gamma+\xi} \)-name \( \dot{D}_\xi \) such that \( \mathbb{P}^{\gamma+\xi} \) forces that \( \dot{D}_\xi \subset \dot{E}_\xi \). An important induction hypothesis is that \( \{ \dot{D}_\eta : \eta < \xi \} \cup \{ \dot{E}_\xi : \xi < \mu_{i_\gamma} \} \cup \mathcal{E} \) is forced to have the finite intersection property.

For each \( \xi < \gamma + \omega_1 \), let \( i_\xi = 0 \) and \( \check{\mathbb{D}}^\xi_i \) be the \( \mathbb{P}^\xi \)-name \( \langle \mathcal{A} \rangle \cap \mathbb{B}_{i, \xi} \) for all \( i \leq \kappa \). The definition of \( \dot{D}_0 \) is simply \( \dot{E}_0 \). By recursion, for each \( \eta < \omega_1 \) and \( \xi = \eta + 1 \), we define \( \dot{D}_\xi \) to be the intersection of \( \dot{D}_\eta \) and \( \dot{E}_\xi \). For limit \( \xi < \omega_1 \), we note that \( \mathbb{P}_{i, \xi} \) forces that \( \mathbb{L}(\langle \mathcal{A} \rangle) \) is isomorphic to \( \mathbb{L}(\langle \dot{D}_\eta \cap \dot{E}_\xi : \eta < \xi \rangle) \). Therefore, we can let \( \dot{D}_\xi \) be a \( \mathbb{P}^{\xi+1} \)-name for the generic real added by \( \mathbb{L}(\langle \dot{D}_\eta \cap \dot{E}_\xi : \eta < \xi \rangle) \). A routine density argument shows that this definition satisfies the induction hypothesis.

The definition of \( i_{\gamma+\omega_1} \) is \( i_\gamma \) and the definition of \( \check{\mathbb{D}}^\gamma_{\gamma+\omega_1} \) is the filter generated by \( \{ \dot{D}_\xi : \xi < \omega_1 \} \). The definition of \( \dot{D}_{\omega_1} \) is \( \dot{L}_{\gamma+\omega_1} \).

Let \( S \) denote the set of \( \eta < \mu_{i_\gamma} \) with uncountable cofinality. We now add additional induction hypotheses:

1. if \( \zeta = \sup(S \cap \xi) < \xi \) and \( \xi = \nu + 1 \), then \( \dot{D}_\xi = \dot{D}_\nu \cap \dot{E}_\xi \), and \( i_\xi = 0 \) and \( \check{\mathbb{D}}^{\gamma+\xi}_i = \langle \mathcal{A} \rangle \) for all \( i \leq \kappa \)
2. if \( \zeta = \sup(S \cap \xi) < \xi \) and \( \xi \) is a limit of countable cofinality, then \( i_\xi = 0 \) and \( \check{\mathbb{D}}^{\gamma+\xi}_i = \langle \mathcal{A} \rangle \) for all \( i \leq \kappa \), and \( \dot{D}_\xi \) is forced by \( \mathbb{P}^{\gamma+\xi+1} \) to be the generic real added by \( \mathbb{L}(\langle \dot{D}_\eta \cap \dot{E}_\xi : \zeta \leq \eta < \xi \rangle) \),
(3) if $\zeta = \text{sup}(S \cap \xi)$ and $\xi = \zeta + \omega_1$, then $i_\xi = i_\gamma$, $\mathcal{D}_i^{\gamma+\xi}$ is the filter generated by $\{\dot{E}_\xi \cap \dot{D}_\eta : \zeta \leq \eta < \xi\}$ and $\dot{D}_\xi$ is $\dot{L}_{\gamma+\xi}$.

(4) if $S \cap \xi$ is cofinal in $\xi$ and $\text{cf}(\xi) > \omega$, then $i_\xi = i_\gamma$ and $\mathcal{D}_i^{\gamma+\xi}$ is the filter generated by $\{\dot{D}_{\gamma+\eta} : \eta \in S \cap \xi\}$ and $\dot{D}_\xi = \dot{L}_{\gamma+\xi}$.

(5) if $S \cap \xi$ is cofinal in $\xi$ and $\text{cf}(\xi) = \omega$, then $i_\xi = 0$ and $\mathcal{D}_i^{\gamma+\xi} = \langle A \rangle$ for all $i \leq \kappa$, and $\dot{D}_\xi$ is forced by $\mathcal{P}_i^{\gamma+\xi+1}$ to be the generic real added by $L(\{\dot{D}_{\eta_n} \cap \dot{E}_\xi : n \in \omega\})$, where $\{\eta_n : n \in \omega\}$ is some increasing cofinal subset of $S \cap (\gamma, \xi)$.

It should be clear that the induction continues to stage $\mu_i$, and that $\mathcal{P}_i^{\gamma+\xi} \in H(\lambda(i_\gamma))$ for all $\xi \leq \mu_i$, with $\beta_i = \gamma$ being the witness to Definition 4.10 for all $\xi$ with $\text{cf}(\xi) > \omega$.

The final definition of the sequence $(\mathcal{D}_i^\delta : i_\delta = i_\gamma \leq i \leq \kappa)$, where $\delta = \gamma + \mu_i$, is that $\mathcal{D}_i^\delta$ is the filter generated by $\{\dot{L}_{\gamma+\xi} : \text{cf}(\xi) > \omega\}$, and for $i_\gamma < i \leq \kappa$, $\mathcal{D}_i^\delta$ is the filter generated by $\mathcal{D}_i^\delta \cup (\mathcal{E} \cap \mathcal{B}_{i,\gamma})$. \hfill \square

**Lemma 5.4.** Suppose that $\mathcal{E}$ is a $(\mathcal{P}_i^\gamma, \lambda(i_\gamma))$-thin filter base. Also assume that $i < \kappa$ and $\alpha \leq \gamma$ and $\mathcal{E}_1 \subset \mathcal{B}_{i,\alpha}$ is a $(\mathcal{P}_i^\alpha, \lambda(i_\gamma))$-thin filter base satisfying that $\langle \mathcal{E} \rangle \cap \mathcal{B}_{i,\alpha} \subset \langle \mathcal{E}_1 \rangle$, then there is a $(\mathcal{P}_i^\gamma, \lambda(i_\gamma))$-thin filter base $\mathcal{E}_2$ such that $\mathcal{E} \cup \mathcal{E}_1 \subset \mathcal{E}_2 \subset \langle \mathcal{E} \cup \mathcal{E}_1 \rangle$.

**Proof.** The first claim is that if $\alpha = \gamma$, then $\mathcal{E}_2$ simply equalling $\mathcal{E} \cup \mathcal{E}_1$ will work. To see this, assume that $i_\gamma \leq j_1 < j_2$ and that for some $p$ and $b \in \mathcal{B}_{j_1,\gamma}$ some $p \models \neg (b \cap (\dot{E} \cap \dot{E}_1)) = 0$ for a pair $\dot{E}, \dot{E}_1 \in \mathcal{B}_{j_2,\gamma}$ with $\dot{E} \in \mathcal{E}$ and $\dot{E}_1 \in \mathcal{E}_1$. If $i \leq j_1$, then $b \cap \dot{E}_1 \in \mathcal{B}_{j_1,\gamma}$. So just use that $\mathcal{E}$ is thin. For $j_1 \leq i$, we proceed by induction on $j_2$. If $j_1 \leq i < j_2$, then $p \models \neg (b \cap \dot{E}_1) \cap \dot{E} = 0$, so again, there is $\dot{E}_2 \in \mathcal{E} \cap \mathcal{B}_{i,\gamma}$ such that $p \models \neg (b \cap \dot{E}_1) \cap \dot{E}_2 = b \cap (\dot{E}_1 \cap \dot{E}_2)$ is empty. Then, by the induction hypothesis, there is an $\dot{E}_3 \in \langle \mathcal{E} \cup \mathcal{E}_1 \rangle \cap \mathcal{B}_{j_1,\gamma}$ such that $p \models \neg b \cap \dot{E}_3$ is empty. Finally, if $j_2 \leq i$, then $\dot{E} \cap \dot{E}_1 \in \langle \mathcal{E}_1 \rangle$, so there is $\dot{E}_3 \in \langle \mathcal{E}_1 \rangle \cap \mathcal{B}_{j_1,\gamma}$ with $p \models \neg b \cap \dot{E}_3 = 0$.

Choose any $\omega$-closed elementary submodel $M$ of $H(2^{\lambda \gamma+})$ containing $\{\mathcal{E}, \mathcal{P}_i^\gamma\}$. We may assume that $\mathcal{E}$ contains all $\dot{y} \in M \cap \mathcal{B}_{i,\gamma}$ such that $1 \models \dot{y} \in \langle \mathcal{E} \cap \mathcal{B}_{i,\gamma} \rangle$. Now we show that $\mathcal{E}$ has the following closure property: if $\dot{E}_0 \in \mathcal{E} \cap \mathcal{B}_{i,\beta}$ and $p \in \mathcal{P}_{i,\gamma}$, there is a $\dot{E}_2 \in \mathcal{B}_{j,\beta}$ such that $p \models \dot{E}_2 = \dot{E}_0$ and $r \models \dot{E}_2 = \omega$ for all $r \perp p$. For each $\ell \in \omega$, choose a maximal antichain $A_\ell \subset M \cap \mathcal{P}_{i,\beta}$ such that for each $q \in A_\ell$

1. either $q \models \ell \in \dot{E}_0$ or $q \models \neg \ell \notin \dot{E}_0$,
2. either $q \perp p$ or every extension of $q$ in $\mathcal{P}_{i,\gamma} \cap M$ is compatible with $p$ (i.e. $q$ is an $M \cap \mathcal{P}_{i,\gamma} \cap M$-reduct of $p$).
We define $\hat{E}_2$ to be the set of all pairs $(\ell, q)$ with $q \in A_\ell \cap p^\perp$ or with $q \Vdash \ell \in \hat{E}_0$. That is, the only pairs $(\ell, q)$ from $\{\ell\} \times A_\ell$ are those $q$ that are compatible with $p$ and force that $q$ is not in $\hat{E}_0$. It is immediate that $1 \Vdash \hat{E}_2 \supset \hat{E}_0$. It should be clear that if $r \perp p$, then $r \Vdash \hat{E}_2 = \omega$. Similarly if $r < p$ and $r < q$ for some $q \in A_\ell$, then $q$ is compatible with $p$ and so $q \Vdash \ell \in \hat{E}_0$.

We may similarly assume that $E_1$ has this same closure property. We let $q <_j p$ denote the relation that $q \in \mathbb{P}_{j, \gamma}$ and every extension of $q$ in $\mathbb{P}_{j, \gamma}$ is compatible with $p$ (i.e. $q$ is a $\mathbb{P}_{j, \gamma}$-reduct of $p$). For any $\dot{y} \in \mathbb{B}_{i, \gamma}$ and $j < i$, let $\dot{y}_{j, \gamma}$ be any nice $\mathbb{P}_{j, \gamma}$-name that is forced to be equal to $\{(\ell, q) : (\exists (\ell, q_e) \in \dot{y}) \ q <_j p\}$.

By the $\alpha = \gamma$ case, it is sufficient to prove that

$\mathcal{E}_2 = \{(\hat{E}_0 \cap \hat{E}_1)_{\mathbb{P}_{j, \gamma}} : \hat{E}_0 \in \mathcal{E} \cap \mathbb{B}_{i, \gamma}, \hat{E}_1 \in \mathcal{E}_1\}$

is $(\mathbb{P}_{\gamma}, (\lambda_{i, \gamma}))$-thin. It is clear that $|\mathcal{E}_2| \leq \mu_{i, \gamma}$. So now suppose that $p \in \mathbb{P}_{i, \gamma}$, $\dot{b} \in \mathbb{B}_{j, \gamma}$ and that $p \Vdash \dot{b} \cap (\hat{E}_0 \cap \hat{E}_1) = \emptyset$ for some $\hat{E}_0 \in \mathcal{E} \cap \mathbb{B}_{i, \gamma}$ and $\hat{E}_1 \in \mathcal{E}_1$. It suffices to produce $\hat{E}_2 \in \mathcal{E}$ and $\hat{E}_3 \in \mathcal{E}_1$ so that $p \Vdash \dot{b} \cap (\hat{E}_0 \cap \hat{E}_1)_{\mathbb{P}_{j, \gamma}} = \emptyset$.

Choose $\hat{E}_2 \in \mathcal{E} \cap \mathbb{B}_{i, \gamma}$ so that $p \Vdash \hat{E}_2 = \hat{E}_0$ and each $r \perp p$ forces that $E_2 = \omega$. Similarly choose $\hat{E}_3 \in \mathcal{E}_1$ so that $p \Vdash \hat{E}_3 = \hat{E}_1$ and each $r \perp p$ forces that $E_3 = \omega$. Suppose that $q <_j p$ and suppose that $q \Vdash \ell \in \hat{b}$. Since $p \Vdash \ell \notin \hat{E}_2 \cap \hat{E}_3$, it follows that $q \perp p$. Therefore, if $q <_j p$ and $q \Vdash \dot{b} \cap (\hat{E}_2 \cap \hat{E}_3)_{\mathbb{P}_{j, \gamma}}$ is empty. This in turn implies that $p$ forces that $\dot{b}$ is disjoint from $\hat{E}_2 \cap \hat{E}_3$.

Let $\mathbb{P}^\gamma \in \mathcal{H}(\bar{\mathcal{Y}})$ and let $\dot{y} \in \mathbb{B}_{\kappa, \gamma}$. For a family $\mathcal{E} \subset \mathbb{B}_{\kappa, \gamma}$ and condition $p \in \mathbb{P}^\gamma$ say that $p$ forces that $\mathcal{E}$ measures $\dot{y}$ if $p \Vdash_{\mathbb{P}^\gamma} \{\dot{y}, \omega \setminus \{\dot{y}\} \cap (\mathcal{E}) \neq \emptyset\}$. Naturally we will just say that $\mathcal{E}$ measures $\dot{y}$ if $1$ forces that $\mathcal{E}$ measures $\dot{y}$.

Given Lemma 5.3, it will now suffice to prove.

**Lemma 5.5.** If $\mathcal{Y} \subset \mathbb{B}_{\kappa, \gamma}$ for some $\mathbb{P}^\gamma \in \mathcal{H}(\bar{\mathcal{X}})$ and $|\mathcal{Y}| \leq \mu_{i, \gamma}$ for some $i, j < \kappa$, then there is a $(\mathbb{P}^\gamma, \vec{\lambda}(i, \gamma))$-thin filter $\mathcal{E} \subset \mathbb{B}_{\kappa, \gamma}$ that measures every element of $\mathcal{Y}$.

In fact, to prove Lemma 5.5, it is evidently sufficient to prove:

**Lemma 5.6.** If $\mathbb{P}^\gamma \in \mathcal{H}(\bar{\mathcal{X}})$, $\dot{y} \in \mathbb{B}_{\kappa, \gamma}$, and if $\mathcal{E}$ is a $(\mathbb{P}^\gamma, \vec{\lambda}(i, \gamma))$-thin filter, then there is a family $\mathcal{E}_1 \supset \mathcal{E}$ measuring $\dot{y}$ that is also a $(\mathbb{P}^\gamma, \vec{\lambda}(i, \gamma))$-thin filter.

**Proof.** Throughout the proof we suppress mention of $\mathbb{P}^\gamma$ and refer instead to component member posets $\mathbb{P}_{i, \alpha}, \hat{Q}_{i, \alpha}$ of $\mathbb{P}^\gamma$. We proceed by
induction on the lexicographic ordering on $\kappa \times \gamma$. That, we assume that $i < \kappa$ is minimal such that the lemma fails for some $\dot{y} \in \mathcal{B}_{i,\gamma}$, and we also assume that $\alpha \leq \gamma$ is minimal such that the lemma fails for some $\dot{y} \in \mathcal{B}_{i,\alpha}$. Fix a well-ordering $\sqsubset$ of $H((2^{\lambda})^+)$ and also assume that $\dot{y}$ is the $\sqsubset$-minimal element of $\mathcal{B}_{i,\alpha}$ for which the lemma fails. Also assume that, for every $j < i$, and every element of $\mathcal{B}_{i,\beta} \cup \mathcal{B}_{j,\alpha}$ is $\sqsubset$-below $\dot{y}$. We can free up the variables $i, \alpha$ by using $i_{\dot{y}}$ and $\alpha_{\dot{y}}$ instead. By the minimality of $\alpha_{\dot{y}}$, it is immediate that $\alpha_{\dot{y}}$ has countable cofinality.

Let $\theta = (2^{\lambda})^+$ and let $\mathcal{M}$ denote the set of elementary submodels $M$ of $H(\theta^+)$ that contain $\mu_{i_{\dot{y}}}, \mathcal{P}^\gamma, \mathcal{E}, \dot{y}$ and so that $M$ has cardinality equal to $\mu_{i_{\dot{y}}}$ and, by our cardinal assumptions, $M^{\lambda_j} \subset M$ for all $j < i_{\dot{y}}$. Naturally this also implies that $M^{\omega} \subset M$. Choose any $M_0 \in \mathcal{M}$ and assume (as in the proof of Lemma 5.4), that $\langle \mathcal{E} \rangle \cap M_0 \cap \mathcal{B}_{i_{\dot{y}},\alpha_{\dot{y}}}$ is a subset of $\mathcal{E}$. By Lemma 5.4, it suffices to prove that the lemma holds $\mathcal{P}^\alpha_{\dot{y}}$. Thus, we may henceforth assume that $\gamma = \alpha_{\dot{y}}$.

**Fact 1.** $1 < \gamma$ and $i_{\gamma} < i_{\dot{y}}$.

**Proof of Fact 1.** The fact that $1 < \gamma$ follows from the fact that $\mathcal{P}^1$ is simply Cohen forcing. That is, it is well-known that $\langle \dot{b} \rangle \cap \mathcal{B}_{j,1}$ is countably generated for all $j < \kappa$. This implies that $\langle \dot{b} \cup (\mathcal{E} \cap \mathcal{B}_{\kappa,1}) \rangle$ is $(\mathcal{P}^1, \mathcal{E}(i_{\gamma}))$-thin for all $\dot{b} \in \mathcal{B}_{\kappa,1}$. Then, by Lemma 5.4, $\langle \dot{b} \cup \mathcal{E} \rangle$ can be extended to a $(\mathcal{P}^1, \mathcal{E}(i_{\gamma}))$-thin filter. Similarly, if $i_{\dot{y}} \leq i_{\gamma}$, then $\langle \dot{y} \rangle \cup (\mathcal{B}_{i_{\dot{y}},\gamma} \cap \mathcal{E})$ is a $(\mathcal{P}^\gamma, \mathcal{E}(i_{\gamma}))$-thin filter. Therefore, by Lemma 5.4, this contradicts that the lemma fails for $\dot{y}$. \[ \square \]

Working in $M_0$ use the well-ordering $\sqsubset$, to perform a transfinite recursion to choose a $(\mathcal{P}^\gamma, \mathcal{E}(i_{\gamma}))$-thin $\mathcal{E}_0 \subset \mathcal{B}_{i_{\dot{y}},\gamma}$ that extends $\mathcal{E} \cap \mathcal{B}_{i_{\dot{y}},\gamma}$. The induction chooses the $\sqsubset$-least $(\mathcal{P}^\gamma, \mathcal{E}(i_{\gamma}))$-thin filter in $M_0$ (which will be in $H(\theta)$) that extends the union of the recursively chosen sequence and also measures the $\sqsubset$-least member of $\mathcal{B}_{i_{\dot{y}},\gamma}$ that is $\sqsubset$-below $\dot{y}$ and is not yet measured. Suppose that $\dot{x} \in \mathcal{B}_{i_{\dot{y}},\gamma}$ is the $\sqsubset$-least that is not measured by $\mathcal{E}_0$. Since $\mathcal{E}_0$ is definable from $\dot{x}$ and $\sqsubset$, it follows that $\mathcal{E}_0 \in M_0$. Since the recursion stopped, it follows that $\dot{x} = \dot{y}$. Therefore $\mathcal{E}_0 \supset (\mathcal{E} \cap \mathcal{B}_{i_{\dot{y}}})$ is $(\mathcal{P}^\gamma, \mathcal{E}(i_{\gamma}))$-thin and measures every element of $M_0 \cap \mathcal{B}_{i_{\dot{y}},\gamma}$ that is $\sqsubset$-below $\dot{y}$.

Let $A_1(M_0, \mathcal{E}_0)$ be the set of all $p \in M_0 \cap \mathcal{P}_{i_{\dot{y}},\gamma}$ that force that $\dot{y}$ measures $\dot{y}$. We may similarly choose $\{M_0, \mathcal{E}_0\} \in M_1 \in \mathcal{M}$ and select $\mathcal{E}_1 \supset \mathcal{E}_0$ just as we did $\mathcal{E}_0$. Similarly, let $A_1(M_1, \mathcal{E}_1)$ be the set of all $p \in M_1 \cap \mathcal{P}_{i_{\dot{y}},\gamma}$ that force that $\dot{y}$ measures $\dot{y}$. Note that $A_1(M_0, \mathcal{E}_0) \subset A_1(M_1, \mathcal{E}_1)$. If $M_1$ can be chosen so that $A_1(M_0, \mathcal{E}_0)$ is not pre-dense
in $A_1(M_1, \mathcal{E}_1)$, then we make such a choice. Suppose that $\rho < \omega_1$ and we have recursively chosen a sequence $\{M_\xi, \mathcal{E}_\xi : \xi < \rho\}$ so that for $\xi < \rho$, $\bigcup\{\mathcal{E}_\eta : \eta < \xi\} \subset \mathcal{E}_\xi \subset M_\xi$, $\{\bigcup\{M_\eta : \eta < \xi\}, \bigcup\{\mathcal{E}_\eta : \eta < \xi\}\} \in M_\xi$, and so that $\mathcal{E}_\xi$ is $(\mathbb{P}_\gamma, \bar{\lambda}(i_\gamma))$-thin and measures every element of $M_\xi \cap \mathbb{B}_{\xi, \gamma}$ that is $\square$-below $\dot{\gamma}$. Suppose further that for each $\xi + 1 < \rho$, $A_1(M_\xi, \mathcal{E}_\xi)$ is not pre-dense in $A_1(M_{\xi+1}, \mathcal{E}_{\xi+1})$. If $\rho$ is a limit, then $\bigcup\{\mathcal{E}_\xi : \xi < \rho\}$ is a $(\mathbb{P}_\gamma, \bar{\lambda}(i_\gamma))$-thin filter base and the properties of $\mathbb{M}$ ensures that there is a suitable $M_\rho \in \mathbb{M}$, and the family $\bigcup\{\mathcal{E}_\xi : \xi < \rho\}$ can be suitably extended to $\mathcal{E}_\rho$ just as $\mathcal{E}_\rho$ was chosen to extend $\mathcal{E}$. If $\rho = \xi + 1$ is a successor, then we extend $(M_\xi, \mathcal{E}_\xi)$ to $(M_\rho, \mathcal{E}_\rho)$ as we did when choosing $(M_1, \mathcal{E}_1)$ to extend $(M_0, \mathcal{E}_0)$, but only if there is such an extension with $A_1(M_\xi, \mathcal{E}_\xi)$ not being pre-dense in $A_1(M_\rho, \mathcal{E}_\rho)$.

Since $\mathbb{P}_{i_\gamma, \gamma}$ is ccc, there is some $\rho + 1 < \omega_1$ when this recursion must stop and for the reason that $A_1(M_\rho, \mathcal{E}_\rho)$ cannot be made larger. Now we work with such an $(M_\rho, \mathcal{E}_\rho)$. Let $A_1 \subset A_1(M_\rho, \mathcal{E}_\rho)$ be an antichain that is pre-dense in $A_1(M_\rho, \mathcal{E}_\rho)$.

We work in the poset $\mathbb{P}_{i_\gamma, \gamma}$. We can replace $\dot{\gamma}$ by any $\dot{x} \in \mathbb{B}_{i_\gamma, \gamma}$ that has the property that $1 \models \dot{x} \in \{\dot{\gamma}, \omega \setminus \dot{\gamma}\}$ since if we measure $\dot{x}$ then we also measure $\dot{\gamma}$. With this reduction then we can assume that no condition forces that $\omega \setminus \dot{\gamma}$ is in the filter generated by $\mathcal{E}$.

**Fact 2.** There is a maximal antichain $A \subset M_\rho \cap \mathbb{P}_{i_\gamma, \gamma}$ extending $A_1$ such that for each $p \in A \setminus A_1$, there is an $i_p < i_\gamma$ and an $\dot{E}_p \in \mathcal{E}_\rho$ such that

1. there is a $b_1 \in \mathbb{B}_{i_\gamma, \gamma} \cap \mathcal{E}_\rho^+$, such that $p \models b_1 \cap \dot{E}_p \cap \dot{\gamma} = \emptyset$, and
2. there is $b_2 \in \mathbb{B}_{i_\gamma, \gamma} \cap \mathcal{E}_\rho^+$ such that $p \models b_2 \cap \dot{E}_p \cap (\omega \setminus \dot{\gamma}) = \emptyset$.

**Proof of Fact 2.** Suppose that $p_1 \in \mathbb{P}_{i_\gamma, \gamma} \cap A_1^\perp$ has no extension $p$ with a suitable pair $i_p, b_1, \dot{E}_1$ as in (1). Define $\dot{E} \in \mathbb{B}_{i_\gamma, \gamma}$ so that $p_1$ forces $\dot{E} = \dot{\gamma}$ and each $q \in \mathbb{P}_{i_\gamma, \gamma} \cap p_1^\perp$ forces that $\dot{E} = \omega$. It is easily checked that $\mathbb{B}_{i_\gamma, \gamma} \cap (\mathcal{E}_\rho \cup \{\dot{E}\})$ is then $(\mathbb{P}_\gamma, \bar{\lambda}(i_\gamma))$-thin and that $p_1$ forces that it measures $\dot{\gamma}$. Therefore we can choose $p_2 < p_1$ so that there are $i_{p_2}, b_1, \dot{E}_{p_2}$ as in (1). Similarly, $p_2$ has an extension $p$ so that there are $i_p, b_2, \dot{E}_p$ as in (2). There is no loss to assuming that $i_p \geq i_{p_2}$ and $p \models \dot{E}_p \subset \dot{E}_{p_2}$. Then $i_p, b_1, \dot{E}_p$ also satisfy (1) for $p$. \qed

Now we choose any $p \in A \setminus A_1$. It suffices to produce an $\dot{E}_p \in \mathbb{B}_{i_\gamma, \gamma}$ that can be added to $\mathcal{E}_\rho$ that measures $\dot{\gamma}$ and satisfies that $q \models \dot{E}_p = \omega$ for all $q \in p^\perp$. This is because we then have that $\mathcal{E}_1 \cup \{\dot{E}_p : p \in A \setminus A_1\}$ is contained in a $\bar{\lambda}(i_\gamma)$-thin filter that measures $\dot{\gamma}$. By symmetry, we may assume that $i_p \leq j_p$. 


Fact 3. There is an $\alpha$ such that $\gamma = \alpha + 1$.

Proof of Fact 3. Otherwise, let $j = i_p$ and for each $r < p$ in $\mathbb{P}_{i_\beta,\gamma}$, choose $\beta \in M_\rho \cap \gamma$ such that $r \in \mathbb{P}_{i_\beta,\gamma}$, and define a name $\check{y}[r]$ in $M_\rho \cap \mathbb{B}_{j,\gamma}$ according to $(\ell, q) \in \check{y}[r]$ providing there is a pair $(\ell, p_\ell) \in \check{y}$ such that $q <_{P_{\ell}} p_\ell$ and $q \upharpoonright \beta$ is in the set $M_\rho \cap \mathbb{P}_{j,\beta} \setminus (r \upharpoonright P_{\ell} \upharpoonright \beta)^+$. This set, namely $\check{y}[r]$, is in $M_\rho$ because $\mathbb{P}_{j,\beta}$ is ccc and $M_{\rho}^\omega \subseteq M_\rho$.

We prove that $r$ forces that $\check{y}[r]$ contains $\check{y}$. Suppose that $r_1 < r$ and there is a pair $(\ell, p_\ell) \in \check{y}$ with $r_1 < p_\ell$. Choose an $r_2 \in \mathbb{P}_{j,\gamma}$ so that $r_2 <_{P_{\ell}} r_1$. It suffices to show $r_2 \VDash \ell \in \check{y}[r]$. Let $q <_{P_{\ell}} q \cap M_\rho$. Then $r_2 \not\Vdash p_\ell$ implies $r_2 \not\Vdash q$. Since $r_2$ was any $\leq$-projection of $r_1$ we can assume that $r_2 < q$. Since $r_2 \Vdash \beta$ is in $(\mathbb{P}_{j,\beta} \cap (r \wedge P_{\ell} \upharpoonright \beta)^+)^+$, it follows that $q \upharpoonright \beta \neq (r \wedge p_\ell \upharpoonright \beta)^+$. This implies that $(\ell, q) \in \check{y}[r]$ and completes the proof that $r_2 \Vdash \ell \in \check{y}[r]$.

Now assume that $\beta < \gamma$ and $r \Vdash \check{b} \cap \check{E} \cap \check{y}$ is empty for some $r < p$ in $\mathbb{P}_{i_\beta,\gamma}$, $\check{b} \in \mathbb{B}_{j,\gamma}$, and $\check{E} \in \mathbb{E}_\rho \cap \mathbb{B}_{i_\gamma,\gamma}$. Let $\hat{x} = ('E \cap \check{y})[r]$ (defined as above for $\check{y}[r]$). We complete the proof of Fact 3 by proving that $r \Vdash \check{b} \cap \hat{x}$ is empty. Since each are in $\mathbb{B}_{j,\gamma}$, we may choose any $r_1 <_{P_{\ell}} r$, and assume that $r_1 \Vdash \ell \in \check{b} \cap \hat{x}$. In addition we can suppose that there is a pair $(\ell, q) \in \hat{x}$ such that $r_1 < q$. The fact that $(\ell, q) \in \hat{x}$ means there is a $p_\ell$ with $(\ell, p_\ell)$ in the name $\check{E} \cap \check{y}$ such that $q <_{P_{\ell}} p_\ell$. Since $r_1 \in \mathbb{P}_{j,\gamma}$ and $r_1 < q$, it follows that $r_1 \not\Vdash p_\ell$. Now it follows that $r_1$ has an extension forcing that $\ell \in \check{b} \cap (\check{E} \cap \check{y})$ which is a contradiction. \qed

Fact 4. $i_y = i_\alpha$ and so also $i_p < i_\alpha$.

Proof of Fact 4. Since $\mathbb{P}_{i_\alpha,\alpha+1} = \mathbb{P}_{i_\alpha}$ for $i < i_\alpha$, we have that $i_\alpha \leq i_y$. Now assume that $i_\alpha < i_y$ and we proceed much as we did in Fact 3 to prove that $i_p$ does not exist. Assume that $r < p$ (in $\mathbb{P}_{i_y,\alpha+1}$) and $r \Vdash \check{b} \cap (\check{E} \cap \check{y})$ is empty for some $\check{E} \in M_\rho \cap \langle \mathbb{E}_\rho \rangle \cap \mathbb{B}_{i_\gamma,\gamma}$ and $\check{b} \in \mathbb{B}_{i_\gamma,\gamma}$. Let $T_\alpha$ be the $\mathbb{P}_{i_y,\alpha}$-name such that $r \Vdash \alpha \Vdash r(\alpha) = T_\alpha \in \mathbb{L}(\mathbb{D}_{i_y}^\alpha)$.

Choose any $M_\rho \cap \mathbb{P}_{i_\alpha,\alpha}$-generic filter $G$ such that $r \Vdash \alpha \in \check{G}^+$. Since $\mathbb{P}_{i_\alpha,\alpha}$ is ccc and $M_{\rho}^\omega \subseteq M_\rho$, it follows that $M_\rho[G]$ is closed under $\omega$-sequences in the model $\mathbb{L}(\theta^+)[G]$.

In this model, define an $\mathbb{L}(\mathbb{D}_{i_y}^\alpha)$-name $\check{x}$. A pair $(\ell, T_\ell) \in \check{x}$ if $t_\alpha \leq \text{stem}(T_\ell) \in T_\ell \leq T_\ell \in \mathbb{L}(\mathbb{D}_{i_y}^\alpha)$ and for each stem$(T_\ell) \leq t \in T_\ell$, there is $q_{\ell,t} \in M_\rho$ such that $t_{\ell,t} = t$, $q_{\ell,t} \Vdash \ell \in (\check{y} \cap \check{E})$, and $q_{\ell,t} \wedge r$ is in $\check{G}^+$. We will show that $r$ forces over the poset $\check{G}^+$ that $\check{x}$ contains $\check{E} \cap \check{y}$ and that $\check{x} \cap \check{b}$ is empty. This will complete the proof since it contradicts the assumption on $i_p$.

To prove that $r$ forces that $\check{x}$ contains $\check{y} \cap \check{E}$, we consider any $r_\ell < r$ in $\check{G}^+$ that forces over $\check{G}^+$ that $\ell \in \check{y} \cap \check{E}$ We may choose $p_\ell \in M_\rho$
such that $r_\ell < p_\ell$ and $p_\ell \models \ell \in \dot{E} \cap \dot{y})$. Since $r_\ell \in \dot{G}^+$, it follows that $p_\ell \wedge r$ is in $\dot{G}^+$. We may assume that $t_{\alpha}^r = t_{\alpha}^{p_\ell}$. To show that $r$ forces that $\ell \in \dot{x}$ we have to show there is a $T_\ell \in \mathbb{L}(\mathcal{D}_{i_\alpha}^\alpha)$ with $t_{\alpha}^{p_\ell} = \text{stem}(T_\ell)$.

Starting with $t = t_{\alpha}^{p_\ell}$, assume that $t \in T_\ell$ with $q_{\ell,t}$ as the witness. Let $L^- = \{ k : \phantom{\ell} t^{-} k \notin T_\ell \}$; it suffices to show that $L^- \notin (\mathcal{D}_{i_\alpha}^\alpha)^+$. By assumption that $q_{\ell,t}$ is the witness, there is an $r_t \prec (q_{\ell,t} \upharpoonright \alpha \wedge r \upharpoonright \alpha)$ in $\mathbb{P}_{i_\alpha,\alpha}$, such that $r_t \models t \in T_\alpha$ and $t = t_{\alpha}^r$. By strengthening $r_t$ we can assume that $r_t$ forces a value $\dot{D} \in \dot{\mathcal{D}}_{i_\alpha}^\alpha$ on $\{ k : t^{-} k \in \dot{T}_\alpha \cap q_{\ell,t}(\alpha) \}$. But now, it follows that $r_t$ forces that $\dot{D}$ is disjoint from $L^-$ since if $r_{t,k} \models k \in \dot{D}$ for some $r_{t,k} < r_t$, then $r_{t,k}$ is the witness to $t^{-} k$ is in $T_\ell$. Since some condition forces that $L^-$ is not in $(\dot{\mathcal{D}}_{i_\alpha}^\alpha)^+$ it follows that $L^-$ is not in $(\dot{\mathcal{D}}_{i_\alpha}^\alpha)^+$.

Finally we must show that $r$ forces over $\dot{G}^+$ that $\dot{b}$ is disjoint from $\dot{x}$. Suppose that $\dot{r} \models \ell \in \dot{b} \cap \dot{x}$ where $\dot{r} <_{i_\alpha} r$ and $\dot{r} \in \dot{G}^+$. We obtain a contradiction by showing that $r \not\models \ell \notin \dot{E} \cap \dot{y}$. We may assume, by possibly strengthening $\dot{r} \models \alpha$, that $t = t_{\alpha}^r$ is a branching node of $T_\ell$. This means that there is some $q_{\ell,t} \in M_\rho$ such that $q_{\ell,t} \models \ell \in \dot{E} \cap \dot{y}$ and $q_{\ell,t} \wedge r \in \dot{G}^+$. Let $\bar{p} = \dot{r} \wedge q_{\ell,t} \wedge r$. Notice that $\bar{p} \models t_{\alpha}^r \subseteq t \in \dot{T}_\alpha$. Since $\bar{p} \in \dot{G}^+$, we have that $\dot{G}$ is disjoint from $\mathbb{P}_{i_\alpha,\alpha} \cap \bar{r}^\perp$. Since $\dot{G}$ is $\mathbb{P}_{i_\alpha,\alpha}$-generic, there is a $\bar{q} \in \dot{G}$ satisfying that $\bar{q} <_{i_\alpha} \bar{p}$. In particular, $\bar{q} < \bar{r}$. But also, it follows that $\bar{q}$ has an extension in $\mathbb{P}_{i_\alpha,\alpha}$ that is below $q_{\ell,t} \wedge r$, which forces that $\ell \in (\dot{E} \cap \dot{y})$.  

**Fact 5.** The character of $\mathcal{D}_{i_\alpha}^\alpha$ is greater than $\mu_{i_\gamma}$.

**Proof of Fact 5.** Since $i_\alpha = i_\gamma > 0$ and $\mathbb{P}^\gamma \in \mathcal{H}(\lambda)$, the cofinality of $\alpha$ is uncountable. It also means that $\mathcal{D}_{i_\alpha}^\alpha$ is forced to have a descending mod finite base with cofinality equal to the cofinality of $\alpha$. As usual, we proceed by contradiction and assume that the character of $\mathcal{D}_{i_\alpha}^\alpha$, and therefore the cofinality of $\alpha$, is less than $\mu_{i_\gamma}$. Choose $\beta_\alpha < \alpha$ as per the definition of $\mathbb{P}^\gamma \in \mathcal{H}(\lambda)$. Choose $\dot{b}_1, \dot{b}_2 \in \mathcal{E}_\rho^+ \cap \mathcal{B}_{i_\rho,\alpha+1} = \mathcal{E}_\rho^+ \cap \mathcal{B}_{i_\alpha,\alpha}$ and $\dot{E}_\rho \in \mathcal{E}_\rho$ as in Fact 2. That is, $p \models \dot{b}_1 \cap \dot{E}_\rho \cap \dot{y} = \dot{b}_2 \cap \dot{E}_\rho \setminus \dot{y} = \emptyset$.

Let $\dot{T}_\alpha$ be a $\mathbb{P}_{i_\alpha,\alpha}$-name such that $p \models [\alpha] \vdash p(\alpha) = \dot{T}_\alpha$. There is no loss of generality to assume that $\text{stem}(\dot{T}_\alpha)$ is forced to be the empty sequence. Since $\mathcal{D}_{i_\alpha}^\alpha$ has a descending mod finite base (contained in $M$) with uncountable cofinality, there is a $\dot{D}_0 \in M \cap \mathcal{B}_{i_\alpha,\alpha}$ such that $p \models$ forces that $\dot{D}_0 \in M \cap \mathcal{D}_{i_\alpha}^\alpha$ and for each $t \in \dot{T}_\alpha$, $(\dot{T}_\alpha)_t$ is almost $\dot{D}_0$-branching in the sense that $\{ k \in \dot{D}_0 : t \setminus k \in \dot{T}_\alpha \}$ contains a cofinite subset of $\dot{D}_0$. Choose also a sequence $\{ \dot{D}_n : 0 < n < \omega \} \subseteq M_{\rho} \cap \mathcal{D}_{i_\alpha}^\alpha$.
so that it is forced (by $1_{\mathbb{P}_{\alpha,\beta}}$) that $\{\dot{D}_n : n \in \omega\}$ is strictly descending mod finite.

Choose $\beta \in M \cap \alpha$ large enough so that

1. $\beta_\alpha < \beta$ and $\{\dot{D}_n : n \in \omega\} \subset \mathbb{B}_{i,\beta}$
2. $p|\alpha \in \mathbb{P}_{i,\beta}$ and $\dot{T}_\alpha$ is a $\mathbb{P}_{i,\beta}$-name, and,
3. for all $(\ell, q) \in \dot{E}_r \cap \dot{y}$, $q|\alpha \in \mathbb{P}_{i,\beta}$, and $q(\alpha)$ is a $\mathbb{P}_{i,\beta}$-name.

If the cofinality of $\alpha$ is greater than $\aleph_1$, then choose $\beta \leq \eta \in M \cap \alpha$ with uncountable cofinality and let $\dot{Q}$ denote $\mathbb{L}(\mathcal{D}_{i,\eta}^\alpha)$. If the cofinality of $\alpha$ is $\aleph_1$, then set $\eta = \beta$. Recall that $i_\eta \leq i_\alpha$ and that $\mathcal{D}_{i,\eta}^\alpha$ is free and has a countable strictly descending mod finite base. Choose a $\mathbb{P}_{i,\eta}$-name $f$ of a bijection on $\omega$ so that $\mathbb{L}(\mathcal{D}_{i,\eta}^\alpha) = \mathbb{L}(\{f(\dot{D}_n) : n \in \omega\})$.

In this case we let $\dot{Q}$ be the $\mathbb{P}_{i,\eta}$-name of $\mathbb{L}(\{\dot{D}_n : n \in \omega\})$. Regardless of our definition of $\dot{Q}$, we have that $q_p = p \upharpoonright \eta \cup \{(\eta, \dot{T}_\alpha)\}$ is an element of $\mathbb{P}_{i,\eta} \ast \dot{Q}$.

Now construct the name $\dot{y}_\eta \in \mathbb{B}_{i,\eta} + 1$ where, for each $(\ell, q) \in (\dot{E} \cap \dot{y})$, $(\ell, q|\eta \cup \{(\eta, q(\alpha))\}) \in \dot{y}_\eta$. It is routine to check that $q_p$ forces, over the poset $\mathbb{P}_{i,\eta} \ast \dot{Q}$ that $\dot{b}_1 \cap \dot{y}_\eta$ is empty. Next, let $\dot{x}_\eta$ be the name where, for each $(\ell, q) \in \dot{E}_p \setminus \dot{y}$, $(\ell, q|\eta \cup \{(\eta, q(\alpha))\}) \in \dot{x}_\eta$, and it also follows that $q_p$ forces, over the poset $\mathbb{P}_{i,\eta} \ast \dot{Q}$, that $\dot{b}_2 \cap \dot{y}_\eta$ is empty.

Let $\varphi$ denote the canonical isomorphism from $\mathbb{P}_{i,\eta} \ast \dot{Q}$ to $\mathbb{P}_{i,\eta} + 1$ and let $\bar{p} = \varphi(q_p)$. Let $\varphi(\dot{y}_\eta)$ denote the name $\{((\ell, \varphi(q)) : (\ell, q) \in \dot{y}_\eta\}$ and similarly define $\varphi(\dot{x}_\eta)$.

Consider any $(\ell, q) \in (\dot{E}_p \cap \dot{y}) \cup (\dot{E}_p \setminus \dot{y})$ and any $r < \bar{p} \wedge p = \bar{p} \cup \{i, \dot{T}_\alpha\}$ such that $r < q$. Let $t = t_r^\ell$ and choose $r_2 < r \upharpoonright \eta \cup \{(i, \dot{T}_\alpha)\}$. If $r_2 \models \ell \in \dot{E}_p$, we can assume that $r_2 < q_\ell$ for some $(\ell, q_\ell) \in (\dot{E}_p \cap \dot{y}) \cup (\dot{E}_p \setminus \dot{y})$. Therefore $(\ell, q_\ell|\eta \cup \{(\eta, q_\ell(\alpha))\})$ is an element of $\dot{y}_\eta \cup \dot{x}_\eta$. This, in turn, implies that $(\eta, \varphi(q_\ell(\alpha))) \in \varphi(\dot{y}_\eta) \cup \varphi(\dot{x}_\eta)$ and proves that $\bar{p} \wedge p$ forces that $\varphi(\dot{y}_\eta) \cup \varphi(\dot{x}_\eta)$ contains $\dot{E}_p$. By the minimality of $\alpha_\eta$, $\bar{p} \wedge p$ forces that $\mathcal{E}_p \cap \{\varphi(\dot{y}_\eta), \varphi(\dot{x}_\eta)\}$ is not empty. However this then implies that $\bar{p} \wedge p$ forces that one of $\dot{b}_1, \dot{b}_2$ is not in $\mathcal{E}_p^+$, and this contradiction completes the proof. \hfill \Box

**Definition 5.7.** For each $t \in \omega^{<\omega}$, define the $\mathbb{P}_{i,\alpha}$-name $\dot{E}^\alpha$ according to the rule that $r \models \ell \in \dot{E}^\alpha$ providing $r \in \mathbb{P}_{i,\alpha}$ forces that there is a $\dot{T}$ with $r \models \dot{T} \in \mathbb{L}(\mathcal{D}_{i,\eta}^\alpha)$, $r \models t = \text{stem}(\dot{T})$, and $r \cup \{(\alpha, \dot{T})\} \models \ell \notin \dot{y}$.

**Fact 6.** There is a $\dot{T}_\alpha \in \mathbb{L}(\dot{D}_{i,\alpha}) \cap M_p$ such that $p|\alpha$ forces the statement: $\dot{T}_\alpha < p(\alpha)$ and $\dot{E}_t \in \mathcal{E}_p$ for all $t$ such that $\text{stem}(\dot{T}) \leq t \in \dot{T}_\alpha$. 
Now we show how to extend $E_\rho \cap \mathbb{B}_{i_\alpha,\gamma}$ so as to measure $\dot{y}$. Choose a $P_{i_\alpha,\alpha}$-name, $\dot{T}_\alpha$ as in Fact 6. Let $\beta = \sup(M_\rho \cap \alpha)$. By Fact 5, $\beta < \alpha$ and by the definition of $H(\bar{X})$, $\dot{L}_\beta \in \dot{\mathcal{D}}^\alpha_{i_\alpha}$, $i_\beta = i_\alpha$, and $M_\rho \cap \dot{\mathcal{D}}^\alpha_{i_\alpha}$ is a subset of $\langle \dot{\mathcal{D}}^\beta_{i_\beta} \rangle$. We also have that the family $\{ \dot{L}_\xi : \text{cf}(\xi) \geq \omega_1 \text{ and } \beta \leq \xi \in M_\rho \cap \beta \}$ is a base for $\dot{\mathcal{D}}^\beta_{i_\beta}$. For convenience let $q < M_\rho, p$ denote the relation that $q$ is an $M_\rho \cap P_{i_\alpha,\alpha+1}$-reduct of $p$. Let $\bar{p}$ be any condition in $P_{i_\beta,\beta+1}$ satisfying that $\bar{p} \upharpoonright \beta = p \upharpoonright \alpha$ and $\bar{p} \upharpoonright \beta \Vdash \text{stem}(\bar{p}(\beta)) = t_\alpha$ (recall that $p \upharpoonright \alpha \Vdash t_\alpha = \text{stem}(p(\alpha))$).

Let us note that for each $q \in M_\rho \cap P_{i_\alpha,\alpha+1}$, $q \upharpoonright \alpha = q \upharpoonright \beta$ and $q \upharpoonright \beta \Vdash q(\alpha)$ is also a $P_{\beta,i_\beta}$-name of an element of $\mathbb{L}(\mathcal{D}^\beta_{i_\beta})$. Let $\dot{x}$ be the following $P_{i_\beta,\beta+1}$-name

$$\dot{x} = \{ (\ell, q \upharpoonright \beta \cup \{ (\beta, q(\alpha)) \}) : (\ell, q) \in \dot{y} \cap M_\rho \text{ and } q < M_\rho, p \}.$$ 

We will complete the proof by showing that there is an extension of $p$ that forces that $E_\rho \cup \{ \omega \setminus \{ \dot{x}[\dot{L}_\beta] \} \}$ measures $\dot{y}$ and that 1 forces that $\langle E_\rho \cup \{ \omega \setminus \{ \dot{x}[\dot{L}_\beta] \} \} \rangle \cap \mathbb{B}_{i_\beta,\beta+1}$ is $\bar{\lambda}(i_\gamma)$-thin. Here $\dot{x}[\dot{L}_\beta]$ abbreviates the $P_{i_\beta,\beta+1}$-name

$$\{ (\ell, r) : (\exists q) (\ell, q) \in \dot{x}, q \upharpoonright \beta = r \upharpoonright \beta, \text{ and } r \Vdash \text{stem}(q(\beta)) \in \dot{L}_\beta^{\omega} \}. $$

The way to think of $\dot{x}[\dot{L}_\beta]$ is that if $\bar{p}$ is in some $P_{i_\alpha,\alpha}$-generic filter $G$, then $\dot{y}[G]$ is now an $\mathbb{L}(\mathcal{D}^\alpha_{i_\alpha})$-name, $L_{\dot{L}_\beta}^{\omega} = (\dot{L}_\beta[G])^{\omega}$ is in $\mathbb{L}(\mathcal{D}^\alpha_{i_\alpha})$, and $\langle \dot{x}[\dot{L}_\beta] \rangle[G]$ is equal to $\{ \ell : L_{\dot{L}_\beta}^{\omega} \not\Vdash \ell \notin \dot{y} \}$. We will use the properties of $\dot{x}$ to help show that $E_\rho \cup \{ \omega \setminus \{ \dot{x}[\dot{L}_\beta] \} \}$ is $\bar{\lambda}(i_\gamma)$-thin. This semantic description of $\dot{x}[\dot{L}_\beta]$ makes clear that $\bar{p} \cup \{ (\alpha, (\dot{L}_\beta)^{\omega}) \} \in P_{i_\alpha,\alpha+1}$ forces that $\dot{x}[\dot{L}_\beta]$ contains $\dot{y}$. This implies that $E_\rho \cup \{ \omega \setminus \{ \dot{x}[\dot{L}_\beta] \} \}$ measures $\dot{y}$. 

**Proof of Fact 6.** By elementarity, there is a maximal antichain of $P_{i_\alpha,\alpha}$ each element of which decides if there is a $\bar{T}$ with $\dot{E}_i \in E_\rho$ for all $t \in \bar{T}$ above $\text{stem}(\bar{T})$. Since $p \in A \setminus A_1$ it follows that there is an $i_p < i_\alpha$ as in condition (2) of Fact 2. Let $t_0 \in \omega^{\omega}$ so that $p \upharpoonright \alpha \Vdash t_0 = \text{stem}(p(\alpha))$. By the maximum principle, there is a $b \in \mathbb{B}_{i_p,\gamma}$ and a $\dot{E}_0 \in E_\rho$ satisfying that $p \Vdash b \cap \dot{E}_0 \cap \dot{y}$ is empty, while $p \Vdash b \cap \dot{E}$ is infinite for all $\dot{E} \in (E_\rho)$. This means that $p$ forces that $b \cap \dot{E}_0$ is an element of $\langle E_\rho \rangle^+$ that is contained in $\omega \setminus \dot{y}$. As in the proof of Lemma 5.4, there is an $\dot{E}_2 \in \langle E_\rho \rangle \cap \mathbb{B}_{i_p,\gamma}$ such that $p$ forces that $b \cap \dot{E}_2$ is contained in $\dot{E}_0$. We also have that $(b \cap \dot{E}_2) \cap \dot{y}$ is forced to be contained in $\omega \setminus \dot{y}$. It now follows that $p \upharpoonright \alpha$ forces that for all $t_0 \leq t \in p(\alpha)$, $p \upharpoonright \alpha$ forces that $\dot{E}_t$ contains $(b \cap \dot{E}_2) \cap \dot{y}$ and so is in $\langle E_\rho \rangle^+$. Since $\dot{E}_t$ is also measured by $E_\rho$, we have that $p \upharpoonright \alpha$ forces that such $\dot{E}_t$ are in $E_\rho$. This completes the proof. \[\square\]
Each element of $\mathcal{E}_\rho$ is in $M_\rho$ and simple elementarity will show that for any condition $q$ in $M_\rho$ that forces $\dot{E} \cap (\omega \setminus \dot{y})$ is infinite, the corresponding $\dot{q} = q \cup \{ (\beta, q(\alpha)) \}$ will also force that $\dot{E} \cap (\omega \setminus \dot{x})$ is infinite. Therefore, it is forced by $\dot{p}$ that $\omega \setminus \dot{x}$ is not measured by $\mathcal{E}_\rho$.

Recall that $q \Vdash \dot{x} = \emptyset$ for all $q \perp \dot{p}$. Additionally, $\mathcal{E}_\rho \cap \mathbb{B}_{i,\beta+1}$ equals $\mathcal{E}_\rho \cap \mathbb{B}_{i,\beta}$. It thus follows from Fact 5 and the minimality of $\alpha_y$, that $(\mathcal{E}_\rho \cap \mathbb{B}_{i,\beta+1}) \cup \{ \omega \setminus \dot{x} \}$ is $(\mathbb{P}_{i,\beta+1}, \bar{x}(i_\gamma))$-thin.

**Claim 1.** If $\dot{b} \in \mathbb{B}_{i,\beta}$ ($i < i_\beta$) and there is an $\dot{E} \in \mathcal{E}_\rho \cap \mathbb{B}_{i,\beta}$ and $q \Vdash \dot{b} \cap (\dot{E} \setminus \dot{x}) = \emptyset$, then $q \Vdash \beta \in (\exists \dot{E} \in \mathcal{E}_\rho) \dot{b} \cap \dot{E} = \emptyset$.

**Proof of Claim:** Let $q$ and $\dot{b}$ be as in the hypothesis of the Claim. Since $\mathcal{E}_\rho \cup \{ \omega \setminus \dot{x} \}$ is $(\mathbb{P}_{i,\beta+1}, \bar{x}(i_\gamma))$-thin, there is an $\dot{E}_1 \in \mathcal{E}_\rho \cap \mathbb{B}_{i,\beta}$ such that $q$ forces that $\dot{b} \cap \dot{E}_1 = \emptyset$. Since each of $\dot{b}$ and $\dot{E}_1$ are in $\mathbb{B}_{i,\beta}$, $q \Vdash \beta$ forces that $\dot{b} \cap \dot{E}_1 = \emptyset$. This proves the claim. \(\square\)

Now to prove that $\mathcal{E}_\rho \cup \{ \omega \setminus (\dot{x}[\dot{L}_\beta]) \}$ is also $(\mathbb{P}_{i,\beta+1}, \bar{x}(i_\gamma))$-thin, we prove that

$$\langle \mathcal{E}_\rho \cup \{ \omega \setminus (\dot{x}[\dot{L}_\beta]) \} \rangle \cap \mathbb{B}_{i,\beta} = \langle \mathcal{E}_\rho \cup \{ \omega \setminus \dot{x} \} \rangle \cap \mathbb{B}_{i,\beta}$$

for all $i < i_\alpha$. Assume that $\dot{b} \in \mathbb{B}_{i,\beta}$ and $q \Vdash \dot{b} \cap \dot{E} \cap (\omega \setminus (\dot{x}[\dot{L}_\beta])) = \emptyset$ for some $q \in \mathbb{P}_{i,\beta+1}$ and $E \in \mathcal{E}_\rho \cap \mathbb{B}_{i,\beta}$. If $q \Vdash \omega \setminus \dot{x} = \omega$, then we can assume that $q < \dot{p}$. We want to prove that there is some $\dot{E}_1 \in \mathcal{E}_\rho$ such that $q \Vdash \dot{b} \cap (\dot{E}_1 \cap (\omega \setminus \dot{x}))$ is finite. Let $t = t_\beta^q$ and let $H$ be the range of $t$.

Let $E_1$ be the $\mathbb{P}_{i,\beta}$-name for $\dot{E} \cap \{ \dot{E}_s : s \in H^{<\omega} \}$. By Fact 6, $p | \alpha \Vdash E_1 \in \mathcal{E}_\rho$, and by definition of $\beta, 1 \in \mathbb{B}_{i,\beta}$. Therefore $\dot{b} \cap \dot{E}_1 \cap (\omega \setminus \dot{x}[\dot{L}_\beta])$ is a $\mathbb{P}_{i,\beta}$-name. Now suppose that $r < \dot{p} | \beta$ and $r \Vdash \ell \in \dot{b} \cap \dot{E}_1$. Then follows that $r \Vdash \ell \in \dot{x}[\dot{L}_\beta]$. Let $r$ be an element of any $\mathbb{P}_{i,\beta}$-generic filter $G_\beta$. We just have to prove that $\ell \in \text{val}_{G_\beta}(\dot{x})$. For each $s \in H^{<\omega}$, we have that $\ell \in \text{val}_{G_\beta}(\dot{E}_s)$ and so we may choose $T_s \in \mathbb{L}[\mathfrak{D}_{i_\alpha}^\alpha] \cap M_\rho$ so that $s = \text{stem}(T_s)$ and $T_s \Vdash \ell \notin \dot{y}$. Choose any $D \in \mathfrak{D}_{i_\alpha}^\alpha \cap M_\rho$ so that each $T_s \in H^{<\omega}$ is almost $D$-branching. For each $s \in H^{<\omega}$ choose $g_s : \omega^{<\omega} \mapsto \omega$ so that for all $\bar{t} \in \omega^{<\omega}$ and $k \in D \setminus g_s(\bar{t})$, $s * (\bar{t} \langle k \rangle)$ is in $T_s$. Now define $T_\beta \in \mathbb{L}[\mathfrak{D}_{i_\alpha}^\beta]$ according to the recursive rule that $t_{\beta}^q = \text{stem}(T_\beta)$ and for all $t_{\beta}^q \in T_\beta$, $\{ k : (t_{\beta}^q) * \bar{t} = k \in T_\beta \}$ is equal to $D \setminus \text{max}\{ g_s(\bar{t}) : s \in H^{<\omega} \}$. It is easily checked that if $G_{\beta+1}$ is a generic filter for $\mathbb{P}_{i,\beta+1}$ such that $G_\beta \subseteq G_{\beta+1}$ and the condition $\{ (\beta, T_\beta) \}$ is in $G_{\beta+1}$, then $L_{\beta} = \text{val}_{G_{\beta+1}}(\dot{L}_\beta)$ has the property that, for each $s \in H^{<\omega}$, $(L_s^{<\omega})_s \subseteq T_s$. To prove that
\( \ell \in \text{val}_{G_\beta}(\dot{x}) \), it suffices to prove that \( L^\omega_\beta \not
models \ell \notin \dot{y} \). Let \( T \) be any extension of \( L^\omega_\beta \) and let \( s \in H^\omega \) be maximal so that \( s \subset \text{stem}(T) \). Since \( T < T_s \), it follows that \( T \models \ell \notin \dot{y} \). This completes the proof. \( \square \)

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