An important and beautiful topic in the theory of complex variables is *Linear Fractional Transformations.* It turns out that some of the theory is accessible here without using complex numbers. Let $\mathcal{LF}$ denote the set of all functions f of the form

$$f(x) = \frac{ax+b}{cx+d},$$

where a, b, c, d satisfy $ad - bc \neq 0$. The number $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ is called the *determinant* of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Notice that the condition implies that c and d cannot both be zero, and similarly not both a and b can be zero.

If $c \neq 0$, then f has a *vertical asymptote* $x = -\frac{d}{c}$ and a *horizontal asymptote* $y = \frac{a}{c}$. We call $(-\frac{d}{c}, \frac{a}{c})$ the *center* of f. The condition that the determinant be non-zero makes the function non-constant and guarantees that f is undefined for at most one real number. There is another way to write $f(x) = \frac{ax+b}{cx+d}$. We can use ordinary polynomial division to see that $f(x) = \frac{a}{c} + \frac{b-ad/c}{cx+d}$.

This problem appeared on the 2002 AMC 10 contest. For how many values of x is $8xy - 12y + 2x - 3 = 0$ true for all values of y. The answer is just 1. In fact $8xy - 12y + 2x - 3 = 4y(2x-3) + (2x-3) = (4y+1)(2x-3)$. As $(4y+1)(2x-3) = 0$ must be true for all y, we must have $2x - 3 = 0$, hence $x = \frac{3}{2}$. The solution to the equation $(4y+1)(2x-3) = 0$ is the union of two lines, $x = 3/2$ and $y = -1/4$. If we translate this be replacing x by $x + 3/2$ and y by $y - 1/4$, we get the equation $xy = 0$. Of course the solution to this one is just the union of the two axes. Let T denote this set. Now change the zero to 0.1 or -0.1. The solution to the two new problems must be very close to the set T. In fact there is a way to make this statement rigorous. We can define the distance $D(A, B)$ between two (closed) subsets A and B of the plane in such a way that

$$\lim_{t \to 0} S_t = T,$$

where $S_t = \{(x, y) | xy = t\}$. We'll leave this discussion for another time. This problem provides some hints about several of the problems below. Here are some problems on $\mathcal{LF}$.

1. Let $\mathcal{S} = \{i(x) = x, \frac{3}{x}, \frac{x-3}{x-2}, \frac{2x-3}{x-1}, \frac{3x-6}{x-3}, \frac{3x-3}{2x-3}\}$. Sketch the graph of each

member of $\mathcal{S}$. Show that each of the six functions in $\mathcal{S}$ has an inverse in $\mathcal{S}$ and that $\mathcal{S}$ is closed under composition. This is enough to see that $\mathcal{S}$ is a group. Use the table below to build the multiplication table for the group. For example, notice that if $f(x) = \frac{2x-3}{x-1}$ and $g(x) = \frac{3}{x}$, then $f \circ g(x) = \frac{3x-6}{x-3}$.

$f \circ g$	x	$\frac{x-3}{x-2}$	$\frac{2x-3}{x-1}$	$\frac{3}{x}$	$\frac{3x-3}{2x-3}$	$\frac{3x-6}{x-3}$
x						
$\frac{x-3}{x-2}$						
$\frac{2x-3}{x-1}$				$\mathbf{\frac{3x-6}{x-3}}$		
$\frac{3}{x}$						
$\frac{3x-3}{2x-3}$						
$\frac{3x-6}{x-3}$						

2. Show that each $f \in \mathcal{LF}$ has $180°$ rotational symmetry about its center. This is the same a symmetry about a point. For example symmetry about $(0,0)$ means that $(x,y) \in S \Leftrightarrow (-x,-y) \in S$.

3. Prove that each $\mathcal{LF}$ function is one-to-one. In other words, if $f \in \mathcal{LF}$ and $x < y$ belong to the domain of f, then $f(x) \neq f(y)$.

4. Prove that each $\mathcal{LF}$ function omits at most one value.

5. Prove that each $\mathcal{LF}$ function has an inverse function. That is, if $f \in \mathcal{LF}$, there exists a function g in $\mathcal{LF}$ such that for all x for which the composition is defined, $f \circ g(x) = g \circ f(x) = x$. The function $i(x) = x$ is so important and ubiquitous that we shall name it i for this discussion.

6. Prove that $\mathcal{LF}$ is a group under composition. That is, prove that $(\mathcal{LF}, \circ)$ is a group. A group is a set S together with an operation $\circ$ which is associative. There is a special member $i \in S$ for which $i \circ x = x \circ i = x$ for all x in S, and finally, every element x in S has an inverse in S. That is, for each $x \in S$ there exists a y in S for which $x \circ y = y \circ x = i$.

7. Show that the subset $\mathcal{L}$ of $\mathcal{LF}$ of lines (ie, $c = 0$) is a subgroup of $\mathcal{LF}$.

8. Let $g(x) = \frac{x+1}{-3x+\sqrt{3}}$. Prove that $h \circ h \circ h \circ h = i$, the identity function and that h^2 and h^3 are not the identity. Of course, by h^2, we mean $h \circ h$, etc.

9. Let $h(x) = \frac{\sqrt{3}x+1}{-3x+1}$. Prove that $g \circ g \circ g = i$, the identity function and that $g \circ g$ is not the identity.

10. Next, let $j = g \circ h$, so $j(x) = g \circ h(x) = \frac{\frac{\sqrt{3}x+1}{-3x+\sqrt{3}}+1}{-3\cdot\frac{\sqrt{3}+1}{-3x+\sqrt{3}}+1} = \frac{(\sqrt{3}-3)x+(\sqrt{3}+1)}{(-3-3\sqrt{3})x+(-3+\sqrt{3})}$. Since $g^3 = i$ and $h^4 = i$, and 3 and 4 are relatively prime, $j^k \neq i$ if $k < 12$ and $j^{12} = i$.

11. Suppose that we are given three points in the plane no two of which are on the same vertical or horizontal line. Then there is exactly one $f \in \mathcal{LF}$ whose graph includes the three points. In other words, if $(a, b), (c, d)$ and (u, v) are any three points in the plane satisfying the conditions, there is exactly one $f \in \mathcal{LF}$ for which $f(a) = b, f(c) = d$ and $f(u) = v$.

12. The idea for this problem is due to Arthur Holshouser. Recall that the function $h(x) = \frac{x^2-4}{x-2}$ has what is called a removable discontinuity at the point $x = 2$. If we define $h(2) = 4$, then the new h is just $h(x) = x+2$. Here, we explore the idea of adding a point ∞ to the real numbers $\mathbf{R}$ and 'retopologising' the new set $\mathbf{R} \cup \{\infty\}$. This new structure is called the one point compactification of the reals. You should think about it this way. All sequences which decrease without bound and all sequences which increase without bound have limit ∞. This new space is actually the same as a circle. We call it homeomorphic with a circle. In the new skeme, lines take ∞ to itself. Suppose $f(x) = \frac{ax+b}{cx+d}$ and $c \neq 0$. Then define $f(\infty) = a/c$ and $f(-d/c) = \infty$. Prove that f is continuous at both the new points in its domain.

13. Let's define the product of two 2×2 matrices. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and let $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then the product $AB = \begin{pmatrix} ae+bg & ce+dg \\ ae+bf & cg+dh \end{pmatrix}$. Next, consider the three matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $J = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$, $K = \begin{pmatrix} 1 & -3 \\ 1 & -2 \end{pmatrix}$. Build the 3×3 multiplication table for I, J and K. Does this look familiar?

14. Next, build the table of matrix products for the given set of 2×2 matrices.

$A \cdot B$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -3 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 3 & -3 \\ 2 & -3 \end{pmatrix}$	$\begin{pmatrix} 3 & -6 \\ 1 & -3 \end{pmatrix}$
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