1 Introduction

The problem set deals with arithmetic expressions. You'll get to deal with some of these in the exercises that follow where you're asked to insert arithmetic operators and parentheses to achieve certain values. We can define an *unambiguously parenthesized arithmetic expression*, which we shorten to *expression* as follows. Any string of symbols that can be *derived* from a start symbol $\langle S \rangle$ using an alphabet $\{a, b, c, \star,), (, \}$ by using one or more of the *productions* that follow:

$$\langle S \rangle \rightarrowtail (\langle S \rangle \star \langle S \rangle)$$

$$\langle S \rangle \rightarrowtail a|b|c$$

For example, $\langle S \rangle \rightarrowtail (\langle S \rangle \star \langle S \rangle)$

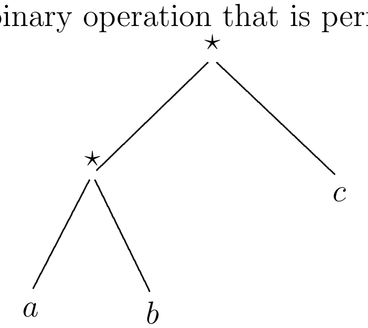
$$\rightarrowtail ((\langle S \rangle \star \langle S \rangle) \star \langle S \rangle)$$

$$\rightarrowtail ((a \star \langle S \rangle) \star \langle S \rangle)$$

$$\rightarrowtail ((a \star b) \star \langle S \rangle)$$

$$\rightarrowtail ((a \star b) \star c)$$

shows that $((a \star b) \star c)$ is an expression. Of course we've simplified enormously, leaving out other usable letters, numbers and operations. In practice we use all these objects. The important thing here is that the notation tells us exactly when to perform each binary operation. Therefore, such expressions lead to what we call binary trees. Notice that the *root* of the tree is labeled with the last binary operation that is performed.



Notice that there are two ways to parenthesize $a \star b \star c$ to get an expression, $(a \star b) \star c$ and $a \star (b \star c)$. Note also that there are five ways to parenthesize

$a \star b \star c \star d$. Can you write them all out? Build the binary tree for each one. Finally, find the number of ways to insert parentheses pairs to get an expressions from $a \star b \star c \star d \star e$. *Reverse Polish Notation* is a listing of the items of an arithmetic expression leaving out the parentheses. To get this, traverse the tree in clockwise order starting at the root, and writing down each item in reverse order. So the tree above gives rise to $a \quad b \quad \star \quad c \quad \star$. Write each of your expressions for $a \star b \star c \star d \star e$ in reverse Polish notation.

2 Build 100 and other Problems

1. Consider each list of numbers below. Insert arithmetic symbols $+, -, \times, \div$ and left and right parentheses, so that the value of each expression has value 100. For example if the list is

$$7 \quad 1 \quad 9 \quad 1 \quad 3 \quad 6 \quad 6,$$

you could write $((7+1)\cdot(9-1))+(3\cdot(6+6))$

(a)

$$11 \quad 4 \quad 7 \quad 3 \quad 7$$

(b)

5 6 2 2 4 6 7 3

(c)

10 1 10 1 5

(d)

2 9 9 5 4 3 3 2 6

(e)

3 7 17 3 2

(f)

5 4 3 3 23 11 2

(g) This puzzle is due to Nandana Marpadaga, Charlotte, NC.

30 10 2 6 4

(h) This puzzle is due to Shauna Sosankin, Charlotte, NC.

96 3 2 4 9

(i) This puzzle is due to Sahana Balakrishnan, Charlotte, NC.

12 10 3 15 3 4

(j) This puzzle is due to Naima Chowdhury, Charlotte, NC.

2 3 5 2 5 8

(k) This puzzle is due to Eshal Chowdhury, Charlotte, NC.

3 5 2 4 3 8 2

(l) This puzzle is due to Rohan Hansalia, Charlotte, NC.

2 25 10 1 16 4

(m) This puzzle is due to Watson Hauck, Charlotte, NC.

2 25 10 1 16 4

(n) This puzzle is due to Nelson Huang, Charlotte, NC.

144 4 3 7 2 3

(o) This puzzle is due to Sanjan Bhoothapuri, Charlotte, NC.

2 25 10 1 16 4

(p) This puzzle is due to Aaliyah Smith, Charlotte, NC.

5 5 30 3

(q) This puzzle is due to Nikhil Mehta, Charlotte, NC.

24 6 12 16 8

(r) This puzzle is due to Agastya Hari Kumar, Charlotte, NC.

13 6 81 3 2 53

(s) This puzzle is due to Alex Mao, Charlotte, NC.

2 25 10 1 16 4

(t) This puzzle is due to Nandana Marpadaga, Charlotte, NC.

30 10 2 6 4

(u) This puzzle is due to Aniruddh Dayananda, Charlotte, NC.

30 29 4 5 2 10 5 2

(v) This puzzle is due to Krish Korrapati, Charlotte, NC.

30 89 9 3 7 5 11

(w) This puzzle is due to Paneet Lingutla, Charlotte, NC.

30 5 6 2 2 4 10

(x) This puzzle is due to Harava Rahardjo

13 8 2 5 1 2 6

(y) This puzzle is due to Aaron Wahid.

6 3 20 4 10 14

(z) This puzzle is due to Tobias Budiman.

66 7 8 24 14

(aa) This puzzle is due to Bear Scofield, Salisbury, NC.

5 6 7 11 2 8

2. For these problems we add two operations to the list, exponentiation ˆ and *concatenation* $*$, $+, -, \times, \div, \hat{}, *$ Concatentation is defined as follows: $a * b = a \cdot 10^{\lfloor \log b \rfloor + 1} + b$ if $b > 0$ and $a * 0 = 10a$. For example, $10 * 12 = 10 \cdot 10^{\lfloor \log 12 \rfloor + 1} + 12 = 10 \cdot 100 + 12 = 1012$. Call a sequence of integers *admissible* if operators and parentheses can be inserted so that the resulting expression is unambiguous and has value 100. For example 1 2 3 4 0 is admissible because $((1 + 2) + (3 + 4)) * 0 = 100$.

   (a)

   10 14 13 12 11

   (b)

   5 3 12 13

   (c)

   3 4 9 10

3. Build the binary tree for each of the expressions in problem 1. And then express each of the arithmetic expressions in problem 1 in Reverse Polish Notation.

4. Show how to replace the operation $\oplus$ in the expression below with each of the symbols $+, -, \times, \div,$ ^ so that the expression

$$9 \oplus 7 \oplus 7 \oplus 1 \oplus 6 \oplus 6 \oplus 1$$

has the value 100. In other words, solve five problems, one for each operation.

5. Let $[n]$ denote the sequence $1 \quad 2 \quad 3 \ldots n$. Find the n for which $[n]$ is admissible.

6. Find the smallest n for which a permutation of $[n]$ is admissible.

7. For each digit d find the shortest sequence of d's which is admissible. For example $5 \quad 5 \quad 5$ is not admissible but $5 \quad 5 \quad 5 \quad 5$ is admissible.

8. The game of 24. Show how to produce the number 24 using the four operations $+, -, \div, \times$ from the numbers $1, 4, 5$, and 6.

9. Four ordinary dice are rolled. The sum S and the product P (or possibly just the product) are given. Your challenge is to determine the values on the four faces. For example if $S = 10$ and $P = 24$, then the four faces are $1, 2, 3, 4$.

   (a) $P = 30$ and $S = 11$.
   (b) $P = 24$ and $S = 11$.
   (c) $P = 24$ and $S = 12$.
   (d) $P = 24$ and $S = 9$.
   (e) $P = 900$.
   (f) $P = 750$.

10. Which values of P by themselves determine the four faces?

11. Are there any values of S which by themselves determine the four faces?

12. Find a pair P, S which do not the four faces.

13. This two-part puzzle is due to Ian Tullis and Wei-Hwa Huang. Combine the four numbers given using the four operations $+, -, \div, \times$, without using parentheses so that the value of the expression is 21. Recall that division and multiplication are evaluated from left to right before + and −, which are then evaluated left to right. For example $9-8\div2\div2=7$. If you're given the set $\{3, 5, 8, 9\}$, you would write $5\times9-8\times3$, which has the operator set $\{-, \times, \times\}$ and this corresponds to the letter F. Once you settle on a way to get 21 with three operations, find the letter that corresponds to that set of operations. Then finally rearrange the set of letters into a single 'symbol' as your final answer.

   (a) $\{3, 5, 8, 9\}$
   (b) $\{2, 3, 17, 24\}$
   (c) $\{4, 8, 8, 9\}$
   (d) $\{1, 1, 1, 24\}$
   (e) $\{2, 2, 2, 2\}$
   (f) $\{1, 2, 4, 8\}$
   (g) $\{1, 4, 4, 4\}$
   (h) $\{2, 2, 7, 7\}$
   (i) $\{4, 4, 7, 8\}$
   (j) $\{6, 13, 18, 18\}$
   (k) $\{3, 3, 9, 9\}$
   (l) $\{2, 5, 15, 15\}$
   (m) $\{1, 2, 4, 6\}$
   (n) $\{2, 3, 5, 10\}$
   (o) $\{5, 10, 13, 17\}$
   (p) $\{11, 11, 11, 11\}$
   (q) $\{2, 2, 16, 17\}$
   (r) $\{2, 5, 6, 7\}$

(s) $\{2, 6, 8, 9\}$

(t) $\{4, 5, 5, 5\}$

(u) $\{5, 9, 9, 12\}$

The letter operator set code is as follows: A, no solution; B, $\{\times, \times, \times\}$; C, $\{\div, \div, \div\}$; D, $\{-, -, -\}$; E, $\{+, +, +\}$; F, $\{-, \times, \times\}$; G, $\{+, \times, \times\}$; H, $\{\div, \times, \times\}$; I, $\{+, \div, \div\}$; J, $\{-, \div, \div\}$; K, $\{\times, \div, \div\}$; L, $\{+, +, \times\}$; M, $\{+, +, \div\}$; N, $\{+, +, -\}$; O, $\{-, -, \times\}$; P, $\{-, -, \div\}$; Q, $\{+, -, -\}$; R, $\{+, -, \times\}$; S, $\{+, \times, \div\}$; T, $\{-, +, \div\}$; U, $\{-, \times, \div\}$;

14. The second part of the puzzle is below.

(a) $\{3, 4, 11, 12\}$

(b) $\{2, 4, 7, 9\}$

(c) $\{4, 12, 13, 14\}$

(d) $\{3, 7, 7, 10\}$

(e) $\{1, 2, 7, 11\}$

(f) $\{6, 6, 6, 9\}$

(g) $\{3, 4, 10, 10\}$

(h) $\{2, 5, 6, 8\}$

(i) $\{3, 5, 5, 8\}$

(j) $\{3, 3, 18, 19\}$

(k) $\{3, 3, 3, 4\}$

(l) $\{3, 8, 14, 16\}$

(m) $\{7, 15, 15, 15\}$

(n) $\{2, 14, 15, 16\}$

(o) $\{2, 6, 7, 9\}$

(p) $\{3, 4, 19, 24\}$

(q) $\{1, 5, 5, 5\}$

(r) $\{3, 9, 12, 16\}$

(s) $\{1, 4, 7, 9\}$

(t) $\{3, 4, 6, 8\}$

(u) $\{1, 2, 7, 13\}$

3 For Super-High Flyers

15. (Hands across the table) Suppose we have $2n$ people sitting around a circular table. We want each of them to shake hands with another person at the table in such a way that none of the handshakes intersect any others. If $n = 1$, then we have two people and of course one way. For 4 people, there are two ways. How about the $n = 3$ problem where there are 6 people. Your assignment here is to make this calculation when $n = 4$.

16. Movie Ticket Problem. This problem was provided by Yao Jiahao, a tenth grader (2018) at Shanghai Foreign Language School. Eight people are in line to buy movie tickets that cost \$5 each. Each person has either a five dollar bill or a ten dollar bill each with probability one half. The ticket office has no way to give change for a \$10 unless someone with a \$5 comes before.

(a) What is the probability that all eight persons can buy their tickets?

(b) Change the problem so that the probability of a person paying with a five is twice as likely as paying with a ten.

(c) Now change the \$10 to \$20 and assume each bill is equally likely.

17. According to the standard convention for exponentiation,

$$2^{2^{2^2}} = 2^{(2^{(2^2)})} = 2^{16} = 65536.$$

If the order in which the exponentiations are performed is changed, how many other values are possible?

18. Show how to replace the operation $\oplus$ in the expression below with each of the symbols $+, -, \times, \div,$ ^ so that the expression

$$9 \oplus 7 \qquad 7 \qquad 1 \qquad 6 \qquad 6 \qquad 1$$

has the value 100. In other words, solve five problems, one for each operation.