# 1 Introduction

Consider a large cube made from unit cubes.[1] Suppose our cube is $n \times n \times n$. Look at the cube from a corner so that you can see three faces. How many unit cubes are in your line of vision? Build a table that shows how many cubes are visible from one corner as a function of $n$.

| $n$ | $n^3$ | *number visible* |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 8 | 7 |
| 3 | 27 | 19 |
| 4 | 64 | 37 |

How does the table continue? Make some guesses and then try to prove your answer. Let's name the number we're looking for. Let $G(n)$ denote the number of cubes visible from a corner of the $n \times n \times n$ cube. Notice that the sequence of differences $G(2) - G(1) = 6; G(3) - G(2) = 19 - 7 = 12; G(4) - G(3) = 37 - 19 = 18$ has an interesting property. The differences are all multiples of 6. When we explore such a sequence in which the sequence of successive differences is eventually constant, we can build a polynomial that produces the sequence. Since the second order differences are constant, we propose that $G(n)$ is a quadratic polynomial, $G(n) = an^2 + bn + c$. We can solve this without great difficulty to get $G(n) = 3n^2 - 3n + 1$. But what do these coefficients have to do with the problem? One way to see this is to extend the chart by one more column that shows the cubes that are not visible.

| $n$ | $n^3$ | *number not visible* | *number visible* |
|---|---|---|---|
| 1 | 1 | 0 | 1 |
| 2 | 8 | 1 | 7 |
| 3 | 27 | 8 | 19 |
| 4 | 64 | 27 | 37 |

Now you can see that we can count the number of visible cubes by counting the invisible ones first. So, in general, $G(n) = n^3 - (n-1)^3 = n^3 - (n^3 - 3n^2 + 3n - 1) = 3n^2 - 3n + 1$. This doesn't completely answer the question however. What are the coefficients telling us? How does the algebra help us reason geometrically? The answer is below. Think of the

[1]Unauthorized reproduction/photocopying prohibited by law' ©

set of cubes in each of the visible faces as $A, B$ and $C$. So each set has $n^2$ elements. Their pairwise intersections, $A \cap B, A \cap C$ and $B \cap C$ are edge cubes, and each of these sets has $n$ members. Finally $A \cap B \cap C = ABC$ is the one corner cube that belongs to all three faces. We have $|A \cup B \cup C| = |A| + |B| + |C| - |AB| - |AC| - |BC| + |ABC| = 3n^2 - 3n + 1$. This idea of over-counting, then removing the extra units, etc. is called the Principle of Inclusion/Exclusion(PIE). You'll find a set of PIE problems in this essay.

But before we leave this rich area, we have one more question to pursue. If we paint the entire outside, how many of the $n^3$ unit cubes receive some paint? Can you write this as a polynomial in $n$ in standard form. What do the coefficients tell you?

## 2 Problems with Cubes

1. The entire outside of an $n \times n \times n$ cube built from unit cubes is painted. Find the number of cubes with some painted faces as a function of $n$. Note that for $n = 3$, the number is $27 - 1 = 26$.

| $n$ | $n^3$ | *number with paint* |
|---|---|---|
| 1 | 1 | 1 |
| 2 | 8 | 8 |
| 3 | 27 | 26 |
| 4 | 64 | 56 |

2. Suppose that every pair of interior faces are glued together so that each pair of faces requires one unit of glue. How many square units of glue is needed? Examine this for $n = 4, 5,$ and 6.

3. What is the fewest cuts needed to separate a wooden $3 \times 3 \times 3$ cube into 27 unit cubes if you're allowed to move blocks of cubes about before cutting? What if the big cube is $4 \times 4 \times 4$?

4. The problem above is just an introduction to the main problem stated here. Suppose $abc$ unit cubes are glued together to build an $a \times b \times c$, $a \leq b \leq c$ block. What is the fewest cuts required to sever the block back into unit cubes? We approach this difficult problem by first assuming $a = b = 1$. Let's build a table.

| $1 \times 1 \times c$ | *cuts rerquired* |
|---|---|
| 1 | 0 |
| 2 | 1 |
| 3 | 2 |
| 4 | 2 |
| $\vdots$ | $\vdots$ |
| $n$ | ? |

5. Suppose all outside edges of an $a \times b,\ a \leq b$ block of unit squares are painted and it turns out that exactly the same number of unit squares have some paint as those that have no paint. Find all ordered pairs $(a, b)$ for which this occurs. Prove that there are no other solutions. Now change the problem to intervals, and paint the two endpoints. In other words, build an interval $[0, a]$ by attaching $a$ unit intervals together, paint the two endpoints, 0 and $a$. Suppose exactly half the intervals have some paint? What is $a$?

6. Suppose all six faces of an $a \times b \times c,\ a \leq b \leq c$ block of unit cubes are painted and it turns out that exactly the same number of unit cubes have some paint as those that have no paint. Find all triplets $(a, b, c)$ for which this occurs. Prove that there are no other solutions. State and solve the planar analog of this problem.

7. Suppose all six faces of an $n \times n \times n$ cube are painted red. Then one of the $n^3$ unit cubes is randomly selected and tossed like a die. What is the probability that the face obtained is painted? Of course, your answer depends on $n$. Try this for $n = 1, 2$, and 3. Then make a conjecture and prove your conjecture.

8. Suppose all six faces of an $a \times b \times c$ block of cubes are painted red. Then one of the $abc$ unit cubes is randomly selected and tossed like a die. Is it possible that the probability of a red face showing up is exactly 2/7? This problem clearly generalizes the previous one. Show that when $a = b = c$, we get from this problem the same result as we got in the one above.

9. Suppose all six faces of an $a \times b \times c,\ a < b < c$, block of cubes are painted red. Then one of the $abc$ unit cubes is randomly selected and

tossed like a die. The probability of a red face showing up is exactly 2/9. What is the least possible volume of the block?

10. Suppose two non-adjacent faces of the big cube are painted red and the other four faces painted black. Let $R$ denote the number of unit cubes with some red faces and $B$ the number of unit cubes with some black faces. Find an $n$ for which $B - R = 390$.

11. (2008 Mathcounts) A $12 \times 12 \times 12$ cube is built using a $10 \times 10 \times 10$ cube and a bunch of $2 \times 2 \times 2$ cubes. How many $2 \times 2 \times 2$ cubes are needed? A $(n+2) \times (n+2) \times (n+2)$ cube is built using a $n \times n \times n$ cube, a bunch of $2 \times 2 \times 2$ cubes and a few unit cubes. How many unit cubes and $2 \times 2 \times 2$ cubes are needed? Your answer may depend on the oddness or evenness of $n$.

12. A mouse eats away at a $3 \times 3 \times 3$ block of cheese that is made from 27 units of cheese. 14 unit cubes are dark cheese and 13 unit cubes are a light colored cheese and they are arranged so that no two cubes of the same cheese have a common face. In other words its colored like a checkerboard. The mouse proceeds always to an adjacent cube. (Adjacent means that the two small cubes share a face.) Can the mouse eat in such a pattern so that she finishes her meal on the centermost cube?

13. A square can be partitioned into four squares by joining the midpoints of opposite sides. If we apply that partitioning again to one of the four subsquares, we get a partition with seven squares. We could also split the original square into nine squares or 16 or 25, etc. What is the largest integer $N$ such that a square cannot be partitioned into $N$ subsquares in this way?

14. You have an unlimited supply of red(R) and blue(B) faces out of which to build cubes. How many distinguishable cubes can you build? Next suppose you have three colors.

15. Bob and Ann play the following game with 8 white unit cubes. Ann wins if she can assemble a $2 \times 2 \times 2$ cube that has only white faces exposed. But Bob gets to paint four of the $8 \cdot 6 = 48$ white faces black. Who wins?

(a) What is the fewest number of faces Bob can paint to deny Ann in the $3 \times 3 \times 3$ game?

(b) What is the fewest number of faces Bob can paint to deny Ann in the $4 \times 4 \times 4$ game?

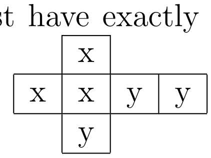
(c) What is the fewest number of faces Bob can paint to deny Ann in the $2 \times 3 \times 4$ game?

16. A truncated octahedron is a geometric solid with 14 faces (6 congruent squares and 8 congruent hexagons). In this particular solid, 2 hexagons and 1 square meet to form each corner. How many corners does this solid have?

17. An $n \times n \times n$ cube is build from $n^3$ unit cubes. A *line* is a line segment in space that includes the center of $n$ of the unit cubes. How many lines are there? Check out the corresponding 2-d problem. Think about tic-tac-toe.

18. **The $3 \times 3 \times 3$ Chameleon Cubes Problem.** You are given 27 unpainted cubes. Can you paint the faces with three colors, red, white, and blue, so that when you're done, you can assemble an all red $3 \times 3 \times 3$ cube, an all white $3 \times 3 \times 3$ cube and an all blue $3 \times 3 \times 3$ cube? Since this problem is much more involved than others, we present the entire solution.

The solution presented here is not ours. We heard it from Dick Stanley, to whom we are grateful. Yes, you can do this in essentially just one way.

Here is one solution. Since the total number of faces we can paint is $27 \cdot 6 = 162$, and since the all-red cube requires $6 \cdot 9 = 54$ red faces as do the other two colors, we must be perfectly efficient in the following sense. Each face we paint red must appear on the outside of the red cube. This implies that there must be exactly one unit cube, the one not visible from the outside, that has no red faces, and similarly exactly one that has no white faces, and one that has no blue faces. These cubes must have exactly three faces of the other two colors. They all look like 

, where $x$ is one color and $y$ is another. Why can't they look like

| | | | |
|---|---|---|---|
| | x | | |
| x | y | y | y |
| | x | | |

, you might ask yourself. The two cubes of this type with red faces account for two of the eight corners. The six other red corner cubes must have faces with the two other colors, so two faces must be one color and one face the third color. Reasoning similarly, there are 6 more cubes with three white and 6 more cubes with three blue faces as shown in the table below. The numbers $a$ and $b$ are 1 and 2 in some order.

| $n$ | $R$ | $W$ | $B$ |
|---|---|---|---|
| 1 | 3 | 3 | 0 |
| 1 | 3 | 0 | 3 |
| 1 | 0 | 3 | 3 |
| 6 | 3 | $a$ | $b$ |
| 6 | $a$ | 3 | $b$ |
| 6 | $a$ | $b$ | 3 |

Notice that exactly 21 cubes have been accounted for, leaving just six more to determine. None of the six can have three faces of the same color. Why? This means they must all have two adjacent faces of each color. Since 12 edge cubes of each color must be available, we can now determine that among the six cubes with three faces of one color, two of another and one of the third color, there are two types, one with two of one color and one with two of the other color. For example, when we look at the six cubes that have three red faces, we'll see three cubes of each pattern

| | | | |
|---|---|---|---|
| | R | | |
| R | R | W | W |
| | B | | |

and

| | | | |
|---|---|---|---|
| | R | | |
| R | R | B | W |
| | B | | |

.

Thus, the complete table is

| $n$ | $R$ | $W$ | $B$ |
|---|---|---|---|
| 1 | 3 | 3 | 0 |
| 1 | 3 | 0 | 3 |
| 1 | 0 | 3 | 3 |
| 3 | 3 | 1 | 2 |
| 3 | 3 | 2 | 1 |
| 3 | 1 | 3 | 2 |
| 3 | 2 | 3 | 1 |
| 3 | 1 | 2 | 3 |
| 3 | 2 | 1 | 3 |
| 6 | 2 | 2 | 2 |

There is another coloring that also works.

| $n$ | $R$ | $W$ | $B$ |
|---|---|---|---|
| 1 | 3 | 3 | 0 |
| 1 | 3 | 0 | 3 |
| 1 | 0 | 3 | 3 |
| 6 | 3 | 1 | 2 |
| 6 | 2 | 3 | 1 |
| 6 | 1 | 2 | 3 |
| 6 | 2 | 2 | 2 |

We'll see shortly that there is a much easier and compelling solution. And the new solution generalizes in several ways.

Some readers may realize that we started with the wrong problem. What if you are given 8 unpainted cubes. Can you paint the faces with two colors, red and blue, so that when you're done, you can assemble both an all red cube and an all blue cube?

These two 3-dimensional problems are just a part of an infinite collection of problems, one for each positive integer $n$. We call this the space problem. **The 3-D Problem.** Color each of the 6 faces of $n^3$ unit cubes one of the $n$ colors $1, 2, 3, \cdots, n$ in such a way that for each color $i \in \{1, 2, 3, \cdots, n\}$ the $n^3$ unit cubes can be stacked to form a $n \times n \times n$ big cube in such a way that the big cube is completely colored $i$.

**The 2-D Problem, also called the planar problem** Color each of the 4 sides of $n^2$ unit squares one of the $n$ colors $1, 2, 3, \cdots, n$ in such a way that for each color $i \in \{1, 2, 3, \cdots, n\}$, the $n^2$ unit squares can be stacked to form a $n \times n$ big square in such a way that the 4 (outer) sides of this big square are completely colored $i$.

Maybe even the two-dimensional problem is the wrong place to start.
**The 1-D Problem, also called the linear problem.** Color each of the 2 end points of $n$ unit intervals one of the $n$ colors $1, 2, 3, \cdots, n$ in such a way that for each color $i \in \{1, 2, 3, \cdots, n\}$, the $n$ unit intervals can be laid side by side to form a big interval of length $n$ in such a way that the two end points of the big interval have the color $i$.

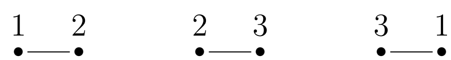
There are several ways to solve these 3 problems. By far the easiest general solution that we know of is to first find the general solution to the 1-dimensional problem. Then we can use any solution to the 1-dimensional problem to solve the other two.

**Solution to Linear Problem.** We want to assign the colors $1, 2, 3, \cdots, n$ to the end points of each of the $n$ unit intervals $[0, 1], [1, 2], \ldots, [n-1, n]$ in such a way that for each color $i \in \{1, 2, 3, \cdots, n\}$, there is an arrangement of intervals so that the endpoints of the union of the intervals both have color $i$. Start by coloring the endpoints at 0 and $n$ with color 1. Then color the right endpoint of $[0, 1]$ and the left endpoint of $[1, 2]$ with color 2, and so on so that the right endpoint of interval $[k-1, k]$ and the left endpoint of $[k, k+1]$ are colored $k+1$, for $k = 1, 2, \ldots n-1$. Now it is clear that we can arrange the intervals so that the interval that is their union has each of the $i$ colors as endpoints.

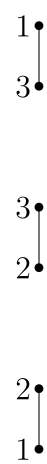
**Solution to the Planar Problem.** We will now illustrate the solution to Problem 2 for $3^2 = 9$ unit squares. The general solution is exactly the same. First, we find any two arbitrary stable arrangements for Problem 3, using $n = 3$.

We then show how we can use these two stable arrangements for each of the two dimensions of Problem 3.

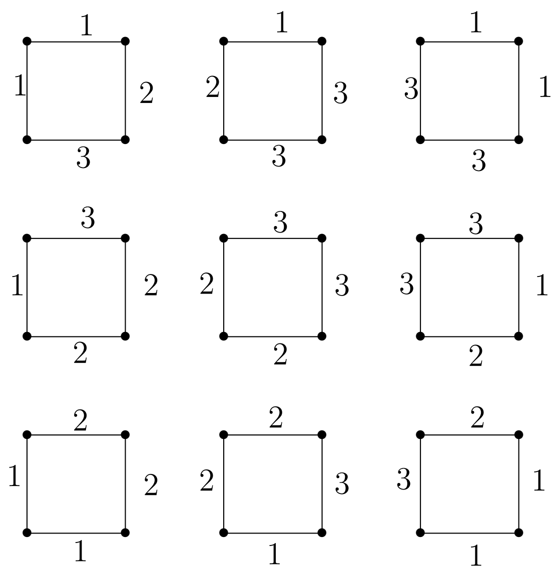
Solution (a)




Solution (b)




We now color the sides of the a squares according to the above stable arrangements (a), (b).




Note that the vertical sides of the columns unit squares are colored according to solution (a). Thus, in column 1, the left side of all squares are colored 1.

Also, in column 1, the right side of all squares are colored 2. Also, in column 2, the left side of all squares are colored 2. Also, in column 2, the right side of all squares are colored 3.

The horizontal sides of the rows of unit squares are colored according to solution (b). Thus, in row 1, the top side of all squares are colored 2. Also, in row 1, the bottom side of all squares are colored 1.

We could achieve this solution in much the same was we achieved the linear solution. Start by coloring the entire outside with color 1, then move the left column of squares to the right side and then move the bottom row to the top. You get a new $3 \times 3$ square the sides of which have not been colored. Use color 2 to do the job. Then move the left column to the right side, and the bottom to the top, once again exposing all uncolored sides. Finish the job using color 3.

We are now ready to attack the four color $4 \times 4 \times 4$ problem. See if you can complete the table below.

| $n$ | $R$ | $W$ | $B$ | $G$ |
|---|---|---|---|---|
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| $n$ | $R$ | $W$ | $B$ | $G$ |
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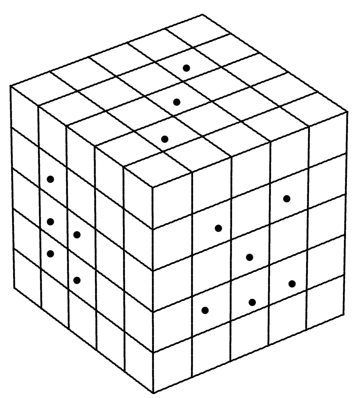
19. Suppose some faces of a large wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. The number of unit cubes with some red paint is found to be exactly 200 larger than the number of cubes with some black paint. How many cubes have no paint at all?

20. Suppose some faces of an $n \times n \times n$ wooden cube are painted red and the rest are painted black. The cube is then cut into unit cubes. Let $R$ denote the number of cubes with some red paint and $B$ the number

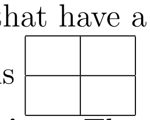
of cubes with some black paint. What is the least value of $n$ for which $B + R$ is a multiple of 100? Find the next five values of $n$ for which $B+R$ is a multiple of 100. In each case decide how the faces of the big cube are painted.

21. (2004 Purple Comet) A cubic block with dimensions $n \times n \times n$ is made up of a collection of $n^3$ unit cubes. What is the smallest value of $n$ so that if the outer two layers of unit cubes are removed from the block, more than half the original unit cubes will still remain?

22. The $5 \times 5 \times 5$ cube shown below is built from 125 unit cubes. The dots on the surface show the places where the big cube is drilled through. When all these 'drilled' cubes have been removed, how many remain?



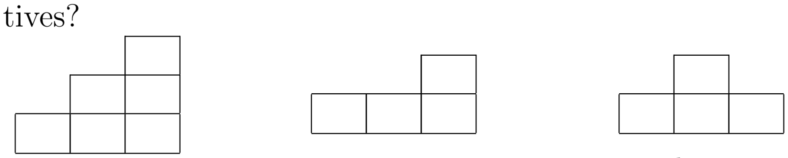
## Using perspectives

23. This problem is about using cubes to build polyhedra that resemble buildings. Students can practice spacial visualization and use their imagination. Here's a sample problem. Find all possible cubical buildings that have a base, and front projection and a right side projection that is 

**Solution.** There are seven solutions. We depict them using the base diagram where the number in each square represents the number of cubes on top of that square. For example $\begin{array}{|c|c|}\hline 2 & 2\\ \hline 2 & 2\\ \hline\end{array}$ is the solution that uses the maximum number of cubes, whereas $\begin{array}{|c|c|}\hline 1 & 2\\ \hline 2 & 1\\ \hline\end{array}$ uses the minimum number of cubes. How many solutions are there altogether? Answer: 7. Two solutions can be built with 6 cubes, four with 7 cubes and one with 8 cubes.

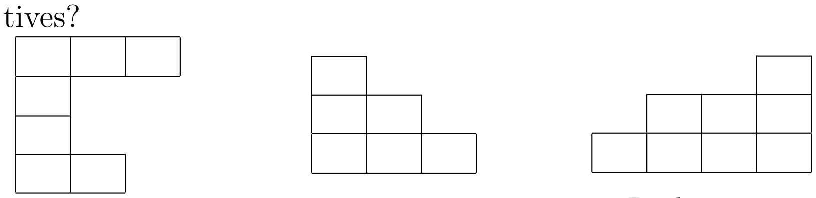
For each problem below, build a model that is consistent with the given perspectives.

(a) Is the number of cubes required determined by the three perspectives?



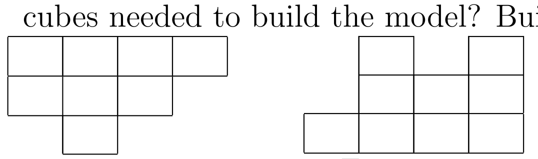
Top view Front view Right view

(b) Is the number of cubes required determined by the three perspectives?



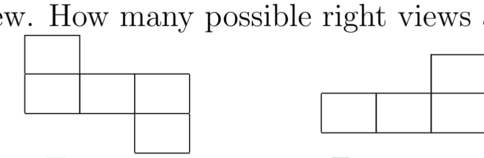
Top view Front view Right view

(c) Is the number of cubes required determined by the three perspectives? What are the maximum number and minimum number of cubes needed to build the model? Build a model for it.



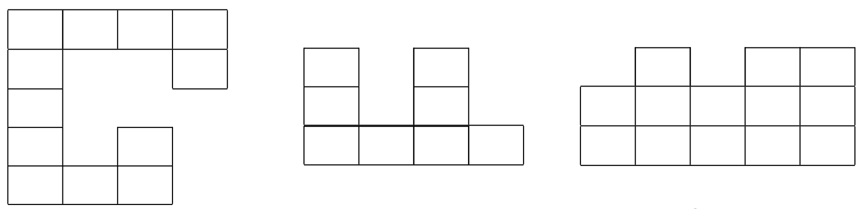
Top view Front view Right view

(d) The top and front projections are given. Build a possible right view. How many possible right views are there?



Top view Front view

(e) Use exactly 20 cubes to make a model from the building plans below. Record the base plan for your building. What are the maximum and minimum numbers of cubes that could be used to build the structure.



Top view Front view Right view

24. A $10 \times 10$ square is decomposed into exactly 75 squares of various (integer) sizes. How many $3 \times 3$ squares are in this decomposition?

25. A $10 \times 10$ square is decomposed into exactly $n$ squares of various (integer) sizes. For which values of $n$ is this possible?

26. Kyle will use four identical unit cubes to create a solid, called a 4-polycube. Each cube must be glued to at least one other cube. Two cubes may only be glued together in such a way that a face of one cube exactly covers a face of the other cube. How many distinct solids could Kyle create? Two solids are considered the same if they can be oriented to they are identical.

27. Is it possible to tile a rectangle with squares all of which are different?

28. Farmer Brown's plot is 225 feet long by 90 feet wide. He is subdividing it into congruent integer sided rectangular plots.

    (a) How many options are there?

    (b) For how many of these options are the rectangles actually squares?

29. Suppose all six faces of an $a \times b \times c,\ a \leq b \leq c$ block of unit cubes are painted. Let $F_{abc}$ be the quotient $M/abc$ where $M$ is the number of cubes with some paint. For which rational numbers $r \in (0,1)$ does there exist a triplet satisfying $r = F_{abc}$?

30. A large wooden cube is painted on some of its six faces and then cut into small identical cubes. The number of small cubes found to have

some paint in 15. How many small cubes have no paint at all? Next change the number 15 to 61 and answer the same question.

31. What is the greatest number of $1\times1\times3$ blocks that can fit in a $5\times5\times10$ box?

32. A $5\times5\times5$ cube is built from 125 unit cubes. Some cubes are removed leaving a smaller cube. How many ways can this be done? (For example, there are 8 ways to remove a particular set of 61 unit cubes to leave a $4\times4\times4$ subcube.

33. A $5\times5\times5$ cube is built from 125 unit cubes. How many rectangular prism subsets does it have.

34. A wooden rectangular block, $4\times5\times6$, is painted red and then cut into several 120 unit cubes. What is the ratio of the number of cubes with two red faces to the number of cubes with three red faces?

35. Twenty-seven identical white cubes are assembled into a single cube, the outside of which is painted black. The cube is then disassembled and the smaller cubes thoroughly shuffled in a bag. A blindfolded man (who cannot feel the paint) reassembles the pieces into a cube. What is the probability that the outside of this cube is completely black?

36. A cube can be partitioned into 8 cubes in an obvious way. A cube can also be partitioned into 27 cubes and into 17 cubes. What is the largest integer $N$ such that a cube cannot be partitioned into $N$ cubes?

37. A polyhedron has faces that are triangles or squares. No two squares share an edge and no two triangles share an edge. What is the ratio of the number of triangular faces to the number of square faces?

38. An $a\times b\times c$, $2\le a\le b\le c$ rectangular block is built from $abc$ unit cubes. From one corner you can see faces of three different sizes. Suppose you can see exactly 36 of the $abc$ cubes. What is $a^2+b^2+c^2$?

39. Corners are sliced off a unit cube so that the six faces each become regular octagons. What is the total volume of the removed tetrahedra?

40. Say you're given the following challenge: create a set of five rectangles that have sides of length $1,2,3,4,5,6,7,8,9$ and 10 units. You can

combine sides in a variety of ways: for example, you could create a set of rectangles with dimensions $1 \times 3, 2 \times 4, 5 \times 7, 6 \times 8$ and $9 \times 10$.

(a) How many different sets of five rectangles are possible?

(b) What are the maximum and minimum values for the total areas of the five rectangles?

(c) What other values for the total areas of the five rectangles are possible?

(d) Which sets of rectangles may be assembled to form a square?

(e) i. Solve the equation

$$ab + cd + ef + gh + ij = 169$$

so that $\{a, b, c, d, e, f, g, h, i, j\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

ii. Build five rectangles using two line segments of each of the lengths 1 through 10 so that you can tile a $13 \times 13$ square.

41. Let $T_n = \{n, n+1, n+2, \ldots, n+9\}$ be a set of 10 consecutive positive integers. A partition of T into two-element subsets is called *duorangment.*

(a) How many duorangments are there?

(b) For each duorangment $D$, let $K_D = \{\{a, b\}, \{c, d\}, \{e, f\}, \{g, h\}, \{i, j\}\}$ denote the sum of the five products. Call $K_D$ the sum of the duorangment. Prove that

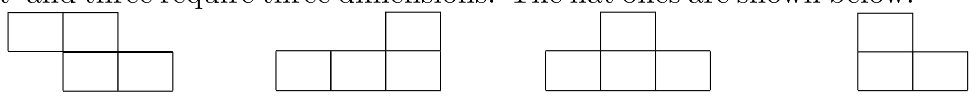
$$5n^2 + 45n + 60 \leq K_D \leq 5n^2 + 45n + 140.$$

(c) Let $D_1, D_2$ and $D_3$ be the duorangements of $T_1$, $T_2$ and $T_3$ respectively. Find all recatangles that can be tiles with the five recangles that can be obtained from $D_1, D_2$ and $D_3$.

42. What if we're building rectangular boxes. Start with 36 struts, four each of lengths 1 through 9. Use these struts to build three boxes. What are the maximum and minimum volumes attainable in this way?

43. The Soma cube was invented by Piet Hein, in the 1930's. It has seven pieces, which are all the ways 3 or 4 cubes can be joined face-to-face, so

that the resulting shape is NOT rectangular. Four of these pieces are 'flat' and three require three dimensions. The flat ones are shown below.



Z shape L shape T shape V shape

The other three are called Y, left hand and right hand.

(a) Try to put the 7 pieces into the cube. (But don't try too long, look at the suggestions below for some hints.)

(b) Look at the pieces.

   i. How many cubes are in each piece?

   ii. How many different pieces could you make with three cubes? With four cubes? With five cubes? (Including the rectangular ones.)

   iii. Which pieces are symmetric? Which have different mirror images?

(c) Here's a sub-problem that might help you as you think about the Soma cube

   i. Look at a checkerboard with two missing corners that are diagonally opposite each other.

   ii. Can you tile the checkerboard with 31 dominos?

   iii. Imagine the dominos are colored so that each domino has a black square and a white square.

   iv. How many black squares are on the checkerboard? How many white squares? How many on each domino? How many are they total in all the 31 dominoes? Can you see why that makes this problem impossible?

(d) Look at $3 \times 3 \times 3$ Soma cube that you are trying to pack. How many corners (vertices) does it have?

(e) Look at the pieces: how many corners can each one fill? What's the maximum number for each piece, and the minimum number?

(f) Note that only ONE piece can fill less than it's maximum corners.

(g) Note that the T piece must either fill no corners, or 2 corners. Can it fill no corners in the solution? Try this out. Where must it go?

(h) Now imagine the big cube is colored like a 3-dimensional checkerboard. Are all the corners the same color? Edges? Facecenters? Center of the whole cube? Let all the corners be colored black. How many other cubes are black?

(i) Look at each piece. If you checkerboard the pieces with two colors, which pieces have an equal number of colored sub-cubes, which have different numbers?

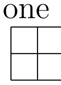
(j) If the T piece has to go on an edge, where can the Y piece go? (The Y piece is the one that looks like a corner, and is one of the ones that require three dimensions.)

(k) The V piece also has limited places it can go, but it is easier to just put the rest of the pieces in (after the T and Y), and save the v for last.

(l) Can you make a table to show where each piece could go in the cube?

(m) Can you guess how many solutions there are in total for the Soma cube? Can you figure an upper-bound for the number of solutions.

(n) There are MANY ways to dissect a cube into pieces like the Soma cube, giving very many possible puzzles. Some are more difficult then Soma, and some are easier. Some have a single solution, while others (like the Soma) that have many solutions.

(o) The last question is much harder than all the rest. Note that we have built all but two of the four cell (quad) pieces. The $2 \times 2$ square can be used to replace one of the other quads pieces. But which ones? Build a square . Notice that the square must have two black and two white cubes no matter how it fits in a Soma cube. Does this mean that it cannot replace the $Y$ or the $T$? Which of the quads can the square replace?

44. A model of a square pyramid is built from unit cubes in such a way that each square layer has one fewer cubes on the side as the layer below it. So, for example, if the bottom layer is $3 \times 3$ then the middle layer would be $2 \times 2$ and the top layer would have just one cube. For this set of problems, suppose the base is $25 \times 25$.

(a) What is the volume of the model? In other words, how many cubes are needed to build the model?

(b) When the model is placed on a table so that the bottom is not visible, what is the surface area of the visible part.

45. Three cubes with volumes 1, 8, and 27 are glued together to obtain a solid polyhedron with minimal surface area. What is that surface area?

46. Four cubes with volumes $0.125, 1, 8$, and 27 are glued together to obtain a solid polyhedron with minimal surface area. What is that surface area?

47. An $a \times b \times c,\ a \leq b \leq c$ blocks has the same volume numerically as surface area. Find all triplets $(a, b, c)$ for which this occurs.

48. Color the surface of a cube of dimension $5 \times 5 \times 5$ red, and then cut the cube into unit cubes. Remove all the unit cubes with no red faces. Use the remaining cubes to build a cuboid (a rectangular brick), keeping the outer surface of the cuboid red. What is the maximum possible volume of the cuboid?

49. The faces of a cube are colored red and blue, one at a time, with equal probability. What is the probability that the resulting cube has at least one vertex $P$ such that all three faces containing $P$ are colored red?

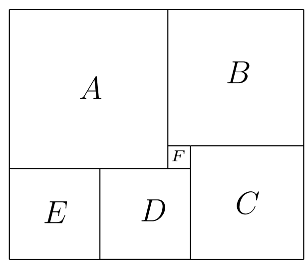
50. Four faces of an $a \times b \times c,\ a \leq b \leq c$ block of unit cubes are painted, not including an opposite pair. The block is cut into $abc$ unit cubes and a cube is selected at random. Let $P$ denote the probability that the selected cube has some paint, and let $Q$ denote the probability that a painted face turns up when the selected cube is rolled like a die. Suppose $P = 5Q$. Find $a, b$, and $c$.

51. A set of cubes each built from $n^3$ unit cubes, has sides $n = k, k+1, k+2, \ldots t$. All six faces of each cube is painted. When all the cubes are cut into unit cubes, the number with paint and the number without paint are found to be the same. What are the sizes of the cubes.

52. A $4 \times 4 \times 4$ cube is made from 32 white unit cubes and 32 black unit cubes. What is the largest possible percent of black surface area?

53. A $4 \times 4 \times 4$ wooden cube is painted on five of its faces and is then cut into 64 unit cubes. One unit cube is randomly selected and rolled. What is the probability that exactly two of the five visible faces are painted?

54. In the diagram, a rectangular sheet of paper is dissected into six squares labelled $A, B, C, D, E$, and $F$. The area of square $B$ is 324. What is the area of the rectangle?



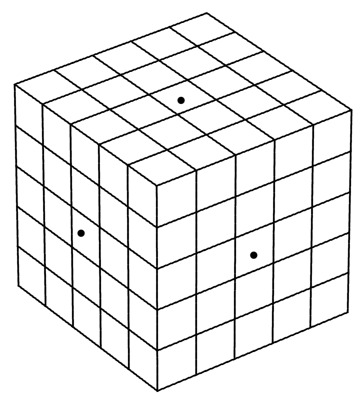

55. An $a \times b \times c$, with $a \leq b \leq c$ block of unit cubes has the interesting property that if each of the numbers $a, b, c$ is increased by 1, the volume of the block doubles. Find all triplets $(a, b, c)$ for which this is the case.

56. The $5 \times 5 \times 5$ cube shown below is built from 125 unit cubes. The dots on the surface show the places where the big cube is drilled through.

 (a) Suppose the cube is located so that each unit cube has a point $P$ at its center, where $P \in \{(x, y, z) \ : \ x, y, z \in \{1, 2, 3, 4, 5\}\}$. Find a relation that defines the cubes that have been drilled out.

 (b) Suppose the cube is located so that each unit cube has a point $P$ at its center, where $P \in \{(x, y, z) \ : \ x, y, z \in \{-2, -1, 0, 1, 2\}\}$. Find an equation that defines the cubes that have been drilled out.

 (c) When all these 'drilled' cubes have been removed, the cube is dipped into a bucket of paint. Then the cubes are cut apart and one is selected at random and rolled. What is the probability that a painted face comes up?



57. Is it possible to paint the faces of $n$ cubes, $n \geq 1$ with three colors, so that (a) all the cubes are painted identically and (b) the probability that when they are all rolled, we get at least one of each color with probability $1/3$? Is it possible that the probability of getting all one color is the same as the probability of getting at least one of each of the others?

58. A cube of SPAM defined by $S = \{(x, y, z) | 0 \leq x, y, z \leq 1\}$ is cut along the planes $x = y, y = z, z = x$. How many pieces are there? (No spam gets moved until all three cuts are made.)

59. What is the largest possible volume of a rectangular box whose diagonal length is 18?

60. How many rectangles are there in a 6 by 6 chessboard that contain neither the cell $(2, 5)$, nor the cell $(4, 3)$? The rows and columns are indexed by 1 through 6.

61. In this problem you're given a cube with an integer value assigned to each face. Our first hurdle is to figure out how to translate back and forth from a physical cube to a flat piece of paper. The mathematical object we deal with is called a *net*. A net is a planar representation

of a three-dimensional polyhedron. A net for a cube is shown below. Think of a cube positioned so that it has a top, a front, left and right sides, a back and a bottom(on the desk).

| | | | |
|---|---|---|---|
| | top | | |
| left | front | right | back |
| | bottom | | |

For example, it could be

| | | | |
|---|---|---|---|
| | 1 | | |
| 2 | 3 | 5 | 4 |
| | 6 | | |

just as the integers 1 through 6 appear on a standard die. Next assign each vertex the product of the numbers in the faces that vertex belongs to. For example, the vertex in the top right corner of the face with 3 would have an assigned value of $1 \cdot 3 \cdot 5 = 15$.

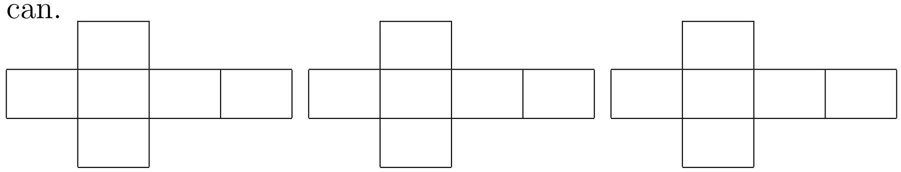
(a) Let $T$ denote the sum of the eight values of the vertices for the die above. Compute $T$ and explain why you get this unusual number. While you're thinking about this problem, imagine what happens to the $T$ value is you interchange the numbers on the top and bottom of your cube leaving the other four numbers in place.

(b) What is the largest possible value of $T$ that can be obtained? Can you prove it?

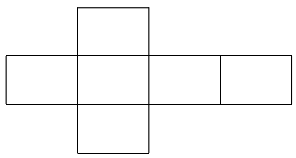
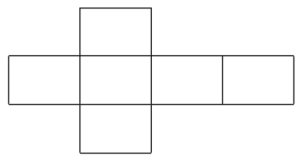
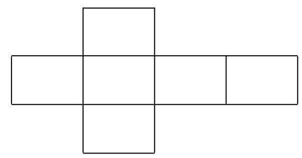
(c) Next we explore the question How many different values of $T$ are obtainable? Assuming that the faces must be labeled with all six digits 1 through 6, how many different values of $T$ can be obtained?

(d) Again each face is assigned a positive integer. Do not assume here that the integers assigned at 1 though 6. Also, we can assign the same integer to more than one face. Suppose the sum $T$ is 70. Can you determine the sum of the six integer faces?

(e) Use the nets provided to find as many different $T$ values as you can.



$T =$ $T =$ $T =$

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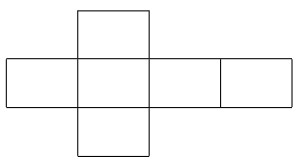 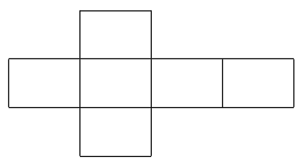 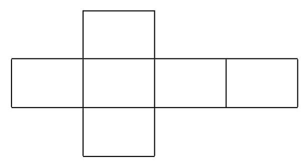

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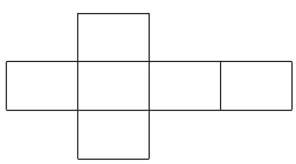
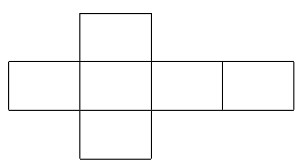
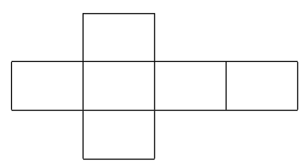
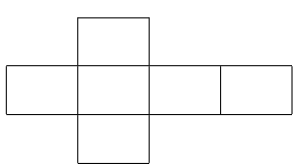
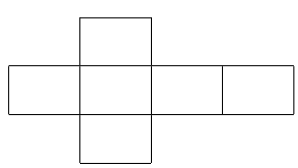
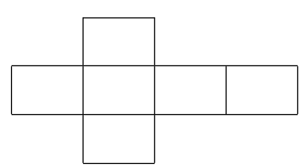
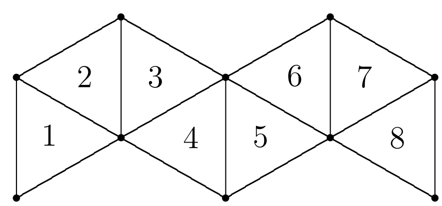
  

$T =$ $T =$ $T =$

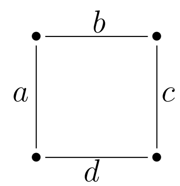
$T =$ $T =$ $T =$

62. Consider next the octahedral net shown below.




Again, assign to each vertex of the polyhedron (here an octagon) the product of the faces adjacent to that vertex, and again let $T$ denote the sum of the vertex values. Find $T$ for the values assigned in the figure. What is the largest value $T$ can have if the numbers 1 to 8 are distributed among the faces of the octahedron?

63. Suppose we want to generalize the cube problem so that we get more or fewer items in the summation. Note that the cube has 8 vertices and there are exactly eight items in the product $(a + b)(c + d)(e + f)$. If we wanted just four items, can we use a square?




As before let us assign to each vertex the product of the edges (faces) which contain it. Thus, we have $ab+bc+cd+da$, which we can factor by grouping:

$$ab+bc+cd+da = b(a+c)+d(a+c) = (b+d)(a+c).$$

64. Now let's go upwards in dimension to the 4-cube. Think of the cube within a cube, or as we show it below, as two cubes in space joined by line segments between corresponding pairs of vertices. The outside cube is the $xyz$ cube, that is the set of coordinates $(x, y, z, 0)$ where the fourth dimension $w$ is zero for all the points in the cube. Then the 'inside' cube is the set $(x, y, z, 1)$. Think of it as a pair of 3-d cubes attached to one another by line segments between corresponding vertices.

| | | | |
|---|---|---|---|
| | $y=1$ | | |
| $x=0$ | $z=0$ | $x=1$ | $z=1$ |
| | $y=0$ | | |

with $w=0$ and

| | | | |
|---|---|---|---|
| | $y=1$ | | |
| $x=0$ | $z=0$ | $x=1$ | $z=1$ |
| | $y=0$ | | |

with $w=1$

Recall that in the case of the cube above, we assigned a value to each 2-face. In the square, we assigned a value to each 1-face (edge). So n the case of four dimensions, we assign a value to each 3-face, that is to each of the eight subsets $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1, w = 0, w = 1$. These are opposites in pairs; ie. $x = 0$ is opposite $x = 1$. Each collection of four 3-faces that do not contain any opposite pairs determines a vertex. For example, the four 3-faces defined by $x = 0, y = 1, z = 0, w = 0$ all contain the vertex $(0, 1, 0, 0)$. If you assign to each 3-face an integer and assign to each vertex the product of the integers assigned to its four 3-faces, the sum of these products factors in the expected way. If

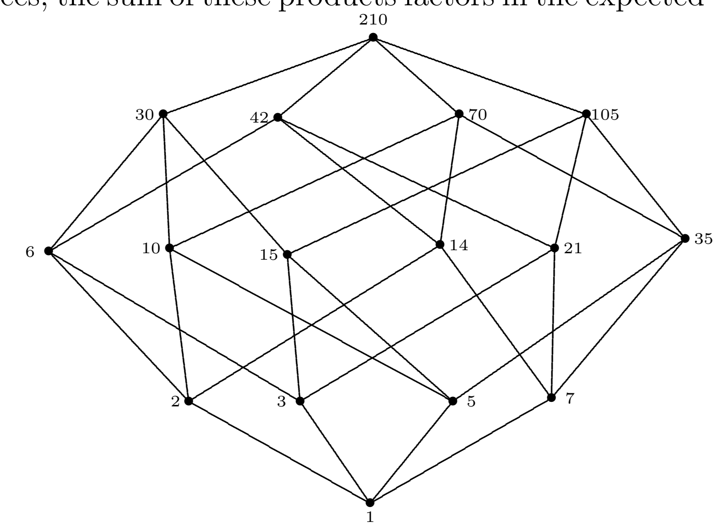



Fig 6a. The Hasse diagram of $D_{210}$

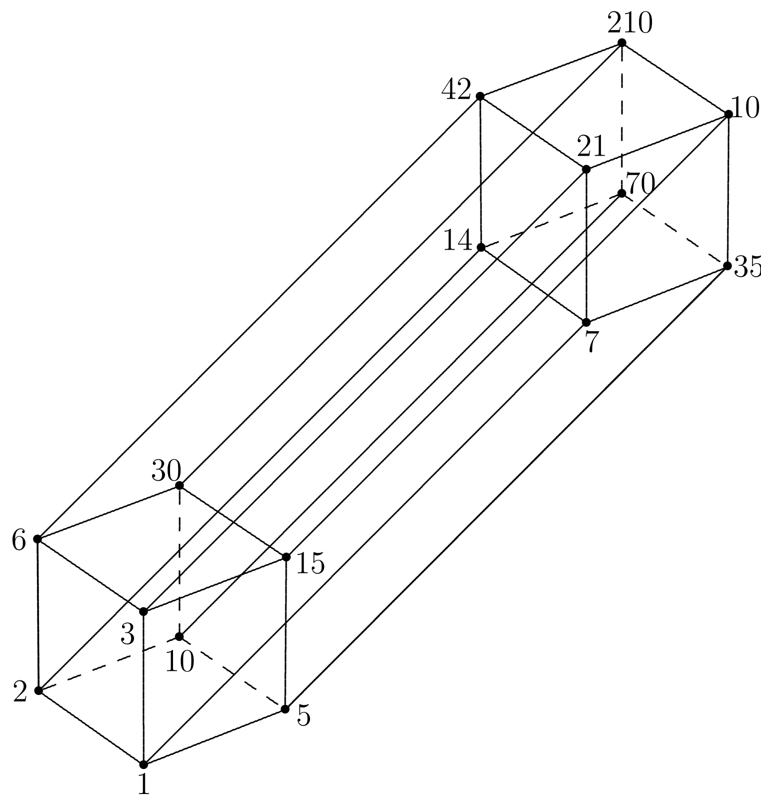



Fig 6b. The Hasse diagram of $D_{210}$

To find the coordinates of a point, say 70, factor it into $2^i 3^j 5^k 7^l$. Then the coordinate is $(i, j, k, l)$. So $70 = (1, 0, 1, 1)$.

65. Is there an $a \times b \times c$, $a \leq b \leq c$ block of cubes with a surface area of exactly 100?

66. Suppose a block of cubes has dimensions $x + 2, 3x - 2$ and $2x + 7$. Can it have a volume of exactly 900?

67. A bug starts at a vertex of the unit cube. Each minute, the bug chooses one of the three adjacent edges to walk along, each with probability 1/3. What is the probability that the bug is back at its starting point after exactly 4 minutes?

68. Each corner of a cube is labelled with a number. In each step, each number is replaced with the average of the labels of the three adjacent

corners. All eight numbers are replaced simultaneously. After ten steps, all labels are the same as their respective initial values. Does it necessarily follow that all eight numbers are equal initially?

69. Each face of two cubes is labelled randomly with a number, 1 to 6, as on a die. What is the probability that the labelings are the same. That is, what is the probability that one die can be rotated so that it is identical with the other.

70. You have many identical cube-shaped wooden blocks. You have four colors of paint to use, and you paint each face of each block a solid color so that each block has at least one face painted with each of the four colors. Find the number of distinguishable ways you could paint the blocks. (Two blocks are distinguishable if you cannot rotate one block so that it looks identical to the other block.)

71. Each face of a cube is painted either red or blue, each with probability 1/2. The color of each face is determined independently. What is the probability that the painted cube can be placed on a horizontal surface so that the four vertical faces are all the same color?

    (A) $\frac{1}{4}$ (B) $\frac{5}{16}$ (C) $\frac{3}{8}$ (D) $\frac{7}{16}$ (E) $\frac{1}{2}$

72. On a standard die one of the dots is removed at random with each dot equally likely to be chosen. The die is then rolled. What is the probability that the top face has an even number of dots?

73. You have many identical cube-shaped wooden blocks. You have four colors of paint to use, and you paint each face of each block a solid color so that each block has at least one face painted with each of the four colors. Find the number of distinguishable ways you could paint the blocks. (Two blocks are distinguishable if you cannot rotate one block so that it looks identical to the other block.)