An Excursion in Transformational Geometry

1 Introduction

There are two versions of this paper, one for teachers and one for students. The one for students leaves out the ideas on how to make best use of the paper. You are reading the teacher version excursiont.pdf This version also includes the solutions to the questions posed. The point of the paper is to enable the teacher to introduce several topics that are relevant to middle and high school mathematics including arithmetic counting problems (how many sets of three digits have a sum of 15), two person game theory (thinking ahead using strategies), and transformational geometry. The compelling aspect of the paper is the *mystery* that students experience when first confronted with the add-to-15 game. They want to resolve it and they don't see how. As the paper develops, they are led to the familiar game Tic-Tac-Toe, and finally they see that the add-to-15 game is just a disguised version of Tic-Tac-Toe. And this leads to the geometric excursion. Another important goal of the paper is to give the student a good idea of the concept of *isomorphism*. The three games discussed here are all isomorphic to one-another.[1]

2 The add-to-15 game.

Two players alternately choose a digit from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The first one to achieve a three-element subset whose sum is 15 wins the game. Here, Ashley and Betsy play: Ashley picks 5, then Betsy picks 3. Then Ashley picks 6 so that she has the subset $\{5, 6\}$. Now Betsy picks 4 to bring her subset to $\{3, 4\}$. Then Ashley picks 8, so now she has $\{5, 6, 8\}$. Betsy picks 1 so now she has $\{1, 3, 4\}$. Then Ashley picks 2 to bring her set to $\{2, 5, 6, 8\}$, and she wins because $2 + 5 + 8 = 15$. Is there something special about the number 15? How would the game be different if the target sum was 16? Play this add-to-15 game a few times with your partner, and see if you can figure out how to win.

1. To help understand the game, make a list of the three-element subsets of $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ whose sum is 15.

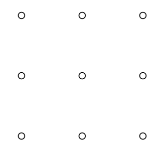
2. Next calculate the number of times each digit appears in one of the sets you found in

    1. Use the table below.

1	2	3	4	5	6	7	8	9

[1] Unauthorized reproduction/photocopying prohibited by law' ©

3. Consider the 3×3 array of dots:



How many straight line segments can you draw that go through three of the dots?

4. Does this remind you of another game?

5. Build a 3×3 magic square using the digits 1 through 9.

6. How does this enable you to play the add-to-15 game with which we started?

3 Word Game

Next, consider a new game that we play with words and letters instead of numbers. You're given nine words, THOU, TANK, TIED, BRIM, WOES, WASP, SHIP, FORM, and HEAR. You and your partner alternately pick a word until at some point someone accumulates three words which all have the same letter. That person wins the game. Play this repeatedly with your partner and see if you can devise a strategy that never loses.

4 Geometry

In this section we turn our attention to the geometry of 3×3 magic squares. Start by taking the one you found above, and see if you can build some other ones.

After some effort you'll find exactly eight of them:

2	7	6
9	5	1
4	3	8

6	1	8
7	5	3
2	9	4

8	3	4
1	5	9
6	7	2

4	9	2
3	5	7
8	1	6

4	3	8
9	5	1
2	7	6

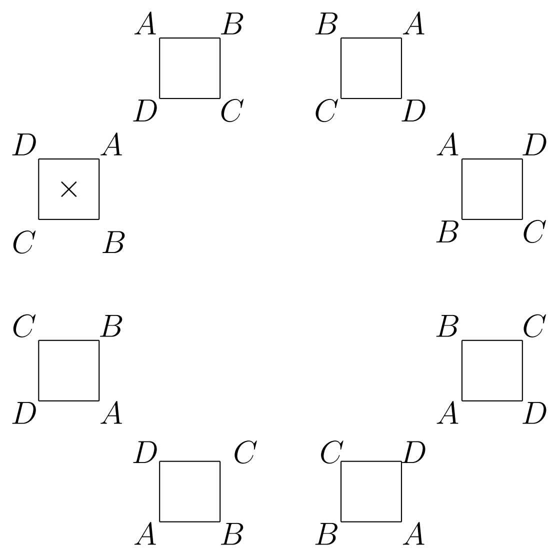
2	9	4
7	5	3
6	1	8

6	7	2
1	5	9
8	3	4

8	1	6
3	5	7
4	9	2

Do you see a pattern? How can you generate all eight of the magic squares starting with just the upper left one? Use the name R to denote the rotation that sends the upper left solution to the next one in that row. That is, R sends $\begin{array}{|c|c|c|}\hline 2&7&6\\\hline 9&5&1\\\hline 4&3&8\\\hline\end{array}$ to $\begin{array}{|c|c|c|}\hline 6&1&8\\\hline 7&5&3\\\hline 2&9&4\\\hline\end{array}$. Let H denote the mapping that sends each magic square solution on the top row to the one just below it on the bottom row. In other words, H is reflection about the middle row. And let V denote reflection about the vertical line through the middle of the square.

1. Draw a line from the square marked with an $\times$ to each of the other seven squares, and label each line with the letter of the transformation that takes the $\times$ square into the other one.



2. **The symmetries of a square.** Build the table of composite motions started below. Note that $R \circ R \circ R \circ R = R^4 = I$ is the identity mapping that leaves the solution unchanged and D_1 and D_2 represent reflections across the main diagonal and the other diagonal respectively. The symbol V represents reflection about the vertical line through the middle of the square. Note that one composition is given, $D_2 \circ H = R^3$. You can verify this by noting that $D_2\left(\begin{array}{|c|c|c|}\hline 2&7&6\\\hline 9&5&1\\\hline 4&3&8\\\hline\end{array}\right) = \begin{array}{|c|c|c|}\hline 8&1&6\\\hline 3&5&7\\\hline 4&9&2\\\hline\end{array}$

   and

   $$H\left(\begin{array}{|c|c|c|}\hline 8&1&6\\\hline 3&5&7\\\hline 4&9&2\\\hline\end{array}\right) = \begin{array}{|c|c|c|}\hline 4&9&2\\\hline 3&5&7\\\hline 8&1&6\\\hline\end{array} = R^3\left(\begin{array}{|c|c|c|}\hline 2&7&6\\\hline 9&5&1\\\hline 4&3&8\\\hline\end{array}\right)$$

$\circ$	I	R	R^2	R^3	H	V	D_1	D_2
I								
R								
R^2								
R^3								
H								
V								
D_1								
D_2					R^3			

When you finish building the table, what patterns do you notice. Is the operation commutative? Is there an identity? If so, which motions have inverses?

3. Identify the symmetries of an equilateral triangle and build the group table for them.
4. Imagine that you have a perfectly square mattress that needs to be turned every year to insure durability. Is there a way to turn the mattress so that you get all eight orientations of the mattress by making this turn seven years running?