ON THE BOUNDING, SPLITTING, AND DISTRIBUTIVITY NUMBERS

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ABSTRACT. The cardinal invariants $\mathfrak{h}, \mathfrak{b}, \mathfrak{s}$ of $\mathcal{P}(\omega)$ are known to satisfy that $\omega_1 \leqslant \mathfrak{h} \leqslant \min\{\mathfrak{b}, \mathfrak{s}\}$. We prove that all inequalities can be strict. We also introduce a new upper bound for $\mathfrak{h}$ and show that it can be less than $\mathfrak{s}$. The key method is to utilize finite support matrix iterations of ccc posets following [4].

1. INTRODUCTION

The cardinal invariants of the continuum discussed in this article are very well known (see [7, van Douwen, p111]) so we just give a brief reminder. They deal with the mod finite ordering of the infinite subsets of the integers. We follow convention and let $[\omega]^\omega$ (or $[\omega]^{\aleph_0}$) denote the family of infinite subsets of ω. A set A is a pseudo-intersection of a family $\mathcal{Y} \subset [\omega]^\omega$ if A is infinite and $A \backslash Y$ is finite for all $Y \in \mathcal{Y}$. The family $\mathcal{Y}$ has the strong finite intersection property (sfip) if every finite subset has infinite intersection and $\mathfrak{p}$ is the minimum cardinal for which there is such a family with no pseudointersection. A family $\mathcal{I} \subset \mathcal{P}(\omega)$ is an ideal if it is closed under finite unions and mod finite subsets. An ideal $\mathcal{I} \subset \mathcal{P}(\omega)$ is dense if every $Y \in [\omega]^\omega$ contains an infinite member of $\mathcal{I}$. A set $S \subset \omega$ is *unsplit* by a family $\mathcal{Y} \subset [\omega]^\omega$ if S is mod finite contained in one member of $\{Y, \omega \backslash Y\}$ for each $Y \in \mathcal{Y}$. The splitting number $\mathfrak{s}$ is the minimum cardinal of a family $\mathcal{Y}$ for which there is no infinite set unsplit by $\mathcal{Y}$ (i.e. every $S \in [\omega]^\omega$ is *split* by some member of $\mathcal{Y}$ and $\mathcal{Y}$ is called a splitting family). The bounding number $\mathfrak{b}$ can easily be defined in these same terms, but it is best defined by the mod finite ordering, $<^*$, on the family of functions ω^ω. The cardinal $\mathfrak{b}$ is the minimum cardinal for which there is a $<^*$-unbounded family $B \subset \omega^\omega$ with $|B| = \mathfrak{b}$.

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The finite support iteration of the standard Hechler poset was shown in [2] to produce models of $\aleph_1 = \mathfrak{s} < \mathfrak{b}$. The consistency of $\aleph_1 = \mathfrak{b} < \mathfrak{s} = \aleph_2$ was established in [17] with a countable support iteration of a special poset we now call $\mathcal{Q}_{Bould}$. It is shown in [12] that one can use Cohen forcing to select ccc subposets of $\mathcal{Q}_{Bould}$ and finite support iterations to obtain models of $\aleph_1 < \mathfrak{b} < \mathfrak{s} = \mathfrak{b}^+$. This result was improved in [5] to show that the gap between $\mathfrak{b}$ and $\mathfrak{s}$ can be made arbitrarily large. The papers [4] and [5] are able to use ccc versions of the well-known Mathias forcing in their iterations in place of those discovered in [12]. The paper [5] also nicely expands on the method of matrix iterated forcing first introduced in [4], as do a number of more recent papers (see [10, 16] and [11] using template forcing). The distributivity number (degree) $\mathfrak{h}$ was first studied in [1]. It equals the minimum number of dense ideals whose intersection is simply the Fréchet ideal $[\omega]^{<\omega}$. It was shown in [1], that $\mathfrak{p} \leqslant \mathfrak{h} \leqslant \min\{\mathfrak{b}, \mathfrak{s}\}$. Our goal is to fully separate all these cardinals. We succeed but confront a new problem since we use the result, also from [1], that $\mathfrak{h} \leqslant \operatorname{cf}(\mathfrak{c})$. The consistency of $\mathfrak{h} < \mathfrak{s} < \mathfrak{b} < \operatorname{cf}(\mathfrak{c})$ has recently been established in [13][1].

2. A new bound on $\mathfrak{h}$

In [1], a family $\mathfrak{A}$ of maximal almost disjoint families of infinite subsets of ω is called a matrix. A matrix $\mathfrak{A}$ is *shattering* if the entire collection $\bigcup \mathfrak{A}$ is splitting. Evidently, if $\{s_\alpha : \alpha < \kappa\}$ is a splitting family, then the family $\mathfrak{A} = \{\{s_\alpha, \omega \setminus s_\alpha\} : \alpha < \kappa\}$ is a shattering matrix. A shattering matrix $\mathfrak{A} = \{\mathcal{A}_\alpha : \alpha < \kappa\}$ is *refining*, if for all $\alpha < \beta < \kappa$, $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ in the natural sense that each member of $\mathcal{A}_\beta$ is mod finite contained in some member of $\mathcal{A}_\alpha$. Finally, a *base matrix* is a refining shattering matrix $\mathfrak{A}$ satisfying that $\bigcup \mathfrak{A}$ is dense in $(\mathcal{P}(\omega)/\operatorname{fin}, \subset^*)$ (i.e. a π-base for ω^*).

We add condition (6) to the following result from [1].

Lemma 2.1. *The value of $\mathfrak{h}$ is the least cardinal κ such that any of the following hold:*

1. *the Boolean algebra $\mathcal{P}(\omega)/\operatorname{fin}$ is not κ-distributive,*
2. *there is a shattering matrix of cardinality κ,*
3. *there is a shattering and refining matrix indexed by κ,*
4. *there is a base matrix of cardinality κ,*
5. *there is a family of κ many nowhere dense subsets of ω^* whose union is dense,*

[1] Our weaker result 4.2 was established in 2018

(6) there is a sequence $\{\mathcal{S}_\alpha : \alpha < \kappa\}$ of splitting families satisfying that no 1-to-1 selection $\langle s_\alpha : \alpha \in \kappa\rangle \in \Pi\{\mathcal{S}_\alpha : \alpha \in \kappa\}$ has a pseudo-intersection.

Proof. Since (1)-(5) are proven in [1], it is sufficient to prove that, for a cardinal κ, (3) and (6) are equivalent. First suppose that $\mathfrak{A} = \{\mathcal{A}_\alpha : \alpha < \kappa\}$ is a refining and shattering matrix. Since the matrix is refining, it follows easily that, for each $\alpha < \kappa$, $\{\mathcal{A}_\beta : \alpha \leqslant \beta < \kappa\}$ is a shattering matrix. Therefore, for each $\alpha < \kappa$, $\mathcal{S}_\alpha = \bigcup\{\mathcal{A}_\beta : \alpha \leqslant \beta\}$ is a splitting family. Similarly, the refining property ensures that if $\langle a_\alpha : \alpha \in \kappa\rangle \in \Pi\{\mathcal{S}_\alpha : \alpha \in \kappa\}$, then $\{a_\alpha : \alpha \in \kappa\}$ has no pseudo-intersection.

Now assume that $\{\mathcal{S}_\alpha : \alpha < \kappa\}$ is a sequence of splitting families as in (6). By [1], it is sufficient to prove that $\mathfrak{h} \leqslant \kappa$, so let us assume that $\kappa \leqslant \mathfrak{h}$. We now make an observation about κ: for each infinite $b \subset \omega$, $\alpha < \kappa$ and family $\mathcal{S}' \subset [\omega]^\omega$ of cardinality less than κ, there is an infinite $a \subset b$ and an $s \in \mathcal{S}_\alpha \backslash \mathcal{S}'$ such that $a \subset s$ and s splits b. We prove this claim. We may ignore all members of $\mathcal{S}'$ that are mod finite disjoint, or mod finite include, b. Since the family $\{\{s' \cap b, b\backslash s'\} : s' \in \mathcal{S}'\}$ is not shattering (as a family of subsets of b) there is an infinite $b' \subset b$ that is not split by $\mathcal{S}'$. Choose any $s \in \mathcal{S}_\alpha$ that splits b' and let $a = s \cap b'$. Evidently, s also splits b. Since the ideal generated by a splitting family is dense, we may choose a maximal almost disjoint family $\mathcal{A}_0$ contained in the ideal generated by $\mathcal{S}_0$. Let s_0 denote any mapping from $\mathcal{A}_0$ into $\mathcal{S}_0$ satisfying that $a \subset s_0(a)$ for all $a \in \mathcal{A}_0$. Suppose that $\alpha < \kappa$ and that we have chosen a refining sequence $\{\mathcal{A}_\gamma : \gamma < \alpha\}$ of maximal almost disjoint families together with mappings $\{s_\gamma : \gamma < \alpha\}$ so that for each $a \in \mathcal{A}_\gamma$, $a \subset s_\gamma(a) \in \mathcal{S}_\gamma$. The extra induction assumption is that for all $a \in \mathcal{A}_\gamma$, $s_\gamma(a)$ is not an element of $\{s_\beta(a') : \beta < \gamma$ and $a \subset^* a' \in \mathcal{A}_\beta\}$. The existence of the family $\mathcal{A}_\alpha$ and the mapping s_α satisfying the induction conditions easily follows from the above Observation. Now we verify that $\mathfrak{A} = \{\mathcal{A}_\alpha : \alpha < \kappa\}$ satisfies that $\bigcup\mathfrak{A}$ is splitting. Fix any infinite $b \subset \omega$ and choose $a_\alpha \in \mathcal{A}_\alpha$, for each $\alpha \in \kappa$ so that $b \cap a_\alpha$ is infinite. By construction, $\{s_\alpha(a_\alpha) : \alpha \in \kappa\}$ is a 1-to-1 selection from $\Pi\{\mathcal{S}_\alpha : \alpha \in \kappa\}$. Since b is therefore not a pseudo-intersection, there is an $\alpha < \kappa$ such that $b\backslash s_\alpha(a_\alpha) \subset b\backslash a_\alpha$ is infinite. $\square$

The following is an immediate corollary to condition (6) in Lemma 2.1 and provide two approaches to bounding the value of $\mathfrak{h}$.

Corollary 2.2 ([1,3]). *(1)* $\mathfrak{h} \leqslant \mathrm{cf}(\mathfrak{c})$.

(2) *A poset $\mathbb{P}$ forces that $\mathfrak{h} \leqslant \kappa$ if $\mathbb{P}$ preserves κ and can be written as an increasing chain $\{\mathbb{P}_\alpha : \alpha < \kappa\}$ of completely embedded posets satisfying that each $\mathbb{P}_{\alpha+1}$ adds a real not added by $\mathbb{P}_\alpha$.*

Proof. For the statement in (1), let $\{\kappa_\alpha : \alpha < \mathrm{cf}(\mathfrak{c})\}$ be increasing and cofinal in $\mathfrak{c}$. Let $\{x_\xi : \xi \in \mathfrak{c}\}$ be an enumeration of $[\omega]^{\aleph_0}$. To apply (6) from Lemma 2.1, let $\mathcal{S}_\alpha = \{x_\xi : (\forall \eta < \kappa_\alpha)\ x_\eta \not\subset^* x_\xi\}$. Since every infinite $Y \subset \omega$ can be refined by an almost disjoint family of cardinality $\mathfrak{c}$, it follows that $\mathcal{S}_\alpha$ is splitting. For the statement in (2), let G be a $\mathbb{P}$-generic filter and, for each $\alpha \in \kappa$, let $G_\alpha = G \cap \mathbb{P}_\alpha$. To apply (6), let $\mathcal{S}_\alpha$ be the set of $x \in [\omega]^{\aleph_0}$ that contain no infinite $y \in V[G_\alpha]$. To see that $\mathcal{S}_\alpha$ is splitting in either case, given any infinite $x \subset \omega$, consider an enumeration $\{x_t : t \in 2^{<\omega}\}$. Then, for all $\alpha \in \kappa$, there is an $f_\alpha \in 2^\omega$ so that $\{x_{f_\alpha \restriction n} : n \in \omega\} \in \mathcal{S}_\alpha$. $\square$

Our introduction of condition (6) in Lemma 2.1 is motivated by the fact that it provides us with a new approach to bounding $\mathfrak{h}$. We introduce the following variant of condition (6) in Lemma 2.1 and note that a shattering refining matrix will fail to satisfy the second condition.

Definition 2.3. *Let $\kappa < \lambda$ be cardinals and say that a family $\{x_\alpha : \alpha < \lambda\}$ of infinite subsets of ω is (κ, λ)-shattering if, for all infinite $b \subset \omega$*

(1) *the set $\{\alpha < \lambda : b \subset^* x_\alpha\}$ has cardinality less than κ, and*
(2) *the set $\{\alpha < \lambda : b \cap x_\alpha =^* \varnothing\}$ has cardinality less than λ.*

Say that a (κ, λ)-shattering family is strongly (κ, λ)-shattering if it contains no splitting family of size less than λ.

Needless to say a (κ, λ)-shattering family is strongly (κ, λ)-shattering if $\lambda = \mathfrak{s}$ and this is the kind of families we are interested in. However it seems likely that producing strongly (κ, λ)-shattering families would be interesting (and as difficult) even without requiring that $\lambda = \mathfrak{s}$. Nevertheless $\mathfrak{s}$ is necessarily bounded by λ as we show next.

Proposition 2.4. *If there is a (κ, λ)-shattering family, then $\mathfrak{h} \leqslant \kappa$ and $\mathfrak{s} \leqslant \lambda$.*

Proof. Let $\mathcal{S} = \{x_\alpha : \alpha < \lambda\}$ be a (κ, λ)-shattering family. Given any infinite $b \subset \omega$, there is a $\beta < \lambda$ such that each of $b \subset^* x_\beta$ and $b \cap x_\beta =^* \varnothing$ fail. This means that $\mathcal{S}$ is splitting. By condition (1) in Definition 2.3 and applying condition (6) of Lemma 2.1 with $\mathcal{S}_\alpha = \mathcal{S}$ for all $\alpha < \kappa$, it follows that $\mathfrak{h} \leqslant \kappa$. $\square$

For any index set I the standard poset for adding Cohen reals, $\mathcal{C}_I$, is the set of all finite functions into 2 with domain a finite subset of I where $p < q$ providing $p \supset q$. If λ is an ordinal, then we may use $\dot{x}_\alpha$

to be the canonical $\mathcal{C}_\lambda$-name $\{(\check{n}, \{\langle \alpha+n, 1\rangle\} : n \in \omega\}$ (i.e. for $s \in \mathcal{C}_\lambda$, $s \Vdash n \in \dot{x}_\alpha$ providing $s(\alpha + n) = 1$).

It is routine to verify that, for any regular cardinal $\lambda > \aleph_1$, forcing with $\mathcal{C}_\lambda$ will naturally add an $(\aleph_1, \lambda)$-shattering family but is is clear that this family would not be strongly $(\aleph_1, \lambda)$-shattering because it has a splitting subfamily of cardinality $\aleph_1$. Nevertheless, it may be possible with further forcing, to have it become strongly (κ, λ)-shattering for some $\aleph_1 \leqslant \kappa < \mathfrak{s}$.

In Theorem 5.7 we will prove that it is consistent with $\aleph_2 < \kappa^+ < \mathfrak{c}$ that there is a strongly (κ, κ^+)-shattering family.

Question 2.1. Assume that $\kappa < \lambda$ are regular cardinals and that there is a strongly (κ, λ)-shattering family. We pose the following questions.
1. Is it consistent that $\kappa^+ < \lambda$?
2. Is it consistent that $\lambda < \mathfrak{b}$?
3. Is it consistent that $\kappa < \mathfrak{b} < \lambda$?

3. Matrix forcing and distinguishing $\mathfrak{h}, \mathfrak{s}, \mathfrak{b}$

In this section we recall the forcing methods for distinguishing $\mathfrak{b}$ and $\mathfrak{s}$ and apply them to prove the main results. We denote by $\mathbb{D}$ the standard (Hechler) poset for adding a dominating real. The poset $\mathbb{D}$ is an ordering on $\omega^{<\omega} \times \omega^\omega$ where $(s, f) < (t, g)$ providing $g \leqslant f$ and s extends t by values that are coordinatewise above g. Given a sfip family $\mathcal{F}$ of subsets of ω, there are two main posets for adding a pseudo-intersection. The Mathias-Prikry style poset is $\mathbb{M}(\mathcal{F})$ that consists of pairs (a, A) where and A is in the filter base generated by $\mathcal{F}$, $a \subset \min(A)$, and $\mathbb{M}(\mathcal{F})$ is ordered by $(a_1, A_1) < (a_2, A_2)$ providing $a_2 \subset a_1 \subset a_2 \cup A_2$ and $A_1 \subset A_2$. When the context is clear, we will let $\dot{x}_{\mathcal{F}}$ denote the canonical name, $\{(\check{n}, (a, \omega \setminus n{+}1)) : n \in a \subset n + 1\}$, which is forced to be the desired pseudo-intersection. When $\mathcal{U}$ is a free ultrafilter on ω, $\mathbb{M}(\mathcal{U})$ was the poset used in [4] and [5] and, in this case $\dot{x}_{\mathcal{U}}$ is unsplit by the set of ground model subsets of ω. When mixed with matrix iteration methods, the ultrafilter $\mathcal{U}$ can be constructed so as to not add a dominating real.

The Laver style poset, $\mathbb{L}(\mathcal{F})$, is also very useful in matrix iterations and is defined as follows. The members of $\mathbb{L}(\mathcal{F})$ are subtrees T of $\omega^{<\omega}$ with a root or stem, $\mathrm{root}(T)$, and for all $\mathrm{root}(T) \subseteq t \in T$, the set $\mathrm{Br}(T, t) = \{j \in \omega : t^\frown j \in T\}$ is an element of the filter generated by $\mathcal{F}$. This poset is ordered by $\subset$. For each $T \in \mathbb{L}(\mathcal{F})$ and $t \in T$, the subtree $T_t = \{t' \in T : t \cup t' \in \omega^{<\omega}\}$ is also a condition. The generic function, $\dot{f}_{\mathbb{L}(\mathcal{F})}$, added by $\mathbb{L}(\mathcal{F})$ can be described by the name of the union of the branch of $\omega^{<\omega}$ named by $\{(\check{t}, (\omega^{<\omega})_t) : t \in \omega^{<\omega}\}$. This poset forces that $\dot{f}_{\mathbb{L}(\mathcal{F})}$ dominates the ground model reals and the range of $\dot{f}_{\mathbb{L}(\mathcal{F})}$ is a pseudo-intersection of $\mathcal{F}$. Again, if $\mathcal{F}$ is an ultrafilter, this pseudo-intersection is not split by any ground model set.

For each sfip family $\mathcal{U}$ on ω, each of the posets $\mathbb{D}$, $\mathbb{M}(\mathcal{U})$, and $\mathbb{L}(\mathcal{U})$ is σ-centered. We just need this for the fact that this ensures that they are upwards ccc.

For a poset P and a set X, a canonical P-name for a subset of X will be a name of the form $\bigcup\{\check{x}\times A_x : x \in X\}$ where, for each $x \in X$, A_x is an antichain of P. An antichain of P is a set whose elements are pairwise incompatible and a subset of P is predense if its downward closure is dense. The incompatibility relation on P is denoted as $\perp_P$. Of course if $\dot{Y}$ is any P-name of a subset of X, there is a canonical name that is forced to equal it. If P is ccc and X is countable, then the set of canonical P-names for subsets of X has cardinality at most $|P|^{\aleph_0}$. When we say that a poset P forces a statement, we intend the meaning that every element (i.e. 1_P) of P forces that statement.

Recall that a poset P is a complete suborder of a poset Q providing $P \subset Q$, $<_P \subset <_Q$, $\perp_P \subset \perp_Q$, and every predense subset of P is predense in Q. We write $P \lessdot Q$ to mean that P is a complete suborder of Q. If G is a Q-generic filter and if $P \lessdot Q$, then $G \cap P$ is a P-generic filter. If we say that Q forces some property concerning the forcing extension by P, we mean that for each Q-generic filter G, that property holds in $V[G \cap P]$.

We say that $p \in P$ is a reduct (or a P-reduct) of $q \in Q$ if every $r \leqslant p$ in P is compatible with q in Q. If $P \lessdot Q$, then every $q \in Q$ has a P-reduct. If $\{P_\alpha : \alpha < \delta\}$ is a $\lessdot$-increasing chain of posets, then the union $P_\delta = \bigcup\{P_\alpha : \alpha < \delta\}$ satisfies that $P_\alpha \lessdot P_\delta$ for all $\alpha < \delta$. Before we recall the definition of a matrix-iteration, we introduce the following generalization used in [9].

Definition 3.1. *Let $\kappa > \omega_1$ be a regular cardinal. For an ordinal ζ, a $\kappa\times\zeta$-matrix of posets is a family $\{P_{\alpha,\xi} : \alpha < \kappa, \xi < \zeta\}$ of ccc posets satisfying, for each $\alpha < \kappa$, and $\xi < \eta < \zeta$:*

(1) $P_{\alpha,\xi} \lessdot P_{\beta,\xi}$ for all $\alpha < \beta < \kappa$,
(2) $P_{\beta,\xi} = \bigcup\{P_{\eta,\xi} : \eta < \beta\}$ for $\beta < \kappa$ with $cf(\beta) > \omega$, and
(3) for some $\gamma < \kappa$, $P_{\beta,\xi} \lessdot P_{\beta,\eta}$ for all $\gamma \leqslant \beta < \kappa$.

Lemma 3.2. *If $\{P_{\alpha,\xi} : \alpha < \kappa, \xi < \zeta\}$ is a $\kappa\times\zeta$-matrix of posets, then there is a sequence $\{P_{\kappa,\xi} : \xi \leqslant \zeta\}$ of ccc posets such that, for each $\xi < \eta \leqslant \zeta$:*

(1) $P_{\kappa,\xi} = \bigcup\{P_{\alpha,\xi} : \alpha < \kappa\}$
(2) $P_{\kappa,\zeta} = \bigcup\{P_{\kappa,\xi} : \xi < \zeta\}$,

(3) for all $\alpha < \kappa$, $P_{\alpha,\xi} \lessdot P_{\kappa,\xi}$, *and*
(4) $P_{\kappa,\xi} \lessdot P_{\kappa,\eta}$.

Proof. Item (3) follows immediately from item (1) of Definition 3.1. To prove (4) it suffices to check that $P_{\alpha,\xi} \lessdot P_{\kappa,\eta}$ for all $\alpha < \kappa$ and $\xi < \eta < \zeta$. Let $\alpha < \kappa$ and $\xi < \eta < \zeta$. Choose $\gamma < \kappa$ as in property (3) of Definition 3.1. Now we have $P_{\alpha,\xi} \lessdot P_{\gamma,\xi} \lessdot P_{\gamma,\eta} \lessdot P_{\kappa,\eta}$. Since $\lessdot$ is a transitive relation, the proof is complete. □

The terminology "matrix iterations" is used in [5], see also forthcoming preprint (F1222) from the second author.

Definition 3.3. *For an infinite cardinal* κ *with uncountable cofinality, and an ordinal* ζ, *a* $\kappa\times\zeta$*-matrix iteration is a family*

$$\langle\langle\mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \zeta\rangle, \langle\dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \zeta\rangle\rangle$$

where, for each $\alpha < \beta \leqslant \kappa$ *and* $\xi < \eta \leqslant \zeta$*:*

(1) $\mathbb{P}_{\beta,\xi}$ *is a ccc poset,*
(2) $\mathbb{P}_{\alpha,\xi} \lessdot \mathbb{P}_{\beta,\xi} \lessdot \mathbb{P}_{\beta,\eta}$,
(3) $\mathbb{P}_{\kappa,\xi}$ *is the union of the chain* $\{\mathbb{P}_{\gamma,\xi} : \gamma < \kappa\}$,
(4) $\dot{\mathbb{Q}}_{\alpha,\xi}$ *is a* $\mathbb{P}_{\alpha,\xi}$*-name of a ccc poset and* $\mathbb{P}_{\alpha,\xi+1} = \mathbb{P}_{\alpha,\xi} * \dot{\mathbb{Q}}_{\alpha,\xi}$,
(5) if η *is a limit, then* $\mathbb{P}_{\beta,\eta} = \bigcup\{\mathbb{P}_{\beta,\gamma} : \gamma < \eta\}$.

One constructs $\kappa\times\zeta$-iterations by recursion on ζ and, for successor steps, by careful choice of the component sequence $\{\dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa\}$. The first important result is that all the work is in the successor steps. The following is from [5, Lemma 3.10]

Lemma 3.4. *If* ζ *is a limit ordinal then a family*

$$\langle\langle\mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \zeta\rangle, \langle\dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \zeta\rangle\rangle$$

is a $\kappa\times\zeta$*-matrix iteration providing for all* $\eta < \zeta$ *and* $\beta \leqslant \kappa$*:*

(1) $\langle\langle\mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \eta\rangle, \langle\dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \eta\rangle\rangle$ *is a* $\kappa\times\eta$*-matrix iteration, and*
(2) $\mathbb{P}_{\beta,\zeta} = \bigcup\{\mathbb{P}_{\beta,\xi} : \xi < \zeta\}$.

The following is well-known, see for example [16, Section 5] and [14].

Proposition 3.5. *For any* ζ *and* $\kappa\times\zeta$*-matrix iteration*

$$\langle\langle\mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \zeta\rangle, \langle\dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \zeta\rangle\rangle$$

the extension

$$\langle\langle\mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \zeta{+}1\rangle, \langle\dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \zeta{+}1\rangle\rangle$$

is a $\kappa\times(\zeta{+}1)$*-matrix iteration if either the following holds:*

(1)$_{\mathbb{Q}}$ *for all* $\alpha \leqslant \kappa$, $\dot{\mathbb{Q}}_{\alpha,\zeta}$ *is the* $\mathbb{P}_{\alpha,\zeta}$*-name for* $\mathbb{D}$,

(2)$_{\mathbb{Q}}$ there is an $\alpha < \kappa$ such that $\dot{\mathbb{Q}}_{\beta,\zeta}$ is the trivial poset for $\beta < \alpha$, $\dot{\mathbb{Q}}_{\alpha,\zeta}$ is a $\mathbb{P}_{\alpha,\zeta}$-name of a σ-centered poset, and $\dot{\mathbb{Q}}_{\beta,\zeta} = \dot{\mathbb{Q}}_{\alpha,\zeta}$ for all $\alpha < \beta \leqslant \kappa$.

Notice that if we define the extension as in $(1)_{\mathbb{Q}}$ then we will be adding a dominating real, but even if $\dot{\mathbb{Q}}_{\alpha,\zeta}$ is forced to equal $\mathbb{D}$ in $(2)_{\mathbb{Q}}$, the real added will only dominate the reals added by $\mathbb{P}_{\alpha,\zeta}$.

Proposition 3.6. [4] *Let M be a model of (a sufficient amount of) set-theory and $P \in M$ be a poset that is also contained in M. Then for any $f \in \omega^\omega$ that is not dominated by any $g \in M \cap \omega^\omega$, P forces that $f \nleq \dot{g}$ for all P-names $\dot{g} \in M$ of elements of ω^ω.*

Proof. Let $p \in P$ and $n \in \omega$. It suffices to prove that there is a $q < p$ in P and a $k > n$ and $m < f(k)$ such that $q \Vdash \dot{g}(k) = m$. Since $p \in M$, we can work in M and define a function $h \in \omega^\omega$ by the rule that, for all $k \in \omega$, there is a $q_k < p$ such that $q_k \Vdash \dot{g}(k) = h(k)$. Choose any $k > n$ so that $h(k) < f(k)$. Then $q_k \Vdash \dot{g}(k) < f(k)$ and proves that $p \nVdash f \leqslant \dot{g}$. □

An analogous result, with the same proof, holds for splitting.

Proposition 3.7. *Let M be a model of (a sufficient amount of) set-theory and $P \in M$ be a poset that is also contained in M. If $x \in [\omega]^\omega$ satisfies that $y \not\subset^* x$ for all $y \in M \cap [\omega]^\omega$, then P forces that $\dot{y} \not\subset^* x$ for all P-names $\dot{y} \in M$ for elements of $[\omega]^\omega$.*

We also use the main construction from [4].

Proposition 3.8. *Suppose that*

$$\langle\langle \mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \zeta\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \zeta\rangle\rangle$$

is a $\kappa\times\zeta$-matrix iteration and that $\{\dot{f}_\alpha : \alpha < \kappa\}$ is a sequence satisfying that, for all $\alpha < \kappa$

(1) $\dot{f}_\alpha$ is a $\mathbb{P}_{\alpha,\zeta}$-name that is forced to be in ω^ω,
(2) for all $\beta < \alpha$ and $\mathbb{P}_{\beta,\zeta}$-name $\dot{g}$ of a member of ω^ω, $\mathbb{P}_{\alpha,\zeta}$ forces that $\dot{f}_\alpha \not<^* \dot{g}$.

Then there is a sequence $\{\dot{\mathcal{U}}_{\alpha,\zeta} : \alpha \leqslant \kappa\}$ such that, for all $\alpha < \kappa$:

(3) $\dot{\mathcal{U}}_{\alpha,\zeta}$ is a $\mathbb{P}_{\alpha,\zeta}$-name of an ultrafilter on ω,
(4) for $\beta < \alpha$, $\dot{\mathcal{U}}_{\beta,\zeta}$ is a subset of $\dot{\mathcal{U}}_{\alpha,\zeta}$
(5) for each $\beta < \alpha$ and each $\mathbb{P}_{\beta,\zeta} * \mathbb{M}(\dot{\mathcal{U}}_{\beta,\zeta})$-name $\dot{g}$ of an element of ω^ω, $\mathbb{P}_{\alpha,\zeta} * \mathbb{M}(\dot{\mathcal{U}}_{\alpha,\zeta})$ forces that $\dot{f}_\alpha \not<^* \dot{g}$, and

(6) *$\langle\langle \mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \zeta+1\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \zeta+1\rangle\rangle$ is a $\kappa\times(\zeta+1)$-matrix iteration, where, for each $\alpha \leqslant \kappa$, $\mathbb{P}_{\alpha,\zeta+1} = \mathbb{P}_{\alpha,\zeta} * \dot{\mathbb{Q}}_{\alpha,\zeta}$ and $\dot{\mathbb{Q}}_{\alpha,\zeta}$ is the $\mathbb{P}_{\alpha,\zeta}$-name for $\mathbb{M}(\dot{\mathcal{U}}_{\alpha,\zeta})$.*

We record two more well-known preparatory preservation results.

Proposition 3.9 ([2]). *Suppose that $M \subset N$ are models of (a sufficient amount of) set-theory and that G is $\mathbb{D}$-generic over N. If $x \in N \cap [\omega]^\omega$ does not include any $y \in M \cap [\omega]^\omega$, it will not include any $y \in M[G] \cap [\omega]^\omega$.*

Proposition 3.10. *Assume that $\{P_\alpha : \alpha \leqslant \delta\}$ is a $<\cdot$-increasing chain of ccc posets with $P_\delta = \bigcup\{P_\alpha : \alpha < \delta\}$. Let G_δ be P_δ-generic. Let $x \in [\omega]^\omega$ and $f \in \omega^\omega$. Then each of the following hold:*

(1) *If $f \not\leqq^* g$ for each $g \in V[G_\alpha]$ for all $\alpha < \delta$, then $f \not\leqq^* g$ for each $g \in V[G_\delta]$.*
(2) *If x does not contain any $y \in [\omega]^\omega \cap V[G_\alpha]$ for all $\alpha < \kappa$, then x does not contain any $y \in [\omega]^\omega \cap V[G_\delta]$.*

Proof. We prove only (1) since the proof of (2) is similar. If δ has uncountable cofinality, then there is nothing to prove since $V[G_\delta] \cap \omega^\omega$ would then equal $\bigcup\{V[G_\alpha] \cap \omega^\omega : \alpha < \delta\}$. Otherwise, consider any P_δ-name $\dot{g}$ and condition $p \in P_\delta$ forcing that $\dot{g} \in \omega^\omega$. We prove that p does not force that $\dot{g}(n) > f(n)$ for all $n > k$. We may assume that $\dot{g}$ is a canonical name, so let $\dot{g} = \bigcup\{\widetilde{(n,m)} \times A_{n,m} : n, m \in \omega\times\omega\}$. Choose any $\alpha < \delta$ so that $p \in P_\alpha$ and work in $V[G_\alpha]$. We define a function $h \in \omega^\omega \cap V[G_\alpha]$. For each $n \in \omega$, we set $h(n)$ to be the minimum m such that there is $q_{n,m} \in A_{n,m}$ having a P_α-reduct $p_{n,m} \in G_\alpha$. Since $A_n = \bigcup\{A_{n,m} : m \in \omega\}$ is predense in P_κ, the set of P_α-reducts of members of A_n is predense in P_α. By hypothesis, there is a $k < n$ such that $h(n) < f(n)$. Since $q_{n,h(n)}$ is compatible with p, this prove that $p \not\Vdash \dot{g}(n) > f(n)$. □

4. Building the models to distinguish $\mathfrak{h}, \mathfrak{b}, \mathfrak{s}$

For simplicity we assume GCH. Let $\aleph_1 \leqslant \mu < \kappa < \lambda$ be regular cardinals and assume that $\theta > \lambda$ is a cardinal with cofinality μ. We will need to enumerate names in order to force that $\mathfrak{p} \geqslant \mu$. For each ccc poset $\tilde{P} \in H(\theta^+)$ let $\{\dot{Y}(\tilde{P}, \xi) : \xi < \theta\}$ be an enumeration of the set of all canonical $\tilde{P}$-names of subsets of ω. Also let $\{S_\xi : \xi < \theta\}$ be an enumeration of all subsets of θ that have cardinality less than μ. For each $\eta < \lambda$, let ζ_η denote the ordinal product $\theta \cdot \eta$.

Theorem 4.1. *There is a ccc poset that forces $\mathfrak{p} = \mathfrak{h} = \mu$, $\mathfrak{b} = \kappa$, $\mathfrak{s} = \lambda$ and $\mathfrak{c} = \theta$.*

Proof. The poset will be obtained by constructing a $\kappa\times\zeta$-matrix iteration where ζ is the ordinal product $\theta\cdot\lambda = \sup\{\zeta_\eta : \eta < \lambda\}$. We begin with the $\kappa\times\kappa$-matrix iteration

$$\langle\langle \mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \kappa\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \kappa\rangle\rangle$$

where, for each $\alpha < \kappa$, $\mathbb{P}_{\alpha,\alpha}$ forces that $\dot{\mathbb{Q}}_{\alpha,\alpha}$ is $\mathbb{D}$, for $\beta < \alpha$, $\dot{\mathbb{Q}}_{\beta,\alpha}$ is the trivial poset, and for $\alpha \leqslant \beta \leqslant \kappa$, $\dot{\mathbb{Q}}_{\beta,\alpha}$ equals $\dot{\mathbb{Q}}_{\alpha,\alpha}$. By Proposition 3.5, there is such a matrix. For each $\alpha < \kappa$, let $\dot{f}_\alpha$ be the canonical name for the dominating real added by $\mathbb{P}_{\alpha,\alpha+1}$. By Propositions 3.6 and 3.10, it follows that for all $\beta < \alpha < \kappa$, $\mathbb{P}_{\alpha,\kappa}$ forces that $\dot{f}_\alpha \not\leq^* \dot{g}$ for all $\mathbb{P}_{\beta,\kappa}$-names $\dot{g}$ of elements of ω^ω.

We omit the routine enumeration details involved in the recursive construction and state the properties we require of our $\kappa\times\zeta$-matrix iteration. Each step of the construction uses either (2) of Proposition 3.5 or Proposition 3.8 to choose the next sequence $\{\dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa\}$. In the case of Proposition 3.5 (2), the preservation of inductive condition (1) follows from Proposition 3.6. The preservation through limit steps follows from Proposition 3.10.

There is a matrix-iteration sequence

$$\langle\langle \mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \zeta\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \zeta\rangle\rangle$$

satisfying each of the following for each $\xi < \zeta$:

(1) for each $\beta < \alpha < \kappa$ and each $\mathbb{P}_{\beta,\xi}$-name $\dot{g}$ for an element of ω^ω, $\mathbb{P}_{\alpha,\xi}$ forces that $\dot{f}_\alpha \not\leq^* \dot{g}$,

(2) for each $\beta < \lambda$ with $\zeta_{\beta+1} \leqslant \xi$ and each $\eta < \theta$, if $\mathbb{P}_{\kappa,\zeta_\beta}$ forces that the family $\mathcal{F}_{\beta,\eta} = \{\dot{Y}(\mathbb{P}_{\kappa,\zeta_\beta}, \gamma) : \gamma \in S_\eta\}$ has the sfip, then there is a $\bar{\eta} < \zeta_{\beta+1}$ and an $\alpha < \kappa$ such that $\dot{\mathbb{Q}}_{\beta,\bar{\eta}}$ equals the $\mathbb{P}_{\alpha,\bar{\eta}}$-name for $\mathbb{M}(\mathcal{F}_{\beta,\eta})$ for all $\alpha \leqslant \beta \leqslant \kappa$,

(3) for each $\beta < \lambda$ such that $\zeta_\beta < \xi$, $\mathbb{P}_{\kappa,\zeta_\beta+1}$ equals $\mathbb{P}_{\kappa,\zeta_\beta} * \mathbb{M}(\dot{\mathcal{U}}_{\kappa,\zeta_\beta})$ and $\dot{\mathcal{U}}_{\kappa,\zeta_\beta}$ is a $\mathbb{P}_{\kappa,\zeta_\beta}$-name of an ultrafilter on ω,

(4) for each $\eta < \lambda$ and each $\alpha < \kappa$ such that $\zeta_\eta < \xi$, then $\dot{\mathbb{Q}}_{\alpha,\zeta_\eta+\alpha}$ is the $\mathbb{P}_{\alpha,\zeta_\eta+\alpha}$-name for $\mathbb{D}$, and $\dot{\mathbb{Q}}_{\beta,\zeta_\eta+\alpha} = \dot{\mathbb{Q}}_{\alpha,\zeta_\eta+\alpha}$ for all $\alpha \leqslant \beta \leqslant \kappa$.

Now we verify that $P = \mathbb{P}_{\kappa,\zeta}$ has the desired properties. Since P is ccc, it preserves cardinals and clearly forces that $\mathfrak{c} = \theta$. It thus follows from Corollary 2.2 that $\mathfrak{p} \leqslant \mathfrak{h} \leqslant \mu = \mathrm{cf}(\mathfrak{c})$. If $\mathcal{Y}$ is a family of fewer than μ many canonical P-names of subsets of ω, then there is an $\alpha < \kappa$ and $\eta < \lambda$ such that $\mathcal{Y}$ is a family of $\mathbb{P}_{\alpha,\zeta_\eta}$-names. It follows that there is a $\beta < \theta$ such that $\mathcal{Y}$ is equal to the set $\{\dot{Y}(\mathbb{P}_{\kappa,\zeta_\beta}, \gamma) : \gamma \in S_\eta\}$. If $\mathbb{P}_{\kappa,\zeta_\beta}$ forces that $\mathcal{Y}$ has the sfip, then inductive condition 2 ensures that there is a P-name for a pseudo-intersection for $\mathcal{Y}$. This shows that P forces that $\mathfrak{p} \geqslant \mu$. It is clear that inductive condition 1 ensures that $\mathfrak{b} \leqslant \kappa$. We check that condition 4 ensure that $\mathfrak{b} \geqslant \kappa$. Suppose that $\mathcal{G}$ is a family of fewer than κ many canonical P-names of members of ω^ω. We again find $\eta < \lambda$ and $\alpha < \kappa$ such that $\mathcal{G}$ is a family of $\mathbb{P}_{\alpha,\zeta_\eta}$-names. Condition 4 forces there is a function that dominates $\mathcal{G}$. Finally we verify that condition 3 ensures that P forces that $\mathfrak{s} = \lambda$. If $\mathcal{S}$ is any family of fewer than λ-many canonical P-names of subsets of ω, then there is an $\eta < \lambda$ such that $\mathcal{S}$ is a family of $\mathbb{P}_{\kappa,\zeta_\eta}$-names. Evidently, $\mathbb{P}_{\kappa,\zeta_\eta+1}$ adds a subset of ω that is not split by $\mathcal{S}$. There are a number of ways to observe that for each $\eta < \lambda$, $\mathbb{P}_{\kappa,\zeta_{\eta+1}}$ adds a real that is Cohen over the extension by $\mathbb{P}_{\kappa,\zeta_\eta}$. This ensures that P forces that $\mathfrak{s} \leqslant \lambda$. $\square$

In the next result we proceed similarly except that we first add κ many Cohen reals and preserve that they are splitting. We then cofinally add dominating reals with Hechler's $\mathbb{D}$ and again use small posets to ensure $\mathfrak{p} \geqslant \mu$. We again mention that this result has been improved in [13], but we include it for completeness.

Theorem 4.2. *There is a ccc poset that forces $\mathfrak{p} = \mathfrak{h} = \mu$, $\mathfrak{s} = \kappa$, $\mathfrak{b} = \lambda$ and $\mathfrak{c} = \theta$.*

Proof. We begin with the $\kappa\times\kappa$-matrix iteration

$$\langle\langle \mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \kappa\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \kappa\rangle\rangle$$

where $\mathbb{P}_{\alpha,\alpha}$ forces that $\dot{\mathbb{Q}}_{\alpha,\alpha}$ is $\mathcal{C}_\omega$, for $\beta < \alpha$, $\dot{\mathbb{Q}}_{\beta,\alpha}$ is the trivial poset, and for $\alpha \leqslant \beta \leqslant \kappa$, $\dot{\mathbb{Q}}_{\beta,\alpha}$ equals $\dot{\mathbb{Q}}_{\alpha,\alpha}$. We let $\dot{x}_\alpha$ denote the canonical Cohen real added by $\mathbb{P}_{\alpha,\alpha+1}$. Of course $\mathbb{P}_{\alpha,\alpha+1}$ forces that neither $\dot{x}_\alpha$ nor its complement include any infinite subsets of ω that have, for any $\beta < \alpha$, a $\mathbb{P}_{\beta,\alpha+1}$-name. By Proposition 3.10, the inductive condition 1 below holds for $\xi = \kappa$.

Then, proceeding as in the proof of Theorem 4.1, we just assert the existence of a $\kappa\times\zeta$-matrix iteration

$$\langle\langle \mathbb{P}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi \leqslant \zeta\rangle, \langle \dot{\mathbb{Q}}_{\alpha,\xi} : \alpha \leqslant \kappa, \xi < \zeta\rangle\rangle$$

satisfying each of the following for each $\kappa \leqslant \xi < \zeta$:

(1) for each $\beta < \alpha < \kappa$, $\mathbb{P}_{\alpha,\xi}$ forces that neither $\dot{x}_\alpha$ nor $\omega\setminus\dot{x}_\alpha$ contains any infinite subset of ω that has a $\mathbb{P}_{\beta,\xi}$-name,
(2) for each $\eta < \lambda$ with $\zeta_{\eta+1} \leqslant \xi$ and each $\delta < \theta$, if $\mathbb{P}_{\kappa,\zeta_\eta}$ forces that the family $\mathcal{F}_{\eta,\delta} = \{\dot{Y}(\mathbb{P}_{\kappa,\zeta_\eta}, \gamma) : \gamma \in S_\delta\}$ has the sfip, then there is a $\bar{\delta} < \zeta_{\eta+1}$ and an $\alpha < \kappa$ such that $\dot{\mathbb{Q}}_{\beta,\bar{\delta}}$ equals the $\mathbb{P}_{\alpha,\bar{\delta}}$-name for $\mathbb{M}(\mathcal{F}_{\eta,\delta})$ for all $\alpha \leqslant \beta \leqslant \kappa$,

(3) for each $\eta < \lambda$ and each $\alpha < \kappa$ such that $\zeta_\eta < \xi$, then $\dot{\mathbb{Q}}_{\alpha,\zeta_\eta+\alpha}$ is the $\mathbb{P}_{\alpha,\zeta_\eta+\alpha}$-name for $\mathbb{M}(\dot{\mathcal{U}}_{\alpha,\zeta_\beta})$ where $\dot{\mathcal{U}}_{\alpha,\zeta_\beta}$ is a $\mathbb{P}_{\alpha,\zeta_\beta}$-name of an ultrafilter on ω, and $\dot{\mathbb{Q}}_{\beta,\zeta_\eta+\alpha} = \dot{\mathbb{Q}}_{\alpha,\zeta_\eta+\alpha}$ for all $\alpha \leqslant \beta \leqslant \kappa$.

(4) for each $\eta < \lambda$ such that $\zeta_\eta < \xi$, $\mathbb{P}_{\kappa,\zeta_\eta+1}$ equals $\mathbb{P}_{\kappa,\zeta_\eta} * \mathbb{D}$,

Evidently conditions (2) and (3) are similar and can be achieved while preserving condition (1) by Proposition 3.5 (2). The fact that $\mathbb{P}_{\kappa,\zeta_\eta} * \mathbb{D}$ preserves condition (1) follows from Proposition 3.9. Condition (1) ensures that $\mathfrak{s} \leqslant \kappa$, and by arguments similar to those in Theorem 4.1, condition (3) ensures that $\mathfrak{s} \geqslant \kappa$. The fact that $\mathfrak{b} = \lambda$ (in fact $\mathfrak{d} = \lambda$) follows easily from condition (4). The facts that that $\mathfrak{c} = \theta$, $\mathfrak{p} \geqslant \mu$ and $\mathfrak{h} = \mu$ are proven exactly as in Theorem 4.1. □

5. On (κ, λ)-shattering

In this section we prove, see Theorem 5.7, that it is consistent that strongly (κ, κ^+)-shattering families exist. We will use the method of matrix of posets from Definition 3.1 in which our main component posets to raise the value of $\mathfrak{s}$ will be the Laver style posets. We recall some notions and results about these studied in [8, 9, 18]. Before proceeding we summarize the rough idea of how we generalize the fundamental preservation technique of a matrix iteration. In a $\kappa \times \kappa^+$-matrix iteration, one may introduce a sequence $\{\dot{a}_\alpha : \alpha < \kappa\}$ of $P_{\kappa,1}$-names that have no infinite pseudointersection. With this fixed enumeration, one then recursively ensures that, for $\gamma < \kappa^+$, no $P_{\alpha,\gamma}$-name will be a subset of $\dot{a}_\beta$ for any $\beta \geqslant \alpha$. In the construction introduced in [9], we instead continually add to the list a $P_{0,\gamma+1}$-name $\dot{a}_\gamma$ and at stage $\mu < \kappa^+$, we adopt a new enumeration of $\{\dot{a}_\alpha : \alpha < \mu\}$ in order-type κ (coherent with previous enumerations) and again ensure that no $P_{\alpha,\mu+1}$-name is a subset of any $\dot{a}_\beta$ for β not listed before α in this new μ-th enumeration. We utilize a $\square$-principle to make these enumerations sufficiently coherent. The greater flexibility in the definition of $\kappa \times \kappa^+$-matrix of posets makes this possible.

Proposition 5.1 ([18, 1.9])**.** *If $P \lessdot P'$ are ccc posets, and $\dot{\mathcal{D}} \subset \dot{\mathcal{E}}$ are, respectively, a P-name and a P'-name, of ultrafilters on ω, then $P * \mathbb{L}(\dot{\mathcal{D}}) \lessdot P' * \mathbb{L}(\dot{\mathcal{E}})$.*

Definition 5.2. *A family $\mathcal{A} \subset [\omega]^\omega$ is thin over a model M if for every I in the ideal generated by $\mathcal{A}$ and every infinite family $\mathcal{F} \in M$ consisting of pairwise disjoint finite sets of bounded size, I is disjoint from some member of $\mathcal{F}$.*

It is routine to prove that, for each limit ordinal δ, $\mathcal{C}_\delta$ forces that the family $\{\dot{x}_\alpha : \alpha \in \delta\}$, as defined above, is thin over the ground model. In fact if $\mathcal{A}$ is thin over some model M, then $\mathcal{C}_\delta$ forces that $\mathcal{A} \cup \{\dot{x}_\alpha : \alpha \in \delta\}$ is also thin over M. This is the notion we use to control that property (1) of the definition of a (κ, κ^+)-shattering sequence will be preserved while at the same time raising the value of $\mathfrak{s}$.

We first note that Proposition 3.7 extends to include this concept.

Proposition 5.3. *Suppose that M is a model of (a sufficient amount of) set-theory and that $\mathcal{A} \subset [\omega]^\omega$ is thin over M. Then for any poset P such that $P \in M$ and $P \subset M$, $\mathcal{A}$ is thin over the forcing extension by P.*

Proof. Let $\{\dot{F}_\ell : \ell \in \omega\}$ be P-names and suppose that $p \in P$ forces that $\{\dot{F}_\ell : \ell \in \omega\}$ are pairwise disjoint subsets of $[\omega]^k$. Also let I be any member of the ideal generated by $\mathcal{A}$. Working in M, recursively choose $q_j < p$ $(j \in \omega)$ and H_j, ℓ_j so that $q_j \Vdash \dot{F}_{\ell_j} = \check{H}_j$ and $H_j \cap \bigcup\{H_i : i < j\} = \emptyset$. The sequence $\{H_j : j \in \omega\}$ is a family in M of pairwise disjoint sets of cardinality k. Therefore there is a j with $H_j \cap I = \emptyset$. This proves that p does not force that I meets every member of $\{\dot{F}_\ell : \ell \in \omega\}$. $\square$

Lemma 5.4 ([9, 3.8])**.** *Let κ be a regular uncountable cardinal and let $\{P_\beta : \beta \leqslant \kappa\}$ be a $<\cdot$-increasing chain of ccc posets with $P_\kappa = \bigcup\{P_\alpha : \alpha < \kappa\}$. Assume that, for each $\beta < \kappa$, $\dot{\mathcal{A}}_\beta$ is a $P_{\beta+1}$-name of a subset of $[\omega]^\omega$ that is forced to be thin over the forcing extension by P_β. Also let $\dot{\mathcal{D}}_0$ be a $P_0 * \mathcal{C}_{\{0\}\times\mathfrak{c}}$-name that is forced to be a Ramsey ultrafilter on ω. Then there is a sequence $\langle \dot{\mathcal{D}}_\beta : 0 < \beta < \kappa\rangle$ such that for all $\alpha < \beta < \kappa$:*

1. *$\dot{\mathcal{D}}_\beta$ is a $P_\beta * \mathcal{C}_{(\beta+1)\times\mathfrak{c}}$-name,*
2. *$\dot{\mathcal{D}}_\alpha$ is a subset of $\dot{\mathcal{D}}_\beta$,*
3. *$P_\beta * \mathcal{C}_{(\beta+1)\times\mathfrak{c}}$ forces that $\dot{\mathcal{D}}_\beta$ is a Ramsey ultrafilter,*
4. *$P_\alpha * \mathcal{C}_{(\alpha+1)\times\mathfrak{c}} * \mathbb{L}(\dot{\mathcal{D}}_\alpha) <\cdot P_\beta * \mathcal{C}_{(\alpha+1)\times\mathfrak{c}} * \mathbb{L}(\dot{\mathcal{D}}_\beta)$, and*
5. *$P_\beta * \mathcal{C}_{(\beta+1)\times\mathfrak{c}} * \mathbb{L}(\dot{\mathcal{D}}_\beta)$ forces that $\dot{\mathcal{A}}_\beta$ is thin over the forcing extension by $P_\alpha * \mathcal{C}_{(\alpha+1)\times\mathfrak{c}} * \mathbb{L}(\dot{\mathcal{D}}_\alpha)$.*

Lemma 5.5 ([9, 2.7])**.** *Assume that $P_{0,0} <\cdot P_{1,0}$ and that $\dot{\mathcal{A}}$ is a $P_{1,0}$-name of a subset of $[\omega]^\omega$. Assume that $\langle P_{0,\xi} : \xi < \delta\rangle$ and $\langle P_{1,\xi} : \xi < \delta\rangle$ are $<\cdot$-chains such that $P_{0,\xi} <\cdot P_{1,\xi}$ for all $\xi < \delta$, and that $P_{1,\xi}$ forces that $\dot{\mathcal{A}}$ is thin over the forcing extension by $P_{0,\xi}$ for all $\xi < \delta$. Then $P_{1,\delta} = \bigcup\{P_{1,\xi} : \xi < \delta\}$ forces that $\mathcal{A}$ is thin over the forcing extension by $P_{0,\delta} = \bigcup\{P_{0,\xi} : \xi < \delta\}$.*

Before proving this next result we recall the notion of a $\square_\kappa$-sequence. For a set C of ordinals, let $\sup(C)$ be the supremum, $\bigcup C$, of C and let $\mathrm{acc}(C)$ denote the set of limit ordinals $\alpha < \sup(C)$ such that $C \cap \alpha$ is cofinal in α. For a limit ordinal α, a set C is a cub in α if $C \subset \alpha = \sup(C)$ and $\mathrm{acc}(C) \subset C$.

Definition 5.6 ([15])**.** *For a cardinal κ, the family $\{C_\alpha : \alpha \in \mathrm{acc}(\kappa^+)\}$ is a $\square_\kappa$-sequence if, for each $\alpha \in \mathrm{acc}(\kappa^+)$:*

(1) C_α is a cub in α,
(2) if $\mathrm{cf}(\alpha) < \kappa$, then $|C_\alpha| < \kappa$,
(3) if $\beta \in \mathrm{acc}(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$.

If there is a $\square_\kappa$-sequence, then $\square_\kappa$ is said to hold.

Theorem 5.7. *It is consistent with $\aleph_1 < \mathfrak{h} < \mathfrak{s} < \mathrm{cf}(\mathfrak{c}) = \mathfrak{c}$ that there is a $(\mathfrak{h}, \mathfrak{s})$-shattering family.*

Proof. We start in a model of GCH satisfying $\square_\kappa$ for some regular cardinal $\kappa > \aleph_1$. Choose any regular $\lambda > \kappa^+$. Fix a $\square_\kappa$-sequence $\{C_\alpha : \alpha \in \mathrm{acc}(\kappa^+)\}$. We may assume that $C_\alpha = \alpha$ for all $\alpha \in \mathrm{acc}(\kappa)$. For each $\alpha \in \mathrm{acc}(\kappa^+)$, let $o(C_\alpha)$ denote the order-type of C_α. When $\mathrm{acc}(C_\alpha)$ is bounded in α with $\eta = \max(\mathrm{acc}(C_\alpha))$, then let $\{\varphi^\alpha_\ell : \ell \in \omega\}$ enumerate $C_\alpha \backslash \eta$ in increasing order.

We will construct a $\kappa \times \kappa^+$-matrix of posets, $\langle P_{\alpha,\xi} : \alpha < \kappa, \xi < \kappa^+ \rangle \in H(\lambda^+)$ and prove that the poset P_{κ,κ^+} as in Lemma 3.2 has the desired properties. For each $\eta < \xi < \kappa^+$, we will also choose an $\iota(\eta, \xi) \in \kappa$ satisfying, as in (3) of the definition of $\kappa \times (\xi+1)$-matrix, that $P_{\alpha,\eta} \lessdot P_{\alpha,\xi}$ for all $\iota(\eta, \xi) \leqslant \alpha < \kappa$. We construct this family by recursion on $\xi < \kappa^+$, and, for each $\xi < \kappa^+$, we let $P_{\kappa,\xi}$ denote the poset $\bigcup\{P_{\alpha,\xi} : \alpha < \kappa\}$ as in Lemma 3.2.

We will recursively define two other families. For each $\alpha < \kappa$ and $\xi < \kappa^+$, we will define a set $\mathrm{supp}(P_{\alpha,\xi}) \subset \xi$ that can be viewed as the union of the supports of the elements of $P_{\alpha,\xi}$ and will satisfy that $\{\mathrm{supp}(P_{\alpha,\xi}) : \alpha < \kappa\}$ is increasing and covers ξ. For each limit $\eta < \kappa^+$ of cofinality less than κ and each $n \in \omega$, we will select a canonical $P_{\kappa,\eta+n+1}$-name, $\dot{a}_{\eta+n}$ of a subset ω that is forced to be Cohen over the forcing extension by $P_{\kappa,\eta}$. While this condition looks awkward, we simply want to avoid this task at limits of cofinality κ. Needing notation for this, let $E = \kappa^+ \backslash \bigcup\{[\eta, \eta + \omega) : \mathrm{cf}(\eta) = \kappa\}$.

For each $\alpha < \kappa$ and $\xi < \eta < \kappa^+$, we define $\mathcal{A}_{\alpha,\xi,\eta}$ to be the family $\{\dot{a}_\gamma : \gamma \in E \cap \eta \backslash \mathrm{supp}(P_{\alpha,\xi})\}$. The intention is that for all $\alpha < \kappa$ and $\xi \leqslant \eta < \kappa^+$, $\mathcal{A}_{\alpha,\xi,\eta}$ is a family of $P_{\kappa,\eta}$-names which is forced by the poset $P_{\kappa,\eta}$ to be thin over the forcing extension by $P_{\alpha,\xi}$. Let us note that if $\alpha < \beta$ and $\xi \leqslant \eta < \kappa^+$, then $\mathcal{A}_{\alpha,\xi,\eta} \backslash \mathcal{A}_{\beta,\eta,\eta}$ should then be a set of $P_{\beta,\eta}$-names. By ensuring that $\mathrm{supp}(P_{\alpha,\xi})$ has cardinality less than κ for all $\alpha < \kappa$ and $\xi < \kappa^+$, this will ensure that the family $\{\dot{a}_\eta : \eta \in E\}$

is (κ, κ^+)-shattering. For each $\eta < \kappa^+$ with cofinality κ we will ensure that $P_{\kappa,\eta+1}$ has the form $P_{\kappa,\eta} * \mathcal{C}_{\kappa\times\lambda}$ and that $P_{\kappa,\eta+2} = P_{\kappa,\eta+1} * \mathbb{L}(\dot{\mathcal{D}}_{\kappa,\eta})$ for a $P_{\kappa,\eta+1}$-name $\dot{\mathcal{D}}_{\kappa,\eta}$ of an ultrafilter on ω. This will ensure that $\mathfrak{c} \geqslant \lambda$ and $\mathfrak{s} = \kappa^+$. The sequence defining $P_{\kappa,\eta+3}$ will be devoted to ensuring that $\mathfrak{p} \geqslant \kappa$.

We start the recursion in a rather trivial fashion. For each $\alpha < \kappa$, $P_{\alpha,0} = \mathcal{C}_\omega$ and, for each $n \in \omega$, $P_{\alpha,n+1} = P_{\alpha,n} * \mathcal{C}_\omega$. We may also let $\iota(n,m) = 0$ for all $n < m < \omega$. For each $n \in \omega$, let $\dot{a}_n$ be the canonical name of the Cohen real added by the second coordinate of $P_{\kappa,n+1} = P_{\kappa,n} * \mathcal{C}_\omega$. For each $\alpha < \kappa$ and $n \in \omega$, define $\mathrm{supp}(P_{\alpha,n})$ to be n. It should be clear that $P_{\kappa,\omega}$ forces that, for each $\alpha < \kappa$ and $n \in \omega$, the family $\{\dot{a}_m : n \leqslant m \in \omega\}$ is thin over the forcing extension by $P_{\alpha,n}$. Assume that P is a poset whose elements are functions with domain a subset of an ordinal ξ. We adopt the notational convention that for a P-name $\dot{Q}$ for a poset, $P *_\xi \dot{Q}$ will denote the representation of $P * \dot{Q}$ whose elements have the form $p \cup \{(\xi, \dot{q})\}$ for $(p, \dot{q}) \in P * \dot{Q}$.

We will prove, by induction on limit $\zeta < \kappa^+$, there is a $\kappa\times\zeta$-matrix $\{P_{\alpha,\xi} : \alpha < \kappa, \xi < \zeta\}$ satisfying conditions (1)-(10):

(1) for all $\alpha < \beta < \kappa$ and $\xi < \eta < \zeta$, if $P_{\alpha,\xi} <\!\cdot\, P_{\beta,\eta}$, then the poset $P_{\beta,\eta}$ forces that the family $\mathcal{A}_{\alpha,\xi,\eta} \setminus \mathcal{A}_{\beta,\eta,\eta}$ is thin over the forcing extension by $P_{\alpha,\xi}$,

(2) for all $\alpha < \kappa$ and $\xi < \zeta$, the elements p of the poset $P_{\alpha,\xi}$ are functions that have a finite domain, $\mathrm{dom}(p)$, contained in ξ,

(3) if $\mathrm{acc}(C_\zeta)$ is cub in ζ and $\eta \in \mathrm{acc}(C_\zeta)$, then
  (a) $P_{n,\zeta}$ is the trivial poset and $\mathrm{supp}(P_{n,\zeta}) = \varnothing$ for $n \in \omega$,
  (b) $P_{\alpha,\zeta} = P_{\alpha,\eta}$ and $\mathrm{supp}(P_{\alpha,\zeta}) = \mathrm{supp}(P_{\alpha,\eta})$ for all $o(C_\eta) \leqslant \alpha < o(C_\eta) + \omega$, and
  (c) $P_{\alpha,\zeta} = \bigcup\{P_{\alpha,\eta} : \eta \in \mathrm{acc}(C_\zeta)\}$ and $\mathrm{supp}(P_{\alpha,\zeta}) = \bigcup\{\mathrm{supp}(P_{\alpha,\eta}) : \eta \in \mathrm{acc}(C_\zeta)\}$, for all $o(C_\zeta) \leqslant \alpha < \kappa$,

  also, let $\iota(\eta,\zeta) = o(C_\eta)$ for all $\eta \in \mathrm{acc}(C_\zeta)$ and, for all $\gamma < \zeta \setminus \mathrm{acc}(C_\zeta)$, let $\iota(\gamma,\zeta) = \iota(\gamma,\eta)$ where $\eta = \min(\mathrm{acc}(C_\zeta)\setminus\gamma)$,

(4) if $\max(\mathrm{acc}(C_\zeta))<\zeta$ then let
$$\iota_\zeta = \max(o(C_\zeta), \sup\{\iota(\varphi^\zeta_\ell, \varphi^\zeta_{\ell'} + n) : \ell \leqslant \ell' < n < \omega\}) \text{ and}$$
  (a) set $P_{\alpha,\zeta} = P_{\alpha,\varphi^\zeta_0}$ and $\mathrm{supp}(P_{\alpha,\zeta}) = \mathrm{supp}(P_{\alpha,\varphi^\zeta_0})$ for all $\alpha < \iota_\zeta$,
  (b) set, for $\iota_\zeta \leqslant \alpha < \kappa$, $P_{\alpha,\zeta} = \bigcup\{P_{\alpha,\varphi^\zeta_\ell+n} : \ell, n \in \omega\}$ and $\mathrm{supp}(P_{\alpha,\zeta}) = \bigcup\{\mathrm{supp}(P_{\alpha,\varphi^\zeta_\ell+n}) : \ell, n \in \omega\}$
  (c) for each $\gamma \in \varphi^\zeta_0$ let $\iota(\gamma,\zeta) = \iota(\gamma,\varphi^\zeta_0)$, let $\iota(\varphi^\zeta_0,\zeta) = o(C_\gamma)$, and for each $\varphi^\zeta_0 < \gamma < \zeta$, $\iota(\gamma,\zeta)$ is the maximum of ι_ζ and $\min\{\iota(\gamma, \varphi^\zeta_\ell+n) : \ell, n \in \omega \text{ and } \gamma < \varphi^\zeta_\ell+n\}$

(5) if $o(C_\zeta) < \kappa$, then for all $\alpha < \kappa$ and $n \in \omega$
   (a) $P_{\alpha,\zeta+n+1} = P_{\alpha,\zeta+n} *_{\zeta+n} \mathcal{C}_\omega$,
   (b) $\dot{a}_{\zeta+n}$ in the canonical $P_{0,\zeta+n} *_{\zeta+n} \mathcal{C}_\omega$-name for the Cohen real added by the second coordinate copy of $\mathcal{C}_\omega$,
   (c) $\mathrm{supp}(P_{\alpha,\zeta+n+1}) = \mathrm{supp}(P_{\alpha,\zeta}) \cup [\zeta, \zeta+n]$, and
   (d) $\iota(\zeta+k, \zeta+n+1) = 0$ for all $k \leqslant n$, and, for all $\gamma < \zeta$, $\iota(\gamma, \zeta+n+1) = \iota(\gamma, \zeta)$,
(6) if $o(C_\zeta) = \kappa$, then for all $\alpha < \kappa$, $P_{\alpha,\zeta+1} = P_{\alpha,\zeta} *_\zeta \mathcal{C}_{\alpha+1\times\lambda}$,
(7) if $o(C_\zeta) = \kappa$, then for all $n \in \omega$ and all $\alpha < \kappa$, $P_{\alpha,\zeta+3+n} = P_{\alpha,\zeta+3}$,
(8) if $o(C_\zeta) = \kappa$, then there is an $\iota_\zeta < \kappa$ such that $P_{\beta,\zeta+2} = P_{\beta,\zeta+1}$ for all $\beta < \iota_\zeta$, and there is a sequence $\langle \dot{\mathcal{D}}_{\alpha,\zeta} : \iota_\zeta \leqslant \alpha < \kappa \rangle$ such that, for each $\iota_\zeta \leqslant \alpha < \kappa$:
   (a) $\dot{\mathcal{D}}_{\alpha,\zeta}$ is a $P_{\alpha,\kappa+1}$-name of a Ramsey ultrafilter on ω,
   (b) for each $\iota_\zeta \leqslant \beta < \alpha$, $\dot{\mathcal{D}}_{\beta,\zeta} \subset \dot{\mathcal{D}}_{\alpha,\zeta}$,
   (c) $P_{\alpha,\zeta+2} = P_{\alpha,\zeta+1} *_{\zeta+1} \mathbb{L}(\dot{\mathcal{D}}_{\alpha,\kappa})$,
(9) if $o(C_\zeta) = \kappa$, then for ι_ζ chosen as in (8)
   (a) for each $\alpha < \iota_\zeta$, $P_{\alpha,\kappa+3} = P_{\alpha,\kappa+2}$,
   (b) $P_{\iota_\zeta,\zeta+3} = P_{\iota_\zeta,\zeta+2} *_{\zeta+2} \dot{Q}_{\iota_\zeta,\zeta+2}$ for some $P_{\iota_\zeta,\zeta}$-name, $\dot{Q}_{\iota_\zeta,\zeta+2}$ in $H(\lambda^+)$ of a finite support product of σ-centered posets,
   (c) for each $\iota_\zeta < \alpha < \kappa$, $P_{\alpha,\zeta+3} = P_{\alpha,\zeta+2} *_{\zeta+2} \dot{Q}_{\iota_\zeta,\zeta+2}$,
(10) if $o(C_\zeta) = \kappa$, then for all $\alpha < \kappa$, $n \in \omega$, and $\gamma < \zeta$, $\mathrm{supp}(P_{\alpha,\zeta+n+1}) = \mathrm{supp}(P_{\alpha,\zeta}) \cup [\zeta, \zeta+n]$, $\iota(\gamma, \zeta+n) = \iota(\gamma, \zeta)$, and $\iota(\zeta+k, \zeta+n) = \iota_\zeta$ for all $k < n \in \omega$,

It should be clear from the properties, and by induction on ζ, that for all $\alpha < \kappa$ and $\xi < \zeta$, each $p \in P_{\alpha,\xi}$ is a function with finite domain contained in $\mathrm{supp}(P_{\alpha,\xi})$. Similarly, it is immediate from the hypotheses that $\mathrm{supp}(P_{\alpha,\xi})$ has cardinality less than κ for all $(\alpha, \xi) \in \kappa\times\kappa^+$.

Before verifying the construction, we first prove, by induction on ζ, that, the conditions (2)-(10) ensure that for all $\xi \leqslant \zeta$ and $\eta \in \mathrm{acc}(C_\xi)$,

Claim (a): $P_{\alpha,\eta} \lessdot P_{\alpha,\xi}$ for all $o(C_\eta) + \omega \leqslant \alpha \in \kappa$,
Claim (b): $P_{\alpha,\eta} = P_{\alpha,\xi}$ for all $\alpha < o(C_\eta) + \omega$

If $o(C_\xi) \leqslant \alpha$, then $P_{\alpha,\eta} \lessdot P_{\alpha,\xi}$ follows immediately from clause 2(c) and, by induction, clauses 3(a). Now assume $\alpha < o(C_\xi) + \omega$. If $\mathrm{acc}(C_\xi)$ is not cofinal in ξ, then, by induction, $P_{\alpha,\eta} = P_{\alpha,\varphi^\xi_0}$ and by clause 3(a) $P_{\alpha,\varphi^\xi_0} = P_{\alpha,\xi}$. If $\mathrm{acc}(C_\xi)$ is cofinal in ξ, then choose $\bar\eta \in \mathrm{acc}(C_\xi)$ so that $o(C_{\bar\eta}) \leqslant \alpha < o(C_{\bar\eta}) + \omega$. By clause 2(b), $P_{\alpha,\xi} = P_{\alpha,\bar\eta}$. By the inductive assumption, $P_{\alpha,\eta} = P_{\alpha,\bar\eta}$ since one of $\eta = \bar\eta, \eta \in \mathrm{acc}(C_{\bar\eta})$ or $\bar\eta \in \mathrm{acc}(C_\eta)$ must hold.

The second thing we check is that the conditions (2)-(10) also ensure that, for each $\zeta < \kappa^+$, $\langle P_{\alpha,\eta} : \alpha < \kappa, \eta < \zeta\rangle$ is a $\kappa\times\zeta$-matrix. We assume, by induction on limit ζ, that for $\gamma < \eta < \zeta$, $\{P_{\alpha,\gamma} : \alpha < \kappa\}$ is a $<\!\cdot$-chain and that $P_{\alpha,\gamma} <\!\cdot\, P_{\alpha,\eta}$ for all η with $\iota(\gamma,\eta) \leqslant \alpha < \kappa$. We check the details for $\zeta+1$ and skip the easy subsequent verification for $\zeta+n$ ($n \in \omega$). Suppose first that $\mathrm{acc}(C_\zeta)$ is cofinal in ζ and let $\iota(\gamma,\zeta) \leqslant \alpha < \kappa$ for some $\gamma < \zeta$. Of course we may assume that $\gamma \notin \mathrm{acc}(C_\zeta)$. Since $\mathrm{acc}(C_\zeta)$ is cofinal in ζ, let $\eta = \min(\mathrm{acc}(C_\zeta)\backslash\gamma)$. By induction, $P_{\alpha,\gamma} <\!\cdot\, P_{\alpha,\eta} <\!\cdot\, P_{\alpha,\zeta}$. Now assume that $\mathrm{acc}(C_\zeta)$ is not cofinal in ζ. If $\gamma \leqslant \varphi^\zeta_0$, then $\iota(\gamma,\zeta) = \iota(\gamma,\varphi^\zeta_0)$, and so we have that $P_{\alpha,\gamma} <\!\cdot\, P_{\alpha,\varphi^\zeta_0} <\!\cdot\, P_{\alpha,\zeta}$. If $\varphi^\zeta_0 < \gamma$, then choose any $\ell \in \omega$ so that $\gamma < \varphi^\zeta_\ell$. By construction, $\iota(\gamma,\zeta) \geqslant \iota(\gamma,\varphi^\zeta_\ell)$ and so, for $\iota(\gamma,\zeta) \leqslant \alpha < \kappa$, $P_{\alpha,\gamma} <\!\cdot\, P_{\alpha,\varphi^\zeta_\ell} <\!\cdot\, P_{\alpha,\zeta}$.

Now we consider the values of $\mathcal{A}_{\alpha,\xi,\eta}$ for $\alpha < \kappa$ and $\omega \leqslant \xi \leqslant \eta$ by examining the names $\dot{a}_\gamma$ for $\gamma \in E$.

By clause (5), $\dot{a}_\gamma$ is a $P_{0,\gamma+1}$-name and γ is in the domain of each $p \in P_{0,\gamma+1}$ appearing in the name. One direction of this next claim is then obvious given that the domain of every element of $P_{\alpha,\xi}$ is a subset of $\mathrm{supp}(P_{\alpha,\xi})$.

Claim (c): $\dot{a}_\gamma$ is a $P_{\alpha,\xi}$-name, if and only if $\gamma \in \mathrm{supp}(P_{\alpha,\xi})$.

Assume that $\gamma \in \mathrm{supp}(P_{\alpha,\xi})$. We prove this by induction on ξ. If ξ is a limit, then $\mathrm{supp}(P_{\alpha,\xi})$ is defined as a union, hence there is an $\eta < \xi$ such that $\gamma \in \mathrm{supp}(P_{\alpha,\eta})$ and $P_{\alpha,\eta} <\!\cdot\, P_{\alpha,\xi}$. If $\xi = \eta + n$ for some limit η and $n \in \omega$, then $P_{\alpha,\eta} <\!\cdot\, P_{\alpha,\xi}$ and so we may assume that $\eta \leqslant \gamma = \eta+k < \eta+n$ and that $o(C_\eta) < \kappa$. Since $P_{0,\eta+k} <\!\cdot\, P_{\alpha,\eta+k} <\!\cdot\, P_{\alpha,\eta+n} = P_{\alpha,\xi}$, it follows that $\dot{a}_\gamma$ is a $P_{\alpha,\xi}$-name.

We prove by induction on ξ (ξ a limit) that for all $\gamma < \xi$:

Claim (d): for all $\alpha < \iota(\gamma{+}1,\xi)$, γ is not in $\mathrm{supp}(P_{\alpha,\xi})$.

First consider the case that $\mathrm{acc}(C_\xi)$ is cofinal in ξ and let η be the minimum element of $\mathrm{acc}(C_\xi)\backslash(\gamma{+}1)$. By definition $\iota(\gamma{+}1,\xi)$ is equal to $\iota(\gamma{+}1,\eta)$ and the claim follows since we have that $\mathrm{supp}(P_{\iota(\gamma+1,\xi),\zeta}) = \mathrm{supp}(P_{\iota(\gamma+1,\xi),\eta})$. Now assume that $\mathrm{acc}(C_\xi)$ is not cofinal in ξ and assume that $\alpha < \iota(\gamma{+}1,\xi)$. We break into cases: $\gamma < \varphi^\xi_0$ and $\varphi^\xi_0 \leqslant \gamma < \xi$. In the first case $\iota(\gamma,\xi) = \iota(\gamma,\varphi^\xi_0)$ and the claim follows by induction and the fact that $\mathrm{supp}(P_{\alpha,\varphi^\xi_0}) = \mathrm{supp}(P_{\alpha,\xi})$ for all $\alpha < \iota(\gamma,\xi)$. Now consider $\varphi^\xi_0 \leqslant \gamma < \xi$. If $\alpha < \iota_\xi$, then $P_{\alpha,\xi} = P_{\alpha,\varphi^\xi_0}$ and, since $\iota_\xi \leqslant \iota(\gamma{+}1,\xi)$, γ is not in $\mathrm{supp}(P_{\alpha,\varphi^\xi_0})$. Otherwise, choose $\ell, n \in \omega$ so that $\iota_\xi \leqslant \alpha < \iota(\gamma{+}1,\xi) = \iota(\gamma{+}1,\varphi^\xi_\ell + n)$ as in the definition of $\iota(\gamma,\xi)$. By the minimality in the choice of $\varphi^\xi_\ell + n$, it follows that γ is not in

$\mathrm{supp}(P_{\alpha,\varphi^{\xi}_{\ell'}+n})$ for all $\ell', n \in \omega$. Since $\mathrm{supp}(P_{\alpha,\xi})$ is the union of all such sets, it follows that γ is not in $\mathrm{supp}(P_{\alpha,\xi})$.

Next we prove, by induction on ζ, that the matrix so chosen will additionally satisfy condition (1). We first find a reformulation of condition (1). Note that by Claim (c), $\mathcal{A}_{\alpha,\xi,\eta} = \{\dot{a}_\gamma : \gamma \in E \cap \eta \setminus \mathrm{supp}(P_{\alpha,\xi})\}$.

Claim (e): For each $\alpha < \kappa$ and $\xi < \eta < \zeta$ and finite subset $\{\gamma_i : i < m\}$ of $E \cap \eta \setminus \mathrm{supp}(P_{\alpha,\xi})$ there is a $\beta < \kappa$ such that $\iota(\xi,\eta) \leqslant \beta$, $\{\gamma_i : i < m\} \subset \mathrm{supp}(P_{\beta,\eta})$ and $P_{\beta,\eta}$ forces that $\{\dot{a}_{\gamma_i} : i < m\}$ is thin over the forcing extension by $P_{\alpha,\xi}$.

Let us verify that Claim (e) follows from condition (1). Let α, ξ, η and $\{\gamma_i : i < m\}$ be as in the statement of Claim (e). Choose $\beta < \kappa$ so that $\iota(\xi,\eta)$ and each $\iota(\gamma_i{+}1,\eta)$ is less than β. Then $P_{\alpha,\xi} <\cdot\, P_{\beta,\eta}$ and $\{\dot{a}_{\gamma_i} : i < m\} \subset \mathcal{A}_{\alpha,\xi,\eta} \setminus \mathcal{A}_{\beta,\eta,\eta}$. This value of β satisfies the conclusion of the Claim.

Now assume that Claim (e) holds and we prove that condition (1) holds. Assume that $P_{\alpha,\xi} <\cdot\, P_{\delta,\eta}$. To prove that $\mathcal{A}_{\alpha,\xi,\eta} \setminus \mathcal{A}_{\delta,\eta,\eta}$ is forced by $P_{\delta,\eta}$ to be thin over the forcing extension by $P_{\alpha,\xi}$, it suffices to prove this for any finite subset of $\mathcal{A}_{\alpha,\xi,\eta} \setminus \mathcal{A}_{\delta,\eta,\eta}$. Thus, let $\{\gamma_i : i < m\}$ be any finite subset of $\mathrm{supp}(P_{\delta,\eta}) \cap E \cap \eta \setminus \mathrm{supp}(P_{\alpha,\xi})$. Choose β as in the conclusion of the Claim. If $\beta \leqslant \delta$, then $P_{\delta,\eta}$ forces that $\{\dot{a}_{\gamma_i} : i < m\}$ is thin over the forcing extension because $P_{\beta,\eta} <\cdot\, P_{\delta,\eta}$ does. Similarly, if $\delta < \beta$, then $P_{\delta,\eta}$ being completely embedded in $P_{\beta,\eta}$ can not force that $\{\dot{a}_{\gamma_i} : i < m\}$ is not thin over the forcing extension by $P_{\alpha,\xi}$.

We assume that $\omega \leqslant \zeta < \kappa^+$ is a limit and that $\langle P_{\alpha,\xi} : \alpha < \kappa, \xi < \zeta\rangle$ have been chosen so that conditions (1)-(10) are satisfied. We prove, by induction on $n \in \omega$, that there is an extension $\langle P_{\alpha,\xi} : \alpha < \kappa, \xi < \zeta + n\rangle$ that also satisfies conditions (1)-(10).

For $n = 1$, we define the sequence $\langle P_{\alpha,\zeta} : \alpha < \kappa\rangle$ according to the requirement of (3) or (4) as appropriate. It follows from Lemma 5.5 that (2) will hold for the extension $\langle P_{\alpha,\xi} : \alpha < \kappa, \xi < \zeta{+}1\rangle$. Conditions (3)-(10) hold since there are no new requirements. We must verify that the condition in Claim (e) holds for $\eta = \zeta$. Let α, ξ and $\{\gamma_i : i < m\}$ be as in the statement of Claim (e) with $\eta = \zeta$. Let $C_\zeta = \{\eta_\beta : \beta < o(C_\zeta)\}$ be an order-preserving enumeration. We first deal with case that $\mathrm{acc}(C_\zeta)$ is cofinal in ζ. Choose any $\beta_0 < \kappa$ large enough so that $\gamma_i \in \mathrm{supp}(P_{\beta_0,\zeta})$ for all $i < m$. Choose $\beta_0 < \beta$ so that $\iota(\xi, \eta_{\beta_0}) \leqslant \beta$. Now we have that $P_{\alpha,\xi} <\cdot\, P_{\beta,\eta_{\beta_0}}$ and $P_{\beta,\eta_{\beta_0}} <\cdot\, P_{\beta,\zeta}$. Applying Claim (e) to η_{β_0}, we have that $P_{\beta,\eta_{\beta_0}}$ forces that $\{\dot{a}_{\gamma_i} : i < m\}$ is thin over the forcing extension by $P_{\alpha,\xi}$. As in the proof of Claim (e), this implies that $P_{\beta,\zeta}$ forces the same thing.

Now assume that $\mathrm{acc}(C_\zeta)$ is not cofinal in ζ. If $\alpha < \iota_\zeta$, then apply Claim (e) to choose β so that P_{β,ι_ζ} forces that $\{\dot{a}_{\gamma_i} : i < m\}$ is not thin over the extension by $P_{\alpha,\xi}$. Since $P_{\beta,\iota_\zeta} <\!\cdot\, P_{\beta,\zeta}$ holds for all β, $P_{\beta,\zeta}$ also forces that $\{\dot{a}_{\gamma_i} : i < m\}$ is not thin over the extension by $P_{\alpha,\xi}$. If $\iota_\zeta \leqslant \alpha$, first choose $\delta < \kappa$ large enough so that $\iota(\xi,\zeta)$ and each $\iota(\gamma_i{+}1,\zeta)$ is less than δ. Since $\{\gamma_i : i < m\}$ is a subset of $\mathrm{supp}(P_{\delta,\zeta})$, we can choose $\ell < \omega$ large enough so that $\{\gamma_i : i < \omega\} \subset \mathrm{supp}(P_{\delta,\varphi^\zeta_\ell})$. Applying Claim (e) to $\eta = \varphi^\zeta_\ell$, we choose β as in the Claim. As we have seen, there is no loss to assuming that $\delta \leqslant \beta$ and, since $P_{\beta,\varphi^\zeta_\ell} <\!\cdot\, P_{\beta,\zeta}$, this completes the proof.

If $o(C_\zeta) < \kappa$, then the construction of $\langle P_{\alpha,\zeta+n} : n \in \omega, \alpha < \kappa\rangle$ is canonical so that conditions (2)-(10) hold. We again verify that Claim (e) holds for all values of η with $\zeta < \eta < \zeta{+}\omega$. Let α, ξ and $\{\gamma_i : i < m\}$ be as in Claim (e) for $\eta = \zeta{+}n$. We may assume that assume that $\{\gamma_i : i < m\} \cap \zeta = \{\gamma_i : i < \bar{m}\}$ for some $\bar{m} \leqslant m$. If $\xi < \zeta$, let $\bar{\xi} = \xi$, otherwise, choose any $\bar{\xi} < \zeta$ so that $P_{\alpha,\zeta} = P_{\alpha,\bar{\xi}}$. Note that $\{\gamma_i : \bar{m} \leqslant i < m\}$ is disjoint from the interval $[\zeta, \xi)$. Choose $\beta < \kappa$ to be greater than $\iota(\bar{\xi},\zeta)$ and each $\iota(\gamma_i{+}1,\zeta)$ $(i < \bar{m})$, and so that $P_{\beta,\zeta}$ forces that $\{\dot{a}_{\gamma_i} : i < \bar{m}\}$ is thin over the extension by $P_{\alpha,\bar{\xi}}$. If $\bar{m} = m$ we are done by the fact that $P_{\alpha,\xi}$ is isomorphic to $P_{\alpha,\bar{\xi}} * \mathcal{C}_\omega$. In fact, we similarly have that $P_{\beta,\xi}$ forces that $\{\dot{a}_{\gamma_i} : i < \bar{m}\}$ is thin over the forcing extension by $P_{\alpha,\xi}$. Since $P_{\beta,\zeta+n}$ forces that $\bigcup\{\dot{a}_{\gamma_i} : \bar{m} \leqslant i < m\}$ is a Cohen real over the forcing extension by $P_{\beta,\xi}$ it also follows that $P_{\beta,\zeta+n}$ forces that $\{\dot{a}_{\gamma_i} : i < m\}$ is thin over the extension by $P_{\alpha,\xi}$.

Now we come to the final case where $o(C_\zeta) = \kappa$ and the main step to the proof. The fact that Claim (e) will hold for $\eta = \zeta + 1$ is proven as above for the case when $o(C_\zeta) < \kappa$ and $\mathrm{acc}(C_\zeta)$ is cofinal in ζ. For values of $n > 3$, there is nothing to prove since $P_{\alpha,\zeta+3+k} = P_{\alpha,\zeta+3}$ for all $k \in \omega$. We also note that $\zeta{+}n \notin E$ for all $n \in \omega$.

At step $\eta = \zeta + 2$ we must take great care to preserve Claim (e) and at step $\zeta + 3$ we make a strategic choice towards ensuring that $\mathfrak{p}$ will equal κ. Indeed, we begin by choosing the lexicographic minimal pair, $(\xi_\zeta, \alpha_\zeta)$, in $\zeta{\times}\kappa$ with the property that there is a family of fewer than κ many canonical $P_{\alpha_\zeta,\xi_\zeta}$-names of subsets of ω and a $p \in P_{\alpha_\zeta,\xi_\zeta}$ that forces over $P_{\kappa,\zeta}$ that there is no pseudo-intersection. If there is no such pair, then let $(\alpha_\zeta, \xi_\zeta) = (\omega, \zeta{+}1)$. Choose ι_ζ so that $P_{\alpha_\zeta,\xi_\zeta} <\!\cdot\, P_{\iota_\zeta,\zeta+1}$.

Assume that $\alpha, \xi, \{\gamma_i : i < m\}$ are as in Claim (e). We first check that if $\xi < \zeta + 2$, then there is nothing new to prove. Indeed, simply choose

$\beta < \kappa$ large enough so that $P_{\beta,\zeta+1}$ has the properties required in Claim (e) for $P_{\alpha,\xi}$. Of course it follows that $P_{\beta,\zeta+2}$ forces that $\{\dot{a}_{\gamma_i} : i < m\}$ is thin over the extension by $P_{\alpha,\xi}$ since $P_{\beta,\zeta+1}$ already forces this.

This means that we need only consider instances of Claim (e) in which $\xi = \zeta + 2$. The analogous statement also holds when we move to $\zeta + 3$. For each $\beta < \kappa$, let

$$T_\beta = E \cap \operatorname{supp}(P_{\beta+1,\zeta}) \setminus \operatorname{supp}(P_{\beta,\zeta})$$

and note that $P_{\beta+1,\zeta+1}$ forces that $\{\dot{a}_\gamma : \gamma \in T_\beta\}$ is thin over the extension by $P_{\beta,\zeta+1}$. Most of the work has been done for us in Lemma 5.4. Except for some minor re-indexing, we can assume that the sequence $\{P_\beta : \beta < \kappa\}$ in the statement of Lemma 5.4 is the sequence $\{P_{\beta,\zeta} : \beta < \kappa\}$. We also have that $P_{\beta,\zeta} * \mathcal{C}_{(\beta+1)\times\mathfrak{c}}$ is isomorphic to $P_{\beta,\zeta+1}$. We can choose any $P_{0,\zeta+1}$-name $\dot{\mathcal{D}}_{0,\zeta}$-name of a Ramsey ultrafilter on ω. The family $\{\dot{a}_\gamma : \gamma \in T_\beta\}$ will play the role of $\dot{\mathcal{A}}_\beta$ in the statement of Lemma 5.4, and we let $\{\dot{\mathcal{D}}_{\beta,\zeta} : 0 < \beta < \kappa\}$ be the sequence as supplied in Lemma 5.4.

Now assume that $\alpha < \kappa$ and that $\{\gamma_i : i < m\} \subset E \cap \zeta \setminus \operatorname{supp}(P_{\alpha,\zeta+1})$. Let $\{\dot{F}_\ell : \ell \in \omega\}$ be any sequence of $P_{\alpha,\zeta+2}$-names of pairwise disjoint elements of $[\omega]^k$ for some $k \in \omega$. We must find a sufficiently large $\beta < \kappa$ so that $P_{\beta,\zeta+2}$ forces that $\dot{a}_{\gamma_0} \cup \cdots \cup \dot{a}_{\gamma_{m-1}}$ is disjoint from $\dot{F}_\ell$ for some $\ell \in \omega$. Let $\{\beta_j : j < \bar{m}\}$ be the set (listed in increasing order) of $\beta < \kappa$ such that $T_\beta \cap \{\gamma_i : i < m\}$ is not empty and let $\beta_m = \beta_{m-1} + 1$. By re-indexing we can assume there is a sequence $\{m_j : j \leqslant \bar{m}\} \subset m+1$ so that $\gamma_i \in T_{\beta_j}$ for $m_j \leqslant i < m_{j+1}$. Although $P_{\beta,\zeta+2} = P_{\beta,\zeta+1}$ for values of $\beta < \iota_\zeta$, we will let $\bar{P}_{\beta,\zeta+2} = P_{\beta,\zeta+1} *_{\zeta+1} \mathbb{L}(\dot{\mathcal{D}}_{\beta,\zeta})$ for $\beta < \iota_\zeta$, and for consistent notation, let $\bar{P}_{\beta,\zeta+2} = P_{\beta,\zeta+2}$ for $\iota_\zeta \leqslant \beta < \kappa$. We note that $\{\dot{F}_\ell : \ell \in \omega\}$ is also sequence of $\bar{P}_{\alpha,\zeta+2}$-names of pairwise disjoint elements of $[\omega]^k$.

For each $j < \bar{m}$, let $\dot{L}_{j+1}$ be the $\bar{P}_{\beta_j+1,\zeta+2}$-name of those ℓ such that $\dot{F}_\ell$ is disjoint from $\bigcup\{\dot{a}_{\gamma_i} : i < m_{j+1}\}$. It follows, by induction on $j < \bar{m}$, that $\bar{P}_{\beta_j+1,\zeta+2}$ forces that $\dot{L}_{j+1}$ is infinite since $\bar{P}_{\beta_j+1,\zeta+2}$ forces that $\{\dot{a}_{\gamma_i} : m_j \leqslant i < m_{j+1}\}$ is thin over the forcing extension by $\bar{P}_{\beta_j,\zeta+2}$. It now follows $\bar{P}_{\beta_m,\zeta+2}$ forces that $\{\dot{a}_{\gamma_i} : i < m\}$ is thin over the forcing extension by $\bar{P}_{\alpha,\zeta+2}$. If $\beta_m < \iota_\zeta$, let $\beta = \iota_\zeta$, otherwise, let $\beta = \beta_m$. It follows that $P_{\beta,\zeta+2}$ forces that $\{\dot{a}_{\gamma_i} : i < m\}$ is thin over the forcing extension by $P_{\alpha,\zeta+2} \lessdot \bar{P}_{\alpha,\zeta+2}$. This completes the verification of Claim (e) for the case $\eta = \zeta+2$ and we now turn to the final case of $\eta = \zeta+3$.

We have chosen the pair $(\alpha_\zeta, \xi_\zeta)$ when choosing ι_ζ. Let $\dot{Q}_{\iota_\zeta,\zeta+2}$ be the $P_{\iota_\zeta,\zeta+2}$-name of the finite support product of all posets of the form

$\mathbb{M}(\mathcal{F})$ where $\mathcal{F}$ is a family of fewer than κ canonical $P_{\alpha_\zeta,\xi_\zeta}$-names of subsets of ω that is forced to have the sfip. Since $P_{\alpha_\zeta,\xi_\zeta} \in H(\lambda^+)$ the set of all such families $\mathcal{F}$ is an element of $H(\lambda^+)$. This is our value of $\dot{Q}_{\iota_\zeta,\zeta+2}$ as in condition (9) for the definition of $P_{\beta,\zeta+3}$ for all $\beta < \kappa$. The fact that Claim (e) holds in this case follows immediately from the induction hypothesis and Proposition 5.3. We also note that $P_{\iota_\zeta,\zeta+3}$ forces that every family of fewer than κ many canonical $P_{\alpha_\zeta,\xi_\zeta}$-names that is forced to have the sfip is also forced, by $P_{\kappa,\zeta+3}$ to have a pseudo-intersection. This means that for values of $\zeta' > \zeta$ with $o(\mathrm{acc}(C_\zeta)) = \kappa$, the pair $(\alpha_\zeta, \xi_\zeta)$ will be lexicographically strictly smaller than the choice for ζ'. In other words, the family $\{(\xi_\zeta, \alpha_\zeta) : \zeta < \kappa^+, \mathrm{cf}(\zeta) = \kappa\}$ is strictly increasing in the lexicographic ordering.

Now we can verify that P_{κ,κ^+} forces that $\mathfrak{p} \geqslant \kappa$. If it does not, then there is a $\delta < \kappa$ and a family, $\{\dot{y}_\gamma : \gamma < \delta\}$ of canonical P_{κ,κ^+}-names of subsets of ω with some $p \in P_{\kappa,\kappa^+}$ forcing that the family has sfip but has no pseudo-intersection. By an easy modification of the names, we can assume that every condition in P_{κ,κ^+} forces that the family $\{\dot{y}_\gamma : \gamma < \delta\}$ is forced to have sfip. Choose any $\xi < \kappa^+$ so that $p \in P_{\kappa,\xi}$ and every $\dot{y}_\gamma$ is a $P_{\kappa,\xi}$-name. Choose $\alpha < \kappa$ large enough so that $p \in P_{\alpha,\xi}$, $\iota(\bar{\zeta}, \xi)$, and each α_γ $(\gamma < \delta)$ is less than α. It follows that $\dot{y}_\gamma$ is a $P_{\alpha,\xi}$-name for all $\gamma < \delta$. Since the family $\{(\xi_\zeta, \alpha_\zeta) : \zeta < \kappa^+, \mathrm{cf}(\zeta) = \kappa\}$ is strictly increasing in the lexicographic ordering, and this ordering on $\kappa^+ \times \kappa$ has order type κ^+, there is a minimal $\zeta < \kappa^+$ (with $\mathrm{cf}(\zeta) = \kappa$) such that $(\xi, \alpha) \leqslant (\xi_\zeta, \alpha_\xi)$. By the assumption on (α, ξ), (ξ_ζ, α_ξ) will be chosen to equal (ξ, α). One of the factors of the poset $\dot{Q}_{\iota_\zeta,\zeta+2}$ will be chosen to be $\mathbb{M}(\{\dot{y}_\gamma : \gamma < \delta\})$. This proves that $P_{\kappa,\zeta+3}$ forces $\{\dot{y}_\gamma : \gamma < \delta\}$ does have a pseudo-intersection.

It should be clear from condition (8) in the construction that P_{κ,κ^+} forces that $\mathfrak{s} \geqslant \kappa^+$. To finish the proof we must show that P_{κ,κ^+} forces that $\{\dot{a}_\gamma : \gamma \in E\}$ is (κ, κ^+)-shattering. Since $\dot{a}_\gamma$ is forced to be a Cohen real over the extension by $P_{\kappa,\gamma}$, condition (2) in the Definition 2.3 of (κ, κ^+)-shattering holds. Finally, we verify condition (1) of Definition 2.3. Choose any P_{κ,κ^+}-name $\dot{b}$ of an infinite subset of ω. Choose any $(\alpha, \xi) \in \kappa\times\kappa^+$ so that $\dot{b}$ is a $P_{\alpha,\xi}$-name. The set $E \cap \mathrm{supp}(P_{\alpha,\xi})$ has cardinality less than κ. For any $\gamma \in E\setminus \mathrm{supp}(P_{\alpha,\xi})$, there is a $(\beta, \zeta) \in \kappa\times\kappa^+$ such that $\{\dot{a}_\gamma\}$ is thin over the forcing extension by $P_{\alpha,\xi}$. It follows trivially that $P_{\beta,\zeta}$ forces that $\dot{b}$ is not a (mod finite) subset of $\dot{a}_\gamma$. $\square$

6. Questions

(1) Is it consistent to have $\omega_1 < \mathfrak{h} < \mathfrak{b} < \mathfrak{s}$ and $\mathfrak{c}$ regular?

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