Supplement to Subsection 2.6.3

As we know, the price of an American put option is the solution of the following LC problem:

\[
\begin{align*}
\min \left( -\frac{\partial P}{\partial t} - L_S P, \ P(S, t) - \max(E - S, 0) \right) &= 0, \quad 0 \leq S, \ t \leq T, \\
P(S, T) &= \max(E - S, 0), \quad 0 \leq S,
\end{align*}
\]

where

\[
L_S = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r.
\]

Let

\[
\begin{align*}
\zeta &= \frac{E^2}{S}, \\
C(\zeta, t) &= \frac{EP(S,t)}{S}.
\end{align*}
\]

Because

\[
\frac{E}{S} \max(E - S, 0) = \max(\zeta - E, 0),
\]

for \(C(\zeta, t)\) the payoff and constraint are \(\max(\zeta - E, 0)\). Noticing

\[
\begin{align*}
\frac{\partial P}{\partial t} &= \frac{S}{E} \frac{\partial C}{\partial t}, \\
\frac{\partial P}{\partial S} &= \frac{1}{E} \left[ C + S \frac{\partial C}{\partial \zeta} \left( -\frac{E^2}{S^2} \right) \right] = \frac{1}{E} \left( C - \zeta \frac{\partial C}{\partial \zeta} \right), \\
\frac{\partial^2 P}{\partial S^2} &= \frac{\zeta^3}{E^3} \frac{\partial^2 C}{\partial \zeta^2},
\end{align*}
\]

we have

\[
\begin{align*}
\frac{\partial P}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + (r - D_0) S \frac{\partial P}{\partial S} - rP \\
= \frac{S}{E} \left\{ \frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 \zeta^2 \frac{\partial^2 C}{\partial \zeta^2} + (D_0 - r) \zeta \frac{\partial C}{\partial \zeta} - D_0 C \right\}.
\end{align*}
\]

Therefore the function \(C(\zeta, t)\) is the solution of the following American call option problem:

\[
\begin{align*}
\min \left( -\frac{\partial C}{\partial t} - L_\zeta C, \ C(\zeta, t) - \max(\zeta - E, 0) \right) &= 0, \quad 0 \leq \zeta, \ t \leq T, \\
C(\zeta, T) &= \max(\zeta - E, 0), \quad 0 \leq \zeta,
\end{align*}
\]

where
\[ \mathbf{L}_s = \frac{1}{2} \sigma^2 z^2 \frac{\partial^2}{\partial z^2} + (D_0 - r) z \frac{\partial}{\partial z} - D_0. \]

Consequently, an American put problem can be converted into an American call problem. However, in the two problems, the state variable and the parameters are different. From the definitions of \( \mathbf{L}_s \) and \( \mathbf{L}_\zeta \), we know that the volatilities of the put and call problems are the same, but the interest rate and the dividend yield of the call problem are equal to the dividend yield and the interest rate of the put problem, respectively. In order to explain these facts, we express the dependences of the options on interest rate and dividend yield explicitly. Let \( P(S, t; b, a) \) denote the price of the put option and \( C(\zeta, t; a, b) \) the price of the call option, where the first and second parameters after the semicolon are the interest rate and the dividend yield, respectively. From the definition of \( \zeta \) and \( C(\zeta, t; a, b) \), we know

\[ P(S, t; b, a) = C(\zeta, t; a, b) \frac{S}{E}, \]

where \( \zeta = \frac{E^2}{S} \). This can also be rewritten as

\[ P(S, t; b, a) = C \left( \frac{E^2}{S}, t; a, b \right) \frac{S}{E}, \]

or

\[ C(\zeta, t; a, b) = P \left( \frac{E^2}{\zeta}, t; b, a \right) \zeta / E, \]

where we used the relation \( E / S = \zeta / E \). In the last relation, we can use \( S \), instead of \( \zeta \), as the state variable. That is, we can write this relation as

\[ C(S, t; a, b) = P \left( \frac{E^2}{S}, t; b, a \right) \frac{S}{E}. \]

Finally, putting them together, we have

\[
\begin{cases}
C(S, t; a, b) = P \left( \frac{E^2}{S}, t; b, a \right) \frac{S}{E}, & \text{or} \\
P(S, t; b, a) = C \left( \frac{E^2}{S}, t; a, b \right) \frac{S}{E}.
\end{cases}
\] (0.1)

For the special case \( S = E \), it becomes

\[ P(E, t; b, a) = C(E, t; a, b). \]

Also, the location of free boundary in the latter problem, \( \zeta_{cf}(t; a, b) \), must be equal to \( E^2 \) divided by the location of free boundary of the former problem, \( E^2 / S_{pf}(t; b, a) \), because \( \zeta = E^2 / S \), i.e.,

\[ \zeta_{cf}(t; a, b) = \frac{E^2}{S_{pf}(t; b, a)} \]

or

\[ S_{cf}(t; a, b) \times S_{pf}(t; b, a) = E^2, \] (0.2)
where in the last relation, instead of \( \zeta_E \), we use \( S_E \) as the name of the function representing the location of the free boundary. From the derivation above we know that for European options, the following relations also hold:

\[
\begin{align*}
\{ c(S, t; a, b) &= p \left( \frac{E^2}{S}, t; b, a \right) S/E, \quad \text{or} \\
p(S, t; b, a) &= c \left( \frac{E^2}{S}, t; a, b \right) S/E. 
\end{align*}
\] (0.3)

The relations (0.1)–(0.3) are called the put–call symmetry relations.

Now let us discuss the financial meaning of the put–call symmetry relations. Suppose that one British pound is worth \( S \) U.S. dollars and that \( E \) U.S. dollars are worth \( \zeta \) British pounds. It is clear that \( \zeta = E^2/S \). Let \( P \) be the value of a put option whose holder can always sell one pound for \( E \) dollars if the holder wants. This means that the payoff and constraint of the put option is \( \max(E - S, 0) \) in dollars. Let \( C \) be the value of a call option whose holder can buy \( E^2 \) dollars by paying \( E \) pounds if the holder wants. This means that the payoff and constraint of the call option are \( \max(E^2/S - E, 0) = \max(\zeta - E, 0) \) in pounds. The holder of the put option has the right to sell one pound for \( E \) U.S. dollars even if \( S \leq E \). The holder of \( 1/E \) units of the call option has the right to buy \( E \) dollars by paying one British pound even if \( \zeta \geq E \). The condition \( S \leq E \) is equivalent to \( E^2/S = \zeta \geq E \). Thus, both the holder of one unit of the put option and the holder of \( 1/E \) units of the call option have the right to exchange one pound for \( E \) dollars even if \( S < E \). The two holders have the same rights, so the value of one unit of the put option and the value of \( 1/E \) units of the call option in U.S. dollars, which is equal to \( S \cdot C/E \), should be equal, i.e.,

\[
P = S \cdot C/E.
\]

Here, we need to notice that \( P \) and \( C \) have different but related volatilities, interest rates, and dividend yields. According to Itô’s lemma, if

\[
dS = \mu S dt + \sigma S dX,
\]

then

\[
d\zeta = (-\mu + \sigma^2)\zeta dt - \sigma \zeta dX.
\]

Hence, the volatilities of \( S \) and \( \zeta = E^2/S \) are the same if the volatilities are constants. Suppose that \( \sigma, r, \) and \( D_0 \) are constant and that the interest rates of the British pound and the U.S. dollar are \( a \) and \( b \), respectively. Then \( r = a \) and \( D_0 = b \) for the call and \( r = b \) and \( D_0 = a \) for the put, and the volatilities are the same. In this case, the relation above can be written as

\[
P(S, t; b, a) = C \left( \frac{E^2}{S}, t; a, b \right) S/E.
\]

The first relation in (0.1) (or (0.3)) actually is another form of the second relation in (0.1) (or (0.3)). Thus from the financial reasoning here, we know that all the relations in (0.1) and (0.3) hold. Because the state variable \( \zeta \) for the call with \( r = a \) and \( D_0 = b \) and the state variable \( S \) for the put with \( r = b \)
and $D_0 = a$ have the relation $\zeta = E^2 / S$, the argument above to obtain (0.2) can still be used here. Hence from the financial reasoning above, we can also have (0.2).

Actually such relations exist for more complicated cases. If $\sigma$ depends upon $S$, then the following relations hold:

$$
\begin{align*}
C(S, t; a, b, \sigma(S)) &= P \left( \frac{E^2}{S}, t; b, a, \sigma(S) \right) S/E, \quad \text{or} \\
P(S, t; b, a, \sigma(S)) &= C \left( \frac{E^2}{S}, t; a, b, \sigma(S) \right) S/E
\end{align*}
$$

and

$$
S_{cf}(t; a, b, \sigma(S)) \times S_{pf}(t; b, a, \sigma(E^2 / S)) = E^2.
$$

Here, the third argument after the semicolon is the function for the volatility.

The symmetry relations can be used when we write codes for pricing American options. Suppose that we need codes for pricing American call and put options and that we already have a code for pricing American call options. If it is very easy for the code to be modified to a code for pricing American put options, then we can have another code for put options by modifying the code we already have. If the code for put options will be quite a different from the code for call options, then we can use the code for call options to find $C(E^2 / S, t; a, b)$ first and then obtain $P(S, t; b, a)$ by using the relation $P(S, t; b, a) = C(E^2 / S, t; a, b) \cdot S/E$.

**Supplement to Section 2.8**

Define

$$
C_\infty(S) = \begin{cases} 
\text{the solution of the free-boundary problem,} & 0 \leq S \leq S_f, \\
S - E, & S_f < S.
\end{cases}
$$

Let us show that $C_\infty(S)$ satisfies the following LC relation:

$$
\min \left( - \frac{1}{2} \sigma^2 S^2 \frac{d^2 C_\infty}{dS^2} + (r - D_0)S \frac{dC_\infty}{dS} - r C_\infty \right),
$$

$$
C_\infty - \max(S - E, 0) = 0,
$$

that is, $C_\infty(S)$ satisfies the LC relation of the perpetual American call option for any $S$.

Here let us verify this conclusion by direct computation. Before doing that, we point out that the following is true: $\alpha_+ \geq 1$ and $1/(1 - 1/\alpha_+) \geq$
max(1, r/D_0), the proof of which is left for readers. It is clear that from 
1/(1 - 1/\alpha_+) \geq \max(1, r/D_0), we can further have \( S_f = E/(1 - 1/\alpha_+) \geq E \max(1, r/D_0). \) For \( S \in (0, E), C_\infty \) satisfies the ODE and is greater than 0, and \( \max(S - E, 0) = 0 \). Thus the LC relation

\[
\min \left( - \left[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C_\infty}{\partial S^2} + (r - D_0) S \frac{\partial C_\infty}{\partial S} - r P_\infty \right] \right) = 0
\]

holds. Now let us check if the LC relation holds for \( S \in (E, S_f) \). Suppose that \( f(x), f'(x), \) and \( f''(x) \) are continuous functions on \([a, b]\). As we known, if \( f(b) = 0 \) and \( f'(b) = 0 \), then the following relation is true: 

\[
f(x) = \frac{1}{2} f''(\xi)(x - b)^2,
\]

where \( x \in [a, b] \) and \( \xi \in [x, b] \). Using this fact, we know that because 

\[
C_\infty(S_f) - (S_f - E) = 0 \quad \text{and} \quad \frac{dC_\infty(S_f)}{dS} - 1 = 0, \quad C_\infty(S) - (S - E) \geq 0 \quad \text{on} \quad (E, S_f) \text{ if } \frac{d^2 C_\infty(S)}{dS^2} \geq 0 \quad \text{on} \quad (E, S_f). \]

From the expression of \( C_\infty \), we have

\[
\frac{d^2 C_\infty}{dS^2} = \frac{\alpha_+ - 1}{S_f} \left( \frac{S}{S_f} \right)^{\alpha_+ - 2}.
\]

Hence from \( \alpha_+ \geq 1 \), we know \( \frac{d^2 C_\infty}{dS^2} \geq 0 \) and the LC relation holds on \((E, S_f)\).

For \( S \in (S_f, \infty) \), because \( S_f \geq E \max(1, r/D_0) \), we have 

\[
C_\infty(S) = S - E = \max(S - E, 0), \quad \text{which means} \quad C_\infty(S) - \max(S - E, 0) = 0,
\]

and

\[
- \frac{\sigma^2 S^2 \frac{\partial^2 C_\infty}{\partial S^2} - (r - D_0) S \frac{\partial C_\infty}{\partial S} + r C_\infty}{2} = D_0 S - r E = D_0 (S - r E/D_0) \geq 0.
\]

Thus the LC relation also holds for \( S \in (S_f, \infty) \). Consequently, we have proved our conclusion for all the cases.

**Supplement to Subsection 2.9.3**

On the \( n \) random variables, we further assume that \( S_1, S_2, \ldots, S_m, m \leq n \), are prices of some assets which can be traded on markets, and that the \( k \)-th asset pays a dividend payment \( D_k dt \) during the time interval \([t, t + dt]\), \( D_k \) being a known function that may depend on \( S_1, S_2, \ldots, S_n \) and \( t \). In order to derive the general PDE for financial derivatives, we suppose that there are \( n - m + 1 \) distinct financial derivatives dependent on \( S_1, S_2, \ldots, S_n \) and \( t \), and let \( V_k \) stand for the value of the \( k \)-th derivative, \( k = 0, 1, \ldots, n - m \). They could have different expiries, different exercise prices, or different payoff
functions. Even some of the derivatives may depend on only some of the random variables. According to the generalized Itô’s lemma, we have

\[ dV_k = f_k dt + \sum_{i=1}^{n} \nu_{i,k} dS_i, \]

where

\[ f_k = \frac{\partial V_k}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V_k}{\partial S_i \partial S_j} b_i b_j \rho_{ij}, \]

and

\[ \nu_{i,k} = \frac{\partial V_k}{\partial S_i}. \]

Furthermore, we suppose that the holder of a derivative might receive some coupon payment during the life of the derivative. Let the coupon payment for the \( k \)-th derivative during the time interval \([t, t + dt]\) be \( K_k dt\), \( K_k \) being a known function that may depend on \( S_1, S_2, \cdots, S_n \) and \( t \). Consider a portfolio consisting of the \( n - m + 1 \) derivatives and the \( m \) assets, whose prices are \( S_1, S_2, \cdots, S_m \):

\[ \Pi = \sum_{k=0}^{n-m} \Delta_k V_k + \sum_{k=n-m+1}^{n} \Delta_k S_{k-n+m}, \]

where \( \Delta_k \) is the amount of the \( k \)-th derivative for \( k = 0, 1, \cdots, n-m \) and the amount of the \( (k-n+m) \)-th asset, for \( k = n-m+1, n-m+2, \cdots, n \). During the time interval \([t, t + dt]\), the holder of this portfolio will earn

\[
\begin{align*}
&\sum_{k=0}^{n-m} \Delta_k (dV_k + K_k dt) + \sum_{k=n-m+1}^{n} \Delta_k (dS_{k-n+m} + D_{k-n+m} dt) \\
&= \sum_{k=0}^{n-m} \Delta_k \left( f_k dt + \sum_{i=1}^{n} \nu_{i,k} dS_i + K_k dt \right) \\
&+ \sum_{k=n-m+1}^{n} \Delta_k (dS_{k-n+m} + D_{k-n+m} dt) \\
&= \sum_{k=0}^{n-m} \Delta_k (f_k + K_k) dt + \sum_{i=1}^{n} \left( \sum_{k=0}^{n-m} \Delta_k \nu_{i,k} \right) dS_i \\
&+ \sum_{i=1}^{m} \Delta_{i+n-m} dS_i + \sum_{k=n-m+1}^{n} \Delta_k D_{k-n+m} dt \\
&= \sum_{k=0}^{n-m} \Delta_k (f_k + K_k) dt + \sum_{i=1}^{n} \left( \sum_{k=0}^{n-m} \Delta_k \nu_{i,k} + \Delta_{i+n-m} \right) dS_i \\
&+ \sum_{i=m+1}^{n} \left( \sum_{k=0}^{n-m} \Delta_k \nu_{i,k} \right) dS_i + \sum_{k=n-m+1}^{n} \Delta_k D_{k-n+m} dt.
\end{align*}
\]
Let us choose $\Delta_k$ so that

$$
\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} + \Delta_{i+n-m} = 0, \quad i = 1, 2, \cdots, m
$$

and

$$
\sum_{k=0}^{n-m} \Delta_k \nu_{i,k} = 0, \quad i = m + 1, m + 2, \cdots, n.
$$

In this case the portfolio is risk-free, so its return rate is $r$, i.e.,

$$
\sum_{k=0}^{n-m} \Delta_k (f_k + K_k) dt + \sum_{k=n-m+1}^{n} \Delta_k D_{k-n+m} dt = r \left[ \sum_{k=0}^{n-m} \Delta_k V_k + \sum_{k=n-m+1}^{n} \Delta_k S_{k-n+m} \right] dt,
$$

or

$$
\sum_{k=0}^{n-m} \Delta_k (f_k + K_k - rV_k) + \sum_{k=n-m+1}^{n} \Delta_k (D_{k-n+m} - rS_{k-n+m}) = 0,
$$

or

$$
\sum_{k=0}^{n-m} \Delta_k (f_k + K_k - rV_k) + \sum_{k=1}^{m} \Delta_{n-m+k} (D_k - rS_k) = 0.
$$

This relation and the relations the chosen $\Delta_k$ satisfy can be written together in a matrix form:

\[
\begin{bmatrix}
\nu_{1,0} & \nu_{1,1} & \cdots & \nu_{1,n-m} & 1 & 0 & \cdots & 0 \\
\nu_{2,0} & \nu_{2,1} & \cdots & \nu_{2,n-m} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
\nu_{m,0} & \nu_{m,1} & \cdots & \nu_{m,n-m} & 0 & 0 & \cdots & 1 \\
\nu_{m+1,0} & \nu_{m+1,1} & \cdots & \nu_{m+1,n-m} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & & \vdots & \vdots & \vdots & \ddots & \vdots \\
\nu_{n,0} & \nu_{n,1} & \cdots & \nu_{n,n-m} & 0 & 0 & \cdots & 0 \\
g_0 & g_1 & \cdots & g_{n-m} & h_1 & h_2 & \cdots & h_m \\
\end{bmatrix}
\begin{bmatrix}
\Delta_0 \\
\Delta_1 \\
\vdots \\
\Delta_{n-m} \\
\Delta_{n-m+1} \\
\Delta_{n-m+2} \\
\vdots \\
\Delta_n \\
\end{bmatrix} = 0,
\]

where $g_k = f_k + K_k - rV_k$, $k = 0, 1, \cdots, n - m$ and $h_k = D_k - rS_k$, $k = 1, 2, \cdots, m$. In order for the system to have a non-trivial solution, the determinant of the matrix must be zero, or the $n + 1$ row vectors of the matrix must be linearly dependent. Therefore, it is expected that the last row can be expressed as a linear combination of the other rows with coefficients $\lambda_1, \lambda_2, \cdots, \lambda_n$:

$$
g_k = \sum_{i=1}^{n} \lambda_i \nu_{i,k}, \quad k = 0, 1, \cdots, n - m
$$
and
\[ h_k = \lambda_k, \quad k = 1, 2, \cdots, m. \]

Using the last \( m \) relations, we can rewritten the first \( n - m + 1 \) relations as

\[
g_k = \sum_{i=1}^{m} h_i \nu_{i,k} - \sum_{i=m+1}^{n} \lambda_i \nu_{i,k} = 0, \quad k = 0, 1, \cdots, n - m,
\]

which means that any derivative satisfies an equation in the form

\[
f + K - rV - \sum_{i=1}^{m} h_i \frac{\partial V}{\partial S_i} - \sum_{i=m+1}^{n} \lambda_i \frac{\partial V}{\partial S_i} = 0,
\]

or

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j \rho_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{m} (rS_i - D_i) \frac{\partial V}{\partial S_i}
- \sum_{i=m+1}^{n} \lambda_i \frac{\partial V}{\partial S_i} - rV + K = 0,
\]

where \( b_i, \rho_{ij} \) are given functions in the models of \( S_i \), \( \lambda_i \) are unknown functions which are independent of \( V_0, V_1, \cdots, V_{n-m} \) and could depend on \( S_1, S_2, \cdots, S_n \) and \( t \), and \( K \) depends on the individual derivative security. Usually \( \lambda_i \) is written in the form:

\[
\tilde{\lambda}_i = \lambda_i b_i - a_i
\]

and \( \lambda_i \) is called the market price of risk for \( S_i \). Using this notation, we finally arrive at

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j \rho_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{m} (rS_i - D_i) \frac{\partial V}{\partial S_i}
+ \sum_{i=m+1}^{n} (a_i - \lambda_i b_i) \frac{\partial V}{\partial S_i} - rV + K = 0.
\]

It is clear that if \( m = n = 1, b_1 = \sigma_1 S_1, D_1 = D_{01} S_1, \) and \( K = 0, \) then this equation becomes the Black–Scholes equation (2.12) after ignoring the subscript 1.

In the last we give some explanation on why \( \lambda_i \) is called the market price of risk for \( S_i \). For simplicity, assume that non of \( S_k, k = 1, 2, \cdots, n, \) is a price. In this case the PDE above becomes

\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_i b_j \rho_{ij} \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} (a_i - \lambda_i b_i) \frac{\partial V}{\partial S_i} - rV + K = 0.
\]

According to Itô’s lemma and using this PDE, we have
\[ dV = \left( \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 V}{\partial S_i \partial S_j} b_i b_j \rho_{ij} \right) dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} dS_i \]
\[ = \left( \sum_{i=1}^{n} (\lambda_i b_i - a_i) \frac{\partial V}{\partial S_i} + rV - K \right) dt + \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} (a_i dt + b_i dX_i) \]

or
\[ dV + K dt - rV dt = \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} b_i (dX_i + \lambda_i dt) . \]

Here, \( dV + K dt \) is the return for the derivative including the coupon payment and \( rV dt \) is the return if the investment is risk-free. Therefore, \( dV + K dt - rV dt \) is the excess return above the risk-free rate during the time interval \([t, t + dt]\). This equals the right-hand side of the equation. Its expectation is \( \sum_{i=1}^{n} \frac{\partial V}{\partial S_i} b_i \lambda_i dt \) because \( E[dX_i] = 0, i = 1, 2, \ldots, n \). Therefore, the term \( \frac{\partial V}{\partial S_i} b_i \lambda_i dt \) may be interpreted as an excess return above the risk-free return for taking the risk \( dX_i \). Consequently, \( \lambda_i \) is a price of risk for \( S_i \) that is associated with \( dX_i \) and is often called the market price of risk for \( S_i \).
Supplement to Subsection 3.2.2

In order to find analytic solutions of European Barrier options, let us consider the following problem:

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV &= 0, \quad 0 \leq S, \quad t \leq T, \\
V(S, T) &= \begin{cases} 
\varphi_1(S), & 0 \leq S \leq B, \\
\varphi_2(S), & B < S,
\end{cases}
\end{align*}
\]

(0.4)

where \( \varphi_1(S) \) and \( \varphi_2(S) \) are continuous functions and \( \varphi_1(B) = \varphi_2(B) \) may not hold. Let us show that if

\[
\varphi_1(S) = -\left( \frac{B}{S} \right)^{2(r-D_0-\sigma^2/2)/\sigma^2} \varphi_2\left( \frac{B^2}{S} \right)
\]
or

\[
\varphi_2(S) = -\left( \frac{B}{S} \right)^{2(r-D_0-\sigma^2/2)/\sigma^2} \varphi_1\left( \frac{B^2}{S} \right),
\]

then the solution \( V(S, t) \) must satisfy the condition \( V(B, t) = 0 \).

As we know, the solution of the above problem can be expressed by

\[
V(S, t) = e^{-r(T-t)} \int_0^\infty V(S', T) G(S', T; S, t) dS',
\]

where

\[
G(S', T; S, t) = \frac{1}{S'\sigma\sqrt{2\pi(T-t)}} e^{-\left[ \ln(S'/S) - (r-D_0-\sigma^2/2)(T-t) \right]^2/2\sigma^2(T-t)}.
\]

Therefore, \( V(B, t) = 0 \) is equivalent to

\[
\begin{align*}
&\int_0^\infty V(S', T) G(S', T; B, t) dS' \\
&= \int_0^B \varphi_1(S') G(S', T; B, t) dS' + \int_B^\infty \varphi_2(S') G(S', T; B, t) dS' \\
&= \int_0^B \varphi_1(S') G(S', T; B, t) dS' \\
&+ \int_B^0 \varphi_2\left( \frac{B^2}{S'} \right) G\left( B^2/S'', T; B, t \right) \left( -\frac{B^2}{S''} \right) dS'' \\
&= \int_B^0 \left[ \varphi_1(S') G(S', T; B, t) + \varphi_2\left( \frac{B^2}{S'} \right) \frac{B^2}{S''} G\left( B^2/S', T; B, t \right) \right] dS' \\
&= 0.
\end{align*}
\]
Thus if
\[ \phi_1 (S') G (S', T; B, t) + \phi_2 \left( \frac{B^2}{S'} \right) \frac{B^2}{S} G (B^2/S', T; B, t) = 0, \]
then \( \nabla (B, t) = 0 \). Because
\begin{align*}
G \left( \frac{B^2}{S'}, T; B, t \right) &= \frac{S'}{B^2} e^{-\frac{1}{2} \left[ \ln(B/S) - (r - D_0 - \sigma^2/2)(T-t) \right]^2 / \sigma^2 (T-t)} \times \\
&= \frac{S'}{B^2} e^{4 \ln(B/S') (r - D_0 - \sigma^2/2)(T-t)/2 \sigma^2 (T-t)} \times \\
&= \left( \frac{B}{S'} \right)^{2(r - D_0 - \sigma^2/2)/\sigma^2 - 2}
\end{align*}
is a function of \( S' \), such a relation exists, i.e., if
\[ \phi_1 (S') = - \left( \frac{B}{S'} \right)^{2(r - D_0 - \sigma^2/2)/\sigma^2} \phi_2 \left( \frac{B^2}{S'} \right), \]
then \( \nabla (B, t) = 0 \) holds. Let \( S' = \frac{B^2}{S''} \), then this relation can also be rewritten as
\[ \phi_1 \left( \frac{B^2}{S''} \right) = - \left( \frac{S''}{B} \right)^{2(r - D_0 - \sigma^2/2)/\sigma^2} \phi_2 \left( S'' \right), \]
or
\[ \phi_2 (S') = - \left( \frac{B}{S'} \right)^{2(r - D_0 - \sigma^2/2)/\sigma^2} \phi_1 \left( \frac{B^2}{S'} \right). \]

Therefore we obtain our conclusion.

Now let us show that for the problem
\[ \begin{align*}
&\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0, \quad B_t \leq S, \quad t \leq T, \\
&V (S, T) = V_T (S), \quad B_t \leq S, \\
&V (B_t, t) = 0, \quad t \leq T,
\end{align*} \tag{0.5} \]
the solution is
\[ V (S, t) = e^{-r(T-t)} \int_{B_t}^{S} V_T (S') G_1 (S', T; S, t, B_t) dS', \]
where
\[ G_1 (S', T; S, t, B_t) = G (S', T; S, t) - (B_t/S')^{2(r - D_0 - \sigma^2/2)/\sigma^2} G (S', T; B_t^2/S, t). \]
Usually $G_1$ is called Green’s function of down-and-out option problems.

Let us set $B = B_t$ and $\varphi_2(S) = V_T(S)$ in (0.4). From the result above we know if

$$\varphi_1(S) = -(B/S)^{2(r-D_0-\sigma^2/2)/\sigma^2} \varphi_2(B^2/S),$$

then the solution of (0.4) on $[B_t, \infty)$ is the solution of (0.5). Thus

$$V(S, t) = e^{-r(T-t)} \left[ - \int_0^{B_t} V_T \left( \frac{B_t^2}{S'} \right) \left( \frac{B_t}{S'} \right)^{2(r-D_0-\sigma^2/2)/\sigma^2} \times G(S', T; S, t) dS' \right. \\
+ \int_{B_t}^{\infty} V_T(S') G(S', T; S, t) dS' \right]$$

$$= e^{-r(T-t)} \int_{B_t}^{\infty} V_T(S') \left[ - \left( \frac{S'}{B_t} \right)^{2(r-D_0-\sigma^2/2)/\sigma^2} \times \frac{G(B_t^2/S', T; S, t)}{G(S', T; B_t^2/S, t)} G(S', T; B_t^2/S, t) \right. \\
+ \left. G(S', T; S, t) \right] dS'$$

$$= e^{-r(T-t)} \int_{B_t}^{\infty} V_T(S') \left[ G(S', T; S, t) \\
- \left( \frac{B_t}{S} \right)^{2(r-D_0-\sigma^2/2)/\sigma^2} G(S', T; B_t^2/S, t) \right] \frac{dS'}{S'}$$

$$= e^{-r(T-t)} \int_{B_t}^{\infty} V_T(S') G_1(S', T; S, t, B_t) dS'.$$

Here we used the relation:

$$\frac{G(B_t^2/S', T; S, t)}{G(S', T; B_t^2/S, t)} = \frac{S'}{B_t^2} e^{-\left[ \ln(B_t^2/S') - (r-D_0-\sigma^2/2)(T-t) \right]^2/2\sigma^2(T-t)}$$

Finally, we have

$$= \frac{S'}{B_t^2} e^{-\left[ \ln(S'/B_t^2) - (r-D_0-\sigma^2/2)(T-t) \right]^2/2\sigma^2(T-t)}$$

$$= \frac{S^2}{B_t^2} e^{4\ln(B_t^2/S') (r-D_0-\sigma^2/2)(T-t)/2\sigma^2(T-t)}.$$
Based on this result we know that if \( B_t \leq E \) and \( S \geq B_t \), then

\[
c_0 (S, t) = e^{-r(T-t)} \int_{B_t}^{\infty} \max (S' - E, 0) G_1 (S', T; S, t, B_t) dS'
= e^{-r(T-t)} \int_{0}^{\infty} \max (S' - E, 0) G_1 (S', T; S, t, B_t) dS' \\
= e^{-r(T-t)} \int_{0}^{\infty} \max (S' - E, 0) G (S', T; S, t) dS' \\
- \left( \frac{B_t}{S} \right)^{2(r-D_0-\sigma^2/2)/\sigma^2} e^{-r(T-t)} \\
\times \int_{0}^{\infty} \max (S' - E, 0) G (S', T; B_t^2/S, t) dS' \\
= c (S, t) - \left( \frac{B_t}{S} \right)^{2(r-D_0-\sigma^2/2)/\sigma^2} c \left( \frac{B_t^2}{S}, t \right). \quad (0.6)
\]

**Supplement to Subsection 3.4.3**

In order to find closed-form solutions of European lookback options, let us consider the following problem:

\[
\begin{aligned}
\frac{\partial W}{\partial t} + \mathbf{L}_\eta W &= 0, \quad 0 \leq \eta, \ t \leq T, \\
W (\eta, T) &= \begin{cases} \\
\varphi_1 (\eta), & 0 \leq \eta \leq 1, \\
\varphi_2 (\eta), & 1 < \eta,
\end{cases}
\end{aligned}
\]

where \( \varphi_1 (\eta) \) and \( \varphi_2 (\eta) \) are continuous functions, \( \varphi_1 (1) = \varphi_2 (1) \) might not hold, and

\[
\mathbf{L}_\eta = \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2}{\partial \eta^2} + (D_0 - r) \eta \frac{\partial}{\partial \eta} - D_0.
\]

Let us show that if

\[
\begin{aligned}
\varphi_1 (1) &= \varphi_2 (1), \\
\frac{d\varphi_1 (\eta)}{d\eta} &= \eta^{2(r-D_0+\sigma^2/2)/\sigma^2} \frac{d\varphi_2 (1/\eta)}{d\eta},
\end{aligned}
\]

or
\[
\left\{
\begin{aligned}
\varphi_2(1) &= \varphi_1(1), \\
\frac{d\varphi_2(\eta)}{d\eta} &= \eta^2(e^{-D_0-\sigma^2/2})\frac{d\varphi_1(1/\eta)}{d\eta},
\end{aligned}
\]

then \( \frac{\partial \bar{W}(1, t)}{\partial \eta} = 0. \)

As we know, the solution of the above problem can be expressed by

\[
W(\eta, t) = e^{-D_0(T-t)} \int_0^\infty \bar{W}(\eta', T) \bar{G}(\eta', T; \eta, t) d\eta',
\]

where

\[
\bar{G}(\eta', T; \eta, t) = \frac{1}{\sigma \sqrt{2\pi (T-t)\eta'}} e^{-\frac{\ln(\eta'/\eta)-(D_0-\sigma^2/2)(T-t)^2}{2\sigma^2(T-t)}}.
\]

Let

\[
\bar{G}(\eta', T; \eta, t) = e^{-\frac{\ln(\eta'/\eta)-(D_0-\sigma^2/2)(T-t)^2}{2\sigma^2(T-t)}}.
\]

For \( \bar{G}(\eta', T; \eta, t) \) the following is true:

\[
-\frac{\partial \bar{G}}{\partial \eta} = \eta' \frac{\partial \bar{G}}{\partial \eta'}.
\]

Then the expression for \( \bar{W}(\eta, t) \) can be rewritten as

\[
W(\eta, t) = \frac{e^{-D_0(T-t)}}{\sigma \sqrt{2\pi (T-t)\eta'}} \int_0^\infty \bar{W}(\eta', T) \bar{G}(\eta', T; \eta, t) d\eta'
\]

and we can have

\[
\frac{\partial \bar{W}(\eta, t)}{\partial \eta} = \frac{e^{-D_0(T-t)}}{\eta \sigma \sqrt{2\pi (T-t)}} \left[ \int_0^\infty \bar{W}(\eta', T) \frac{\partial \bar{G}(\eta', T; \eta, t)}{\partial \eta'} d\eta' \right] \bigg|_0^\infty
\]

\[
- \int_0^\infty \bar{G}(\eta', T; \eta, t) \frac{\partial \bar{W}(\eta', T)}{\partial \eta'} d\eta' \bigg|_0^\infty
\]

\[
- \frac{e^{-D_0(T-t)}}{\eta \sigma \sqrt{2\pi (T-t)}} \int_0^\infty \bar{G}(\eta', T; \eta, t) \frac{\partial \bar{W}(\eta', T)}{\partial \eta'} d\eta'.
\]

Thus, \( \frac{\partial \bar{W}(1, t)}{\partial \eta} = 0 \) is equivalent to that the last integral above at \( \eta = 1 \) is equal to zero, i.e.,
\[
\begin{align*}
&\int_0^\infty \mathcal{G}(\eta', T; 1, t) \frac{\partial \mathcal{W}(\eta', T)}{\partial \eta'} d\eta' \\
&= \int_0^1 \mathcal{G}(\eta', T; 1, t) \frac{\partial \mathcal{W}(\eta', T)}{\partial \eta'} d\eta' + \int_1^\infty \mathcal{G}(\eta', T; 1, t) \frac{\partial \mathcal{W}(\eta', T)}{\partial \eta'} d\eta' \\
&\quad + \int_1^\infty \mathcal{G}(\eta', T; 1, t) \frac{\partial \mathcal{W}(\eta', T)}{\partial \eta'} d\eta' \\
&\quad + \int_1^\infty \mathcal{G}(\eta', T; 1, t) \frac{\partial \mathcal{W}(\eta', T)}{\partial \eta'} d\eta' \\
&= \int_0^1 \mathcal{G}(\eta', T; 1, t) \frac{d\varphi_1(\eta')}{d\eta'} d\eta' + [\varphi_2(1) - \varphi_1(1)] \mathcal{G}(1, T; 1, t) \\
&\quad + \int_1^\infty \mathcal{G}(\eta', T; 1, t) \frac{d\varphi_2(\eta')}{d\eta'} d\eta' \\
&= [\varphi_2(1) - \varphi_1(1)] \mathcal{G}(1, T; 1, t) + \int_0^1 \mathcal{G}(\eta', T; 1, t) \frac{d\varphi_1(\eta')}{d\eta'} d\eta' \\
&\quad + \int_1^\infty \mathcal{G}(\eta', T; 1, t) \frac{d\varphi_2(\eta')}{d\eta'} d\eta' \\
&= 0.
\end{align*}
\]

Consequently, if the two conditions

\[\varphi_1(1) = \varphi_2(1)\]

and

\[\frac{d\varphi_1(\eta')}{d\eta'} = \frac{\mathcal{G}(1/\eta', T; 1, t)}{\mathcal{G}(\eta', T; 1, t)} \frac{d\varphi_2(1/\eta')}{d\eta'}\]

hold, then

\[\frac{\partial \mathcal{W}(1, t)}{\partial \eta} = 0.\]

Because

\[\frac{\mathcal{G}(1/\eta', T; 1, t)}{\mathcal{G}(\eta', T; 1, t)} = \frac{e^{-\ln \eta' + (D_0 - r - \sigma^2/2)(T-t)/2\sigma^2(T-t)}}{e^{-\ln \eta' - (D_0 - r - \sigma^2/2)(T-t)/2\sigma^2(T-t)}}\]

\[= e^{-4\ln \eta'(D_0 - r - \sigma^2/2)(T-t)/2\sigma^2(T-t)} = (\eta')^{2(r-D_0+\sigma^2/2)/\sigma^2},\]

the second condition above can be rewritten as

\[\frac{d\varphi_1(\eta')}{d\eta'} = (\eta')^{2(r-D_0+\sigma^2/2)/\sigma^2} \frac{d\varphi_2(1/\eta')}{d\eta'}.\]

Thus when
\[
\begin{aligned}
\begin{cases}
\varphi_1 (1) = \varphi_2 (1), \\
\frac{d\varphi_1 (\eta)}{d\eta} = \eta^2 (r-D_0 + \sigma^2 / 2) / \sigma^2 \frac{d\varphi_2 (1/\eta)}{d\eta}
\end{cases}
\end{aligned}
\]

\[
\frac{\partial W (1, t)}{\partial \eta} = 0
\] holds. Let \( \xi = \frac{1}{\eta} \). Then from

\[
d\xi = -\frac{1}{\eta^2} d\eta = -\xi^2 d\eta
\]

and the relation

\[
\frac{d\varphi_2 (\eta)}{d\eta} = \eta^2 (r-D_0 + \sigma^2 / 2) / \sigma^2 \frac{d\varphi_1 (1/\eta)}{d\eta},
\]

we have

\[
\frac{d\varphi_2 (\eta)}{d\eta} = \frac{d\varphi_2 (1/\xi)}{-\xi^{-2} d\xi} = \xi^{-2} (r-D_0 + \sigma^2 / 2) / \sigma^2 \frac{d\varphi_1 (\xi)}{-\xi^{-2} d\xi}.
\]

Thus

\[
\frac{d\varphi_1 (\xi)}{d\xi} = \xi^2 (r-D_0 + \sigma^2 / 2) / \sigma^2 \frac{d\varphi_2 (1/\xi)}{d\xi},
\]

or

\[
\frac{d\varphi_1 (\eta)}{d\eta} = \eta^2 (r-D_0 + \sigma^2 / 2) / \sigma^2 \frac{d\varphi_2 (1/\eta)}{d\eta}.
\]

Therefore, from the result we have obtained we can further have that

\[
\begin{aligned}
\begin{cases}
\varphi_2 (1) = \varphi_1 (1), \\
\frac{d\varphi_2 (\eta)}{d\eta} = \eta^2 (r-D_0 + \sigma^2 / 2) / \sigma^2 \frac{d\varphi_1 (1/\eta)}{d\eta}
\end{cases}
\end{aligned}
\]

then \( \frac{\partial W (1, t)}{\partial \eta} = 0 \) also holds.

**Remark** The condition \( \frac{\partial W (1, t)}{\partial \eta} = 0 \) is also equivalent to

\[
\int_0^\infty \left[ \varphi_1 (\eta') \frac{\partial G (\eta', T; 1, t)}{\partial \eta'} - \varphi_2 (1/\eta') \frac{\partial G (1/\eta', T; 1, t)}{\partial \eta'} \right] d\eta' = 0.
\]

However, we cannot find the relation between \( \varphi_1 (\eta') \) and \( \varphi_2 (\eta') \) by assuming

\[
\varphi_1 (\eta') \frac{\partial G (\eta', T; 1, t)}{\partial \eta'} - \varphi_2 (1/\eta') \frac{\partial G (1/\eta', T; 1, t)}{\partial \eta'} = 0
\]

because \( \frac{\partial G (\eta', T; 1, t)}{\partial \eta'} \) depends on not only \( \eta' \) but also \( T - t \).
Let \( c_{ls}(S, L, t) \) denote the price of a European lookback strike call option. As we know, set \( W = \frac{c_{ls}(S, L, t)}{S} \) and \( \eta = \frac{L}{S} \), then \( W(\eta, t) \) satisfies
\[
\begin{cases}
\frac{\partial W}{\partial t} + \mathbf{L}_\eta W = 0, & 0 \leq \eta \leq 1, \quad t \leq T, \\
W(\eta, t) = \max(\alpha - \eta, 0), & 0 \leq \eta \leq 1, \\
\frac{\partial W}{\partial t}(1, t) = 0, & t \leq T,
\end{cases}
\tag{0.8}
\]
where \( \alpha \leq 1 \).

From the result we just obtain, we know that if \( \varphi_1(\eta) = \max(\alpha - \eta, 0) \) and \( \varphi_2(\eta) \) satisfies
\[
\frac{d\varphi_2(\eta)}{d\eta} = \eta^{2(r - D_0 + \sigma^2/2)/\sigma^2} \frac{d\varphi_1(1/\eta)}{d\eta}
\]
with \( \varphi_2(1) = \varphi_1(1) \), then the solution of (0.7) for \( \eta \in [0, 1] \) is the solution of (0.8). In order to find such a solution, first let us find \( \varphi_2(\eta) \). In this case \( \frac{d\varphi_1(1/\eta)}{d\eta} = 0 \) for \( \eta \in (1, 1/\alpha) \) and \( \frac{d\varphi_1(1/\eta)}{d\eta} = 1/\eta^2 \) for \( \eta \in (1/\alpha, \infty) \).

Because \( \varphi_1(1) = 0 \), we have \( \varphi_2(\eta) = 0 \) for \( \eta \in (1, 1/\alpha) \). When \( \eta \in (1/\alpha, \infty) \),
\[
\varphi_2(\eta) = \varphi_2(1/\alpha) + \int_{1/\alpha}^{\eta} \frac{d\varphi_2(\eta)}{d\eta} d\eta
\]
\[
= \int_{1/\alpha}^{\eta} \eta^{2(r - D_0 + \sigma^2/2)/\sigma^2 - 2} d\eta
\]
\[
= \frac{(\eta)^2(r - D_0)/\sigma^2 - \alpha^{-2}(r - D_0)/\sigma^2}{2(r - D_0)/\sigma^2}.
\]

Here we assume \( r - D_0 \neq 0 \). Therefore if set \( \tau = T - t \), then for \( W(\eta, t) \) we have
\[
W(\eta, t)
= e^{-D_0 \tau} \left[ \int_0^1 \varphi_1(\eta') \tilde{G}(\eta', T; \eta, t) d\eta' + \int_1^\infty \varphi_2(\eta') \tilde{G}(\eta', T; \eta, t) d\eta' \right]
= e^{-D_0 \tau} \left[ \int_0^{\alpha - \eta} \varphi_1(\eta') \tilde{G}(\eta', T; \eta, t) d\eta' + \int_1^\infty \varphi_2(\eta') \tilde{G}(\eta', T; \eta, t) d\eta' \right]
= e^{-D_0 \tau} \left[ \alpha N \left( \frac{\ln \frac{\alpha}{\eta e(D_0 - r)\tau} + \sigma^2 \tau/2}{\sigma \sqrt{\tau}} \right) \right.
\]
\[
- \eta e(D_0 - r)\tau N \left( \frac{\ln \frac{\alpha}{\eta e(D_0 - r)\tau} - \sigma^2 \tau/2}{\sigma \sqrt{\tau}} \right) \right].
\]
\[
\begin{align*}
\frac{\sigma^2}{2(r - D_0)} & \int_{1/\alpha}^{\infty} \left[ (\eta')^{2(r-D_0)/\sigma^2} - e^{-2(r-D_0)/\sigma^2} \right] \tilde{G}(\eta', T; \eta, t) d\eta' \\
= e^{-D_0\tau} & \left[ \alpha N \left( \frac{\ln(\alpha/\eta) + (r - D_0 + \sigma^2/2) \tau}{\sigma\sqrt{\tau}} \right) \\
& - \eta e^{(D_0 - r)\tau} N \left( \frac{\ln(\alpha/\eta) + (r - D_0 - \sigma^2/2) \tau}{\sigma\sqrt{\tau}} \right) \\
& + \frac{\sigma^2}{2(r - D_0)} \eta e^{2(r-D_0)/\sigma^2} e^{4(r-D_0)/\sigma^2} e^{-2(r-D_0)/\sigma^2} \sigma^2\tau/2 \\
& \times N \left( \frac{\ln(\alpha e^{(D_0 - r)\tau}) - \sigma^2\tau/2 + 2(r - D_0) \tau}{\sigma\sqrt{\tau}} \right) \\
& - \frac{\sigma^2}{2(r - D_0)} e^{-2(r-D_0)/\sigma^2} N \left( \frac{\ln(\alpha e^{(D_0 - r)\tau}) - \sigma^2\tau/2}{\sigma\sqrt{\tau}} \right) \right]
\end{align*}
\]

and if set
\[
\mu = r - D_0 - \sigma^2/2,
\]
then for \(c_{ls}(S, L, t)\), we have
\[
\begin{align*}
c_{ls}(S, L, t) &= SW(\eta, t) = e^{-r(T-t)}Se^{r(T-t)}W(\eta, t) \\
= e^{-r(T-t)} & S \left[ \frac{\sigma^2}{2(r - D_0)} \left( \frac{L}{S} \right)^{2(r-D_0)/\sigma^2} N \left( \frac{\ln(\alpha L/S) + \mu(T-t)}{\sigma\sqrt{T-t}} \right) \\
& - \frac{\sigma^2}{2(r - D_0)} e^{-2(r-D_0)/\sigma^2} e^{(r-D_0)(T-t)} N \left( \frac{\ln(\alpha L/S) - (\mu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \\
& + \alpha e^{(r-D_0)(T-t)} N \left( \frac{\ln(\alpha S/L) + (\mu + \sigma^2)(T-t)}{\sigma\sqrt{T-t}} \right) \\
& - \frac{L}{S} N \left( \frac{\ln(\alpha S/L) + \mu(T-t)}{\sigma\sqrt{T-t}} \right) \right].
\end{align*}
\]
When \( r - D_0 = 0 \), this expression cannot be used because \( r - D_0 \) appears on denominators. In this case for \( \eta \in (1/\alpha, \infty) \),
\[
\frac{d\varphi_2 (\eta)}{d\eta} = \eta^{-1}
\]
and
\[
\varphi_2 (\eta) = \ln \eta + \ln \alpha.
\]
Thus we have
\[
\int_{1/\alpha}^{\infty} \varphi_2 (\eta') \tilde{G} (\eta', T; \eta, t) \, d\eta'
= \int_{1/\alpha}^{\infty} (\ln \eta' + \ln \alpha) \tilde{G} (\eta', T; \eta, t) \, d\eta'
= \frac{\sigma}{\sqrt{2\pi}} e^{-(\ln(\alpha \eta) - \sigma^2 \eta/2)^2/2\sigma^2 \eta} + (\ln \eta - \sigma^2 \eta/2) \frac{N (\ln(\alpha \eta) - \sigma^2 \eta/2)}{\sigma \sqrt{\eta}}
+ \ln \alpha N \left( \frac{\ln(\alpha \eta) - \sigma^2 \eta/2}{\sigma \sqrt{\eta}} \right)
= \frac{\sigma}{\sqrt{2\pi}} e^{-(\ln(\alpha \eta) - \sigma^2 \eta/2)^2/2\sigma^2 \eta} + [\ln(\alpha \eta) - \sigma^2 \eta/2] N \left( \frac{\ln(\alpha \eta) - \sigma^2 \eta/2}{\sigma \sqrt{\eta}} \right)
\]
and the expression usable in practice for \( c_{ls} (S, L, t) \) is
\[
c_{ls} (S, L, t)
= e^{-(T-t)} S \left\{ [\ln (\alpha S/L) - \sigma^2 (T-t)/2] \frac{N (\ln (\alpha S/L) - \sigma^2 (T-t)/2)}{\sigma \sqrt{T-t}} \right.
+ \frac{\sigma}{\sqrt{2\pi}} e^{-[\ln(\alpha S/L) - \sigma^2 (T-t)/2]^2/2\sigma^2 (T-t)}
+ \alpha N \left( \frac{\ln (\alpha S/L) + \mu (T-t)}{\sigma \sqrt{T-t}} \right)
\left. - \frac{L}{S} N \left( \frac{\ln (\alpha S/L) + \mu (T-t)}{\sigma \sqrt{T-t}} \right) \right\}.
\]
This expression can also be obtained by finding the following limit:
\[
\lim_{r \to -D_0} \frac{\sigma^2}{2 (r - D_0)} \left[ \left( \frac{L}{S} \right)^{2(r-D_0)/\sigma^2} N \left( \frac{\ln (\alpha L/S) + \mu (T-t)}{\sigma \sqrt{T-t}} \right) \right.
\left. - \alpha^{-2(r-D_0)/\sigma^2} e^{(r-D_0)(T-t)} N \left( \frac{\ln (\alpha L/S) - (\mu + \sigma^2) (T-t)}{\sigma \sqrt{T-t}} \right) \right].
\]