# Just the Factors, Ma'am

1   Introduction

The purpose of this note is to find and study a method for determining and counting all the positive integer divisors of a positive integer.[1] Let N be a given positive integer. We say d is a divisor of N and write $d|N$ if N/d is a positive integer. Thus, for example, $2 \mid 6$. Denote by D_N the set of all positive integer divisors of N. For example $D_6 = \{1, 2, 3, 6\}$. There are four parts to this note. In the first part, we count the divisors of a given positive integer N based on its prime factorization. In the second part, we construct all the divisors, and in the third part we discuss the 'geometry' of the D_N. In part four, we discuss applications to contest problems.

0	1	10	2	100	11	1000	3	20
1	2	3	4	5	6	7	8	9
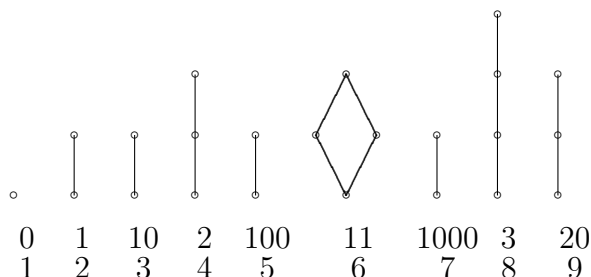
Study the figure above. There are two things to work out. One is how does the set of dots and segments represent a number, and the other is how do the digit strings just above the digits 1 through 9 represent those digits. Take some time now before reading more to figure this out.

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2   Counting the divisors of N

First consider the example $N = 72$. To find the number of divisors of 72, note that the prime factorization of 72 is given by $72 = 2^3 3^2$. Each divisor d of 72 must be of the form $d = 2^i 3^j$ where $0 \le i \le 3$ and $0 \le j \le 2$. Otherwise, $2^3 3^2 / d$ could not be an integer, by the Fundamental Theorem of Arithmetic (the theorem that guarantees the unique factorization into primes of each positive integer). So there are 4 choices for the exponent i and 3 choices for j. Hence there are $4 \cdot 3 = 12$ divisors of 72. Reasoning similarly, we can see that for any integer

$$N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

the number of divisors is

$$\Pi_{i=1}^k (e_i + 1) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1).$$

3   Constructing the divisors of N

In part 1 we found the number of members of D_N for any positive integer N. In this part, we seek the list of divisors themselves. Again we start with the prime factorization of N. Suppose $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. If $k = 2$ the listing is straightforward. In this case, we build a table by first listing the powers of p_1 across the top of the table and the powers of p_2 down the side, thus obtaining an $(e_1 + 1) \cdot (e_2 + 1)$ matrix of divisors. Again we use $N = 72$ as an example. Notice that each entry in the table is the product of its row label and its column label.

	2^0	2^1	2^2	2^3
3^0	1	2	4	8
3^1	3	6	12	24
3^2	9	18	36	72

What do we when the number of prime factors of N is more than 2? If $k = 3$ we can construct $e_3 + 1$ matrices of divisors, one for each power of p_3. For example, if $N = 360 = 2^3 3^2 5$, we construct one 3×4 matrix for 5^0 and one for 5^1. The result is a $3 \times 4 \times 2$ matrix of divisors. The two 3×4 matrices

are shown below.

5^0	2^0	2^1	2^2	2^3
3^0	1	2	4	8
3^1	3	6	12	24
3^2	9	18	36	72

and

5^1	2^0	2^1	2^2	2^3
3^0	5	10	20	40
3^1	15	30	60	120
3^2	45	90	180	360

For larger values of k we can create multiple copies of the matrix associated with the number $N = p_1^{e_1} p_2^{e_2} \cdots p_{k-1}^{e_{k-1}}$.

4   The geometry of D_N

To investigate the geometry of D_N, we first explore the relation 'divides'. Recall that $a|b$ means that a and b are positive integers for which b/a is an integer. The relation '|' has several important properties, three of which are crucial to our discussion.

1. Reflexivity. For any positive integer a, $a|a$.

2. Antisymmetry. For any pair of positive integers a, b, if $a|b$ and $b|a$, then $a = b$.

3. Transitivity. For any three positive integers, a, b, c, if $a|b$ and $b|c$, then $a|c$.

These properties are easy to prove. The first says that each integer is a divisor of itself; that is, a/a is an integer. The second says that no two different integers can be divisors of one another. This is true since a larger integer can never be a divisor of a smaller one. The third property follows from the arithmetic $b/a \cdot c/b = c/a$ together with the property that the product of two positive integers is a positive integer. Any set S with a relation $\preceq$ defined on it that satisfies all three of the properties above is called a *partially ordered set*, or a *poset*. A branch of discrete mathematics studies the properties of posets, $(S, \preceq)$.

    Each finite poset has a unique directed graph representation. This pictorial representation is what we mean by the geometry of D_N. To construct

the directed graph of a poset $(S, \preceq)$, draw a vertex (dot) for each member of S. Then connect two vertices a and b with a directed edge (an arrow) if $a \preceq b$. Of course, in our case D_N this means we connect a to b if $a|b$. The case D_6 is easy to draw:
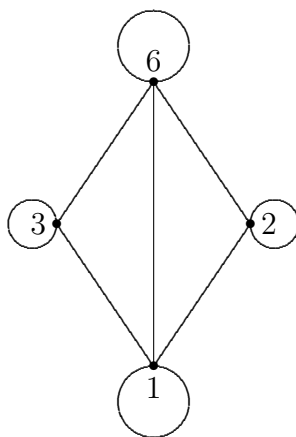

Fig. 1 The digraph of D_6

The circles at each of the four vertices are called *loops*. They are included as directed edges because each number is a divisor of itself. The reader should imagine that all the non-loop edges are upwardly directed. The directed edge from 1 to 6 indicates that 1|6. But since we know that the vertices of D_6 satisfy all three properties required of a poset, we can leave off both (a) the loops, which are implied by the reflexive property, and (b) the edges that are implied by the transitivity condition. The 'slimmed down' representation, called the Hasse diagram, is much easier to understand. It captures all the essential information without cluttering up the scene. The Hasse diagram of D_6 is shown below.
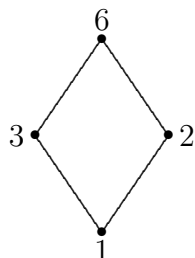
Fig. 2, the Hasse diagram of D_6

In general, the Hasse diagram for D_N has only those non-loop edges which are not implied by transitivity, that is, those edges from a to b for which b is a prime number multiple of a. The Hasse diagrams of D_{72}, D_{30}, and D_{60} are shown below.
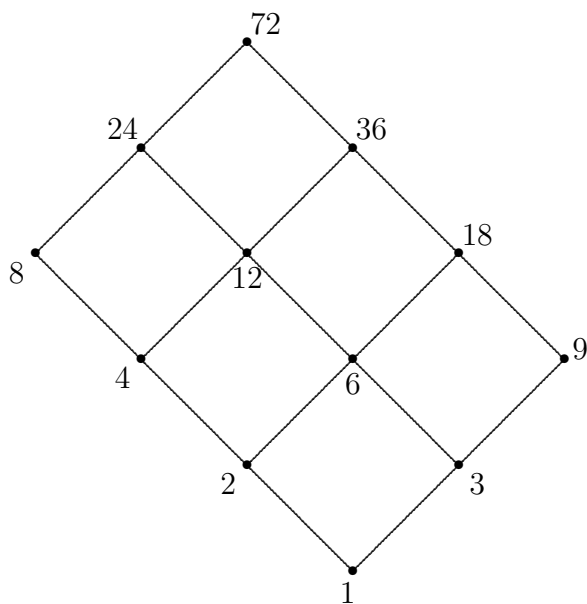
Fig. 3 The Hasse diagram of D_{72}

Notice that each prime divisor of 30 can be considered a direction, multiplication by 2 moves us to the left ($\nwarrow$), by 3 moves us upward ($\uparrow$) and, by 5 moves us to the right ($\nearrow$). Also note that if a and b are divisors of 30 then $a|b$ if and only if there is a sequence of upwardly directed edges starting at a and ending at b. For example, $1|30$ and $(1,3),(3,15),(15,30)$ are all directed edges in the digraph of D_{30}. On the other hand, we say 2 and 15 are incomparable because neither divides the other, and indeed there is no upwardly directed sequence of edges from either one to the other.
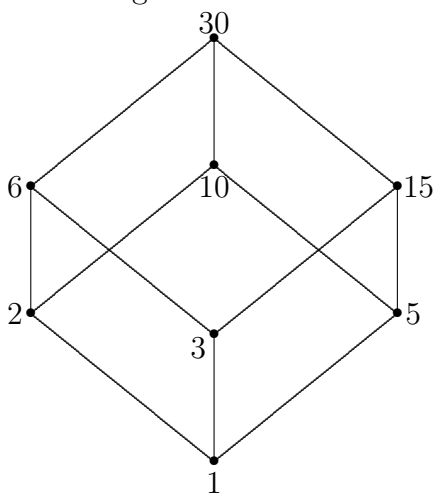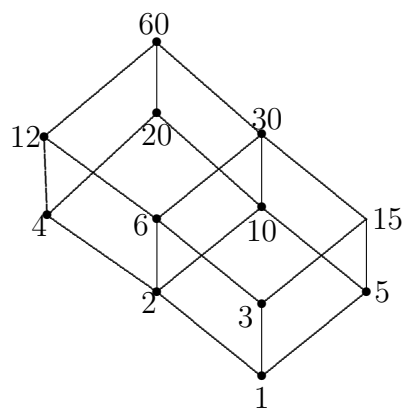

Fig. 4 The Hasse diagram of D_{30}

What would the divisors of 60 look like if we build such a diagram for them? Try to construct it before you look at D_{60}.

Fig. 5 The Hasse diagram of D_{60}

Consider the same (lattice/Hasse) diagram for the divisors of 210. We can draw this in several ways. The first one (Fig. 6a) places each divisor of 210 at a level determined by its number of prime divisors. The second one (Fig. 6b) emphasizes the 'degrees of freedom'. These two diagrams are representations of a four dimensional cube, not surprising since the Hasse diagram for D_{30} is a three-dimensional cube. A mathematical way to say the two digraphs are the same is to say they are *isomorphic*. This means that they have the same number of vertices and the same number of edges and that a correspondence between the vertices also serves as a correspondence between the edges. Note that the digraphs in 6a and 6b have the required number of vertices (16) and the required number of edges (32). Can you find an N such that the Hasse diagram of D_N is a representation of a five-dimensional cube. Such a digraph must have $2^5 = 32$ vertices, and $2 \cdot 32 + 16 = 80$ edges.
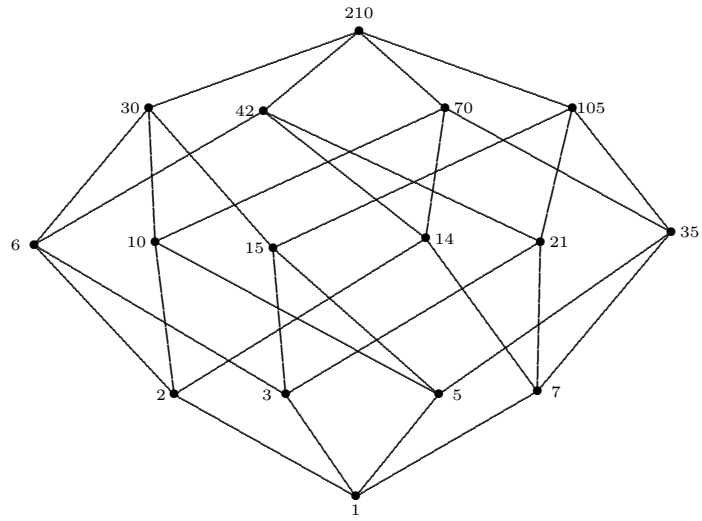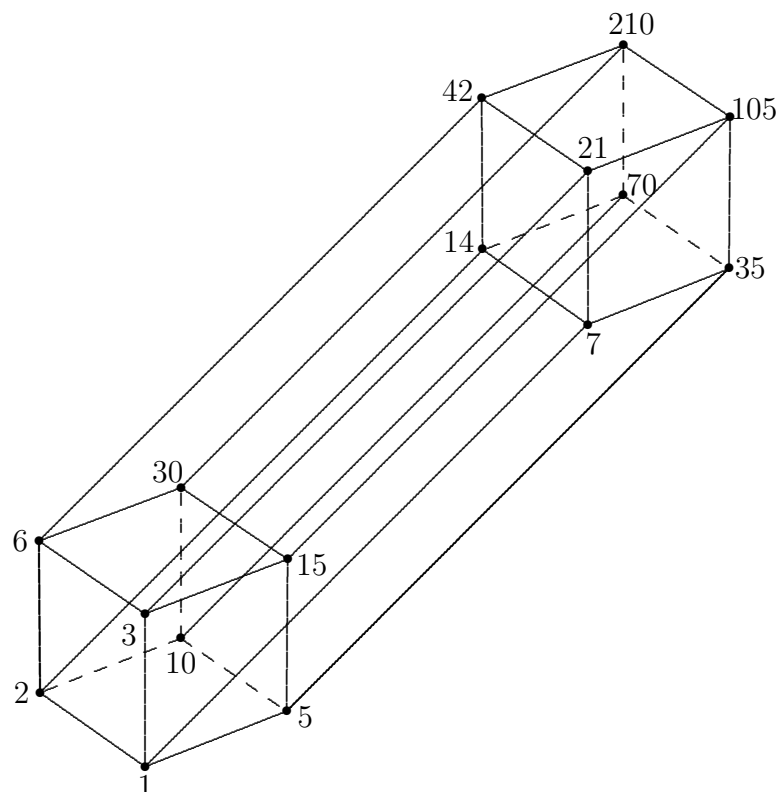
Fig 6a. The Hasse diagram of D_{210}

Fig 6b. The Hasse diagram of D_{210}

How can the geometry help us do number theory? One way to use the

geometry is in the calculation of the GCF and LCM of two members of D_N. Note that each element d of D_N generates a downward 'cone' of divisors and an upward cone of multiples. We can denote these cones by $F(d)$ and $M(d)$ respectively. Then the $GCF(d, e) = \max\{F(d) \cap F(e)\}$ and $LCM(d, e) = \min\{M(d) \cap M(e)\}$.

5   Multiplicative Functions

Certain ideas are so important that make it worth naming them with functions, whose names are then reserved just for their use. Multiplicative func-

tions are such. A function f from the natural numbers to the integers is called multiplicative if $f(a \cdot b) = f(a) \cdot f(b)$ whenever a and b are relatively prime natural numbers. There are three very useful multiplicative functions:

- $\tau(n) =$ the number of positive divisors of n,

- $\sigma(n) =$ the sum of the positive divisors of n, and

- $\phi(n) =$ the number of positive integers less than n that are relatively prime to n. This is called Euler's **totient** function.

There is a lovely theorem that can be used to calculate values of the Euler function.

Theorem. If an integer N has prime factorization $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, then the number of positive integers less than N that are relatively prime to N is given by

$$\phi(N) = N \left(1 - \frac{1}{p_1} \right) \left(1 - \frac{1}{p_2} \right) \cdots \left(1 - \frac{1}{p_k} \right)$$
$$= (p_1^{e_1} - p_1^{e_1-1})(p_2^{e_2} - p_2^{e_2-1}) \cdots (p_k^{e_k} - p_k^{e_m-1}).$$

6   Problems from Competitions

The following problems come from MathCounts and the American Mathematics Competitions.

1. Find the number of three digit divisors of 3600.

2. How many positive integers less than 50 have an odd number of positive integer divisors?

3. (The Locker Problem) A high school with 1000 lockers and 1000 students tries the following experiment. All lockers are initially closed. Then student number 1 opens all the lockers. Then student number 2 closes the even numbered lockers. Then student number 3 changes the status of all the lockers numbered with multiples of 3. This continues with each student changing the status of all the lockers which are numbered by multiples of his or her number. Which lockers are closed after all the 1000 students have done their jobs?

4. If N is the cube of a positive integer, which of the following could be the number of positive integer divisors of N?

   (A) 200   **(B)** 201   **(C)** 202   **(D)** 203   **(E)** 204

5. Let
$$N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1.$$
How many positive integers are factors of N?

6. A teacher rolls four dice and announces both the sum S and the product P. Students then try to determine the four dice values a, b, c, and d. Find an ordered pair (S, P) for which there is more than one set of possible values.

7. How many of the positive integer divisors of $N = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3 \cdot 11^3$ have exactly 12 positive integer divisors?

8. How many ordered pairs (x, y) of positive integers satisfy

$$xy + x + y = 199?$$

9. How many positive integers less than 400 have exactly 6 positive integer divisors?

10. The product of four distinct positive integers, a, b, c, and d is 8!. The numbers also satisfy

$$ab + a + b = 391 \tag{1}$$
$$bc + b + c = 199. \tag{2}$$

What is d?

11. How many multiples of 30 have exactly 30 divisors?

12. Find the number of odd divisors of 13!.

13. Find the smallest positive integer with exactly 60 divisors.

14. (2008 Purple Comet) Find the smallest positive integer the product of whose digits is 9!.

15. (1996 AHSME) If n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors, then how many positive divisors does $6n$ have?

16. Find the sum of all the divisors of $N = 2^2 \cdot 3^3 \cdot 5^4$. In other words, find $\sigma(N)$. Also, find $\tau(N)$ and $\phi(N)$.

17. (2010 Mathcounts Competition, Target 7) Find the product P of all the divisors of 6^3, and express your answer in the form 6^t, for some integer t.

18. What is the sum of the prime number divisors of 159999?

19. Find the largest 10-digit number with distinct digits that is divisible by 11.

20. (This was problem 25 on the Sprint Round of the National MATH-COUNTS competition in 2011.) Each number in the set $\{5, 9, 10, 13, 14, 18, 20, 21, 25, 29\}$ was obtained by adding two number from the set $\{a, b, c, d, e\}$ where $a < b < c < d < e$. What is the value of c?

21. (This was problem 8 on the Target Round of the National MATH-COUNTS competition in 2011.) How many positive integers less than 2011 cannot be expressed as the difference of the squares of two positive integers?

22. Consider the number $N = 63000 = 2^3 \cdot 3^2 \cdot 5^3 \cdot 7$.

    (a) How many divisors does N have? That is find $\tau(N)$.

    (b) How many single digit divisors does N have?

    (c) How many two-digit divisors does N have?

    (d) How many three-digit divisors does N have?

    (e) How many four-digit divisors does N have?

    (f) How many five-digit divisors does N have?

23. The integer N has both 12 and 15 as divisors. Also, N has exactly 16 divisors. What is N?

24. How many multiples of 30 have exactly 3 prime factors and exactly 60 factors.

25. How many whole numbers n satisfying $100 \leq n \leq 1000$ have the same number of odd divisors as even divisors?

26. What is the sum of the digits in the decimal representation of $N = \sqrt{25^{64} \cdot 64^{25}}$?

27. What is the fewest number of factors in the product 27! that have to be removed so that the product of the remaining numbers is a perfect square?

28. Euclid asks his friends to guess the value of a positive integer n that he has chosen. Archimedes guesses that n is a multiple of 10. Euler guesses that n is a multiple of 12. Fermat guesses that n is a multiple of 15. Gauss guesses that n is a multiple of 18. Hilbert guesses that n is a multiple of 30. Exactly two of the guesses are correct. Which persons guessed correctly?

29. The number $9! = 362880 = 2^7 \cdot 3^4 \cdot 5 \cdot 7$ has $(7+1)(4+1)(1+1)(1+1) = 160$ divisors from 1 to 9! itself.

   (a) How many of the 160 divisors of 9! are odd numbers?

   (b) What is the sum of all the divisors of 9!?

   (c) What is the 12th largest among these divisors?

30. The number 6 can be written in two ways as the sum of consecutive positive integers: $6 = 6$ and $6 = 1 + 2 + 3$. The number 7 can also be written in two ways, but 8 can be written in only one way. How many ways can the number 5005 be written as a sum of consecutive positive integers?

31. For each of the numbers 3600, 5005, and 6^{10}, find the sum of the divisors and the sum of the reciprocals of the divisors. In other words, find $\sigma(N)$. Also, find $\tau(N)$ and $\phi(N)$.

32. The number N is the product of two primes. The sum of the positive divisors of N that are less than N is 2014. Find N.

33. Let N be a positive integer with divisor set $D(N)$. Let d be randomly selected from $D(N)$. Suppose $P(d \leq 8) = 3/17$. What is the remainder when N is divided by 1000?

34. Let $a_{10} = 10$ and for each $n > 10$, let $a_n = 100a_{n-1} + n$, Find the remainder when a_n is divided by 99.

35. What is the smallest prime number p greater than 200 that is the sum of two squares?

36. What is the sum of all positive integers less than 100 that have exactly twelve divisors?

37. What is the largest prime factor of $3^{14} + 3^{13} - 12$?

38. What is that largest positive integer n for which $n^3 + 100$ is divisible by $n + 10$?

39. By a proper divisor of a natural number we mean a positive integral divisor other than 1 and the number itself. A natural number greater than 1 will be called nice if it is equal to the product of its distinct proper divisors. What is the sum of the first ten nice numbers?

40. Let m/n, in lowest terms, be the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}. Find $m + n$.

41. Find the sum of all positive rational numbers that are less than 10 and that have denominator 30 when written in lowest terms.

42. Let $n = 2^{31}3^{19}$. How many positive integer divisors of n^2 are less than n but do not divide n?

43. This was sprint 26 on the 2019 MATHCOUNTS national. Let $P(n)$ denote the product of the divisors of the positive integer n. Find the integer k for which $P(P(P(P(10)))) = 10^k$. What is the largest prime factor of k?