Homework Problems (Part I) for

“Derivative Securities and
Difference Methods”
Problems

1. What is the difference between taking a long position in a forward contract and in a call option?

2. Suppose the futures price of gold is currently $324 per ounce. An investor takes a short position in a futures contract for the delivery of 1,000 ounces. How much does the investor gain or lose if the price of gold at the end of the contract is (a) $310 per ounce; (b) $340 per ounce?

3. An investor holds a European call option on a stock with an exercise price of $88 and the option costs $3.50. For what value of the stock at maturity will the investor exercise the option, and for what value of the stock at maturity will the investor make a profit?

4. An investor holds a European put option for a stock with an exercise price of $88 and the option costs $3.50. Find the gain or loss to the investor if the stock price at maturity is (a) $93.50; (b) $81.50.

5. A company will receive a certain amount of foreign currency in one year. To reduce the risk of the changes in the exchange rate, what type of contract is appropriate for hedging?

6. Suppose a fund manager holds 10 million shares of IBM stock and would like to use options to reduce risk. What action is suitable for reducing the risk of decline of the stock price in the next three months?

7. A stock price is $67 just before a dividend of $1.50 is paid. What is the stock price immediately after the payment?
Problems

1. a) Show
\[ \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = 1. \]

b) Show that
\[ \int_{-\infty}^{\infty} \frac{1}{b\sqrt{2\pi}} e^{-(x-a)^2/2b^2} dx = 1 \]
holds for any \( a \) and \( b \). (Because this is true and the integrand is always positive, it can be a probability density function.)

c) If the probability density function of a random variable \( x \) is
\[ \frac{1}{b\sqrt{2\pi}} e^{-(x-a)^2/2b^2}, \]
then it is called a normal random variable. Show \( \text{E}[x] = a \) and \( \text{Var}[x] = b^2 \).

2. Define \( dX = \phi \sqrt{dt} \), where \( \phi \) is a standardized normal random variable and its probability density function is
\[ \frac{1}{\sqrt{2\pi}} e^{-\phi^2/2}, \quad -\infty < \phi < \infty. \]
Find \( \text{E}[dX] \), \( \text{Var}[dX] \), \( \text{E}[(dX)^2] \), and \( \text{Var}[(dX)^2] \).

3. a) Suppose that \( S_1 \) and \( S_2 \) are two independent normal random variables. The mean and variance of \( S_1 \) are \( \mu_1 \) and \( \sigma_1^2 \) and for \( S_2 \) they are \( \mu_2 \) and \( \sigma_2^2 \). Find the probability density function of the random variable \( S_1 + S_2 \) and using this function, show that \( S_1 + S_2 \) is a normal random variable with mean \( \mu_1 + \mu_2 \) and variance \( \sigma_1^2 + \sigma_2^2 \).

\[^1\text{You have to show directly the relation} \]
b) Suppose that $\Delta t = t/n$ and $\phi_i, i = 1, 2, \ldots, n$, are independent standardized normal random variables. Show that

$$X(t) = \lim_{n \to \infty} \left( \phi_1 \sqrt{\Delta t} + \phi_2 \sqrt{\Delta t} + \cdots + \phi_n \sqrt{\Delta t} \right)$$

is a normal random variable with mean zero and variance $t$.

c) Define $dX = X(t + dt) - X(t)$. Show that it is a normal random variable with mean zero and variance $dt$.

d) Suppose $S(t) = e^{\mu t + \sigma X(t)}$. Show that $d\ln S(t) = \mu dt + \sigma dX$ without using Itô's lemma. (This result shows that $S(t) = e^{\mu t + \sigma X(t)}$ is a solution of the equation $d\ln S(t) = \mu dt + \sigma dX$.)

4. Suppose $dS = a(S,t)dt + b(S,t)dX,$

where $dX$ is a Wiener process. Let $f$ be a function of $S$ and $t$. Show that

$$df = \partial f \partial S dS + \frac{1}{2} b^2 \left( \partial^2 f \partial S^2 \right) dt + b \frac{\partial f}{\partial S} dX$$

This result is usually referred to as Itô's lemma.

5. Suppose that a random variable satisfies

$$dS = \mu S dt + \sigma S dX,$$

where $dX$ is a Wiener process. Find the stochastic equation for $\xi = 1/S$ by using Itô’s lemma and determine the mean and variance of $d\xi$.

6. Suppose that $S$ satisfies

$$dS = \mu S dt + \sigma S dX.$$

a) Let $F = e^{(r - D_0)(T-t)} S$, which is called the forward/futures price, and $f = S e^{-D_0(T-t)} - Ke^{-r(T-t)}$, which is the value of a forward/futures contract. Here $K$ is a constant and we assume that $r$ and $D_0$ are constant. By Itô’s lemma, show that $F$ and $f$ satisfy

$$\frac{1}{\sqrt{2\pi \sigma_1}} \int_{-\infty}^{\infty} e^{S_1 - (S_1 - \mu_1)^2/2\sigma_1^2} dS_1 \cdot \frac{1}{\sqrt{2\pi \sigma_2}} \int_{-\infty}^{\infty} e^{S_2 - (S_2 - \mu_2)^2/2\sigma_2^2} dS_2$$

$$= \frac{1}{\sqrt{2\pi \left( \sigma_1^2 + \sigma_2^2 \right)}} \int_{-\infty}^{\infty} e^{S - (S - \mu_1 - \mu_2)^2/2(\sigma_1^2 + \sigma_2^2)} dS$$

if such a conclusion is used.

A problem with * in this book means that you can find the answer in this book. It is suggested that a student should first read and understand the corresponding material and then do the problem without looking at the book.
Problems 7

\[ dF = (\mu - r + D_0)F dt + \sigma F dX \]

and

\[
df = \left[ (\mu + D_0) \left( f + Ke^{-r(T-t)} \right) - r Ke^{-r(T-t)} \right] dt + \sigma \left[ f + Ke^{-r(T-t)} \right] dX
\]

respectively.

b) Define \( \xi_{10} = \frac{Se^{-D_0(T-t)}}{Ee^{-r(T-t)}} \) and \( \xi_{01} = \frac{Ee^{-r(T-t)}}{Se^{-D_0(T-t)}} \). Show

\[ d\xi_{10} = (\mu - r + D_0)\xi_{10} dt + \sigma \xi_{10} dX \]

and

\[ d\xi_{01} = (-\mu + r - D_0 + \sigma^2)\xi_{01} dt - \sigma \xi_{01} dX. \]

7. Suppose that \( S \) satisfies

\[ dS = a(S,t) dt + b(S,t) dX. \]

Show that for any functions \( f_1(S,t) \) and \( f_2(S,t) \), the following is true:

\[ d(f_1f_2) = f_1 df_2 + f_2 df_1 + b_1 \frac{\partial f_1}{\partial S} \frac{\partial f_2}{\partial S} dt. \]

8. Suppose that \( S \) satisfies

\[ dS = \mu S dt + \sigma S dX, \quad 0 \leq S < \infty, \]

where \( \mu, \sigma \) are positive constants and \( dX \) is a Wiener process. Let

\[ \xi = \frac{S}{S + P_m}, \]

where \( P_m \) is a positive constant. The range of \( \xi \) is \([0,1)\). The stochastic differential equation for \( \xi \) is in the form:

\[ d\xi = a(\xi) dt + b(\xi) dX. \]

Find the concrete expressions for \( a(\xi) \) and \( b(\xi) \) by Itô’s lemma and show

\[ \begin{cases} a(0) = 0, & a(1) = 0, \\ b(0) = 0, & b(1) = 0. \end{cases} \]
9. Consider a random variable $r$ satisfying the stochastic differential equation

$$dr = (\mu - \gamma r)dt + wdX, \quad -\infty < r < \infty,$$

where $\mu, \gamma, w$ are positive constants and $dX$ is a Wiener process. Define

$$\xi = \frac{r}{|r| + P_m}, \quad P_m > 0,$$

which transforms the domain $(-\infty, \infty)$ for $r$ into $(-1, 1)$ for $\xi$. Suppose the stochastic equation for the new random variable $\xi$ is

$$d\xi = a(\xi)dt + b(\xi)dX.$$

Find the concrete expressions of $a(\xi)$ and $b(\xi)$ and show that $a(\xi)$ and $b(\xi)$ fulfill the conditions

$$\begin{cases} a(-1) = 0, \\ b(-1) = 0, \end{cases} \quad \begin{cases} a(1) = 0, \\ b(1) = 0. \end{cases}$$

10. Suppose that $S$ has the probability density function

$$G(S) = \frac{1}{\sqrt{2\pi b S}} e^{-[\ln(S/a) + b^2/2] / 2b^2}.$$

Let $\xi = \frac{1}{S}$. Find the probability density function for $\xi$, $E[\xi]$ and $\text{Var}[\xi]$.

11. a) Show that if an investment is risk-free, then theoretically its return rate must be the spot interest rate.

b) Using this fact and Itô’s lemma, derive the Black–Scholes equation.

12. Suppose that $\xi$ satisfy the stochastic differential equation:

$$d\xi = a(\xi, t)dt + b(\xi, t)dX,$$

where $dX$ are the Wiener processes. Let $S(\xi)$ be the price of a stock which pays dividends $D(S(\xi), t)dt$ during the time period $[t, t + dt]$ and $f(\xi, t)$ represent the value of a derivative security.

a) Setting a portfolio $H = f(\xi, t) - \Delta S(\xi)$ and using Itô’s lemma, derive a PDE for $f(\xi, t)$.

b) Assume $f(\xi, t) = V(\xi, t)$, $S(\xi) = e^\xi$ and $D(S(\xi), t) = D_0 e^\xi$. Find the PDE for $V(\xi, t)$.

c) Assume $f(\xi, t) = V(\xi, t)/\xi$, $S(\xi) = 1/\xi$ and $D(S(\xi), t) = D_0/\xi$. Find the PDE for $V(\xi, t)$.

d) Assume $f(\xi, t) = P_m V(\xi, t)/(1 - \xi)$, $S(\xi) = P_m \xi/(1 - \xi)$ and $D(S(\xi), t) = D_0 P_m \xi/(1 - \xi)$. Find the PDE for $V(\xi, t)$.

13. As we know, $f = Se^{-D_0(T-t)} - Ke^{-r(T-t)}$ is the value of a forward/futures contract. If we set $F = e^{(r-D_0)(T-t)} S$, then for $f$ we have another expression: $f = e^{-r(T-t)} (Se^{(r-D_0)(T-t)} - K) = e^{-r(T-t)} (F - K)$. For $S$, we assume $dS = \mu Sdt + \sigma SdX$. 

a) Consider an option on a forward/futures and let the price of such an option be $V_1(f,t)$. Derive the PDE for $V_1$ by using Itô’s lemma. (Hint: Set $\Pi = V_1(f,t) - \Delta f$.)

b) Consider an option on a forward/futures and let the price of such an option be $V(F,t)$. Derive the PDE for $V$ by using Itô’s lemma. (Hint: Set $\Pi = V(F,t) - \Delta e^{-r(T-t)}(F - K)$.)

c) Show that the PDE in b) can be derived from the PDE in a) by using the following transform:

\[
\begin{align*}
F &= e^{r(T-t)}f + K, \\
t &= t, \\
V_1(f,t) &= V(F(f,t),t) = V(e^{r(T-t)}f + K,t).
\end{align*}
\]

14. Find the solution of the form
a) $V(S,t) = V(S)$,
b) $V(S,t) = A(t)B(S)$
for the Black–Scholes equation.

15. Show by substitution that
a) $V(S,t) = Se^{-D_0(T-t)}$,
b) $V(S,t) = e^{-r(T-t)}$
are solutions of the Black–Scholes equation. What do these solutions represent?

16. Suppose $V(S,t)$ is the solution of the problem

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV &= 0, \\
0 &\leq S, \quad t \leq T, \\
V(S,T) &= V_T(S), \quad 0 \leq S.
\end{align*}
\]

Let $x = \frac{\sqrt{2}}{\sigma} \left[ \ln S + (r - D_0 - \sigma^2/2)(T - t) \right]$ and $\tau = T - t$. Show that $u(x,\tau)$ is the solution of the problem

\[
\begin{align*}
\frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2}, \\
-\infty &< x < \infty, \quad 0 \leq \tau, \\
u(x,0) &= V_T \left(e^{\sigma x/\sqrt{2}}\right), \quad -\infty < x < \infty.
\end{align*}
\]

17. Suppose $V(S,t)$ is the solution of the problem

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} - rV &= 0, \\
0 &\leq S, \quad t \leq T, \\
V(S,T) &= V_T(S), \quad 0 \leq S.
\end{align*}
\]
Let \( x = \ln S + (r - D_0 - \sigma^2/2)(T - t), \) \( \bar{\tau} = \sigma^2(T - t)/2 \) and \( V(S, t) = e^{-r(T-t)}u(x, \bar{\tau}) \). Show that \( u(x, \bar{\tau}) \) is the solution of the problem

\[
\begin{cases}
\frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \leq \bar{\tau}, \\
u(x, 0) = V_T(e^x), & -\infty < x < \infty.
\end{cases}
\]

18. Consider problem \( A \):

\[
\begin{cases}
\frac{\partial V}{\partial t} + a(t)S^2 \frac{\partial^2 V}{\partial S^2} + b(t)S \frac{\partial V}{\partial S} - r(t)V = 0, & 0 \leq S, \quad t \leq T, \\
V(S, T) = V_T(S), & 0 \leq S
\end{cases}
\]

and define

\[
\alpha(t) = \int_t^T a(s)ds,
\]

\[
\beta(t) = \int_t^T b(s)ds
\]

and

\[
\gamma(t) = \int_t^T r(s)ds.
\]

Assume that for this problem the uniqueness of solution is proved. Show that

a) Let \( x = \ln S + \beta(t) - \alpha(t), \bar{\tau} = \alpha(t) \) and \( V(S, t) = e^{-\gamma(t)}u(x, \bar{\tau}) \), then \( u(x, \bar{\tau}) \) is the solution of the problem:

\[
\begin{cases}
\frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty, \quad 0 \leq \bar{\tau}, \\
u(x, 0) = V_T(e^x), & -\infty < x < \infty.
\end{cases}
\]

b) \( V(S, t) \) must be in the form

\[
V(S, t) = e^{-\gamma(t)}u(\ln S + \beta(t) - \alpha(t), \alpha(t))
\]
or

\[
V(S, t) = e^{-\gamma(t)}\tilde{u}(Se^{\beta(t)}, \alpha(t)).
\]

c) If

\[
V(S, t) = e^{-r(T-t)}\tilde{u}(Se^{b(T-t)}, a(T - t))
\]
is the solution of problem \( A \) with constant coefficients, then

\[
V(S, t) = e^{-\gamma(t)}\tilde{u}(Se^{\beta(t)}, \alpha(t))
\]
is the solution of problem \( A \) with time-dependent coefficients.
19. *Suppose \( V(S, t) \) is the solution of the problem

\[
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(S) S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - r V &= 0, \quad 0 \leq S, \quad t \leq T, \\
V(S, T) &= V_T(S), \quad 0 \leq S.
\end{aligned}
\]

Let \( \xi = \frac{S}{S + P_m}, \quad \tau = T - t, \) and \( V(S, t) = (S + P_m) \tilde{V}(\xi, \tau) \), where \( P_m \) is a positive constant.

a) Show that \( \tilde{V}(\xi, \tau) \) is the solution of the problem

\[
\begin{aligned}
\frac{\partial \tilde{V}}{\partial \tau} &= \frac{1}{2} \bar{\sigma}^2(\xi) (1 - \xi)^2 \frac{\partial^2 \tilde{V}}{\partial \xi^2} + (r - D_0)(1 - \xi) \frac{\partial \tilde{V}}{\partial \xi} \\
&\quad - [r(1 - \xi) + D_0 \xi] \tilde{V}, \quad 0 \leq \xi \leq 1, \quad 0 \leq \tau,
\end{aligned}
\]

\[
\tilde{V}(\xi, 0) = \frac{1 - \xi}{P_m} V_T \left( \frac{P_m \xi}{1 - \xi} \right), \quad 0 \leq \xi \leq 1,
\]

where \( \bar{\sigma}(\xi) = \sigma \left( \frac{P_m \xi}{1 - \xi} \right) \).

b) What are the advantages of reformulating the problem on a finite domain?

20. *Find an integral expression of the solution of the following problem

\[
\begin{aligned}
\frac{\partial u}{\partial \tau} &= \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad 0 \leq \tau, \\
u(x, 0) &= u_0(x), \quad -\infty < x < \infty.
\end{aligned}
\]

21. *Using the results given in Problems 16 and 20, show that the solution of the following problem

\[
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2(S) S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - r V &= 0, \quad 0 \leq S, \quad t \leq T, \\
V(S, T) &= V_T(S), \quad 0 \leq S
\end{aligned}
\]

is

\[
V(S, t) = e^{-(T-t)} \int_0^\infty V_T(S') G(S', T; S, t) dS',
\]

where

\[
G(S', T; S, t) = \frac{1}{\sqrt{2\pi(T-t)S'}} e^{-\left[ \ln S' - \ln S - (r - D_0 - \sigma^2/2)(T-t)/2\sigma^2(T-t) \right]^2 / 2\sigma^2(T-t)}.
\]
22. Suppose that \( S \) is a random variable which is defined on \([0, \infty)\) and whose probability density function is

\[
G(S) = \frac{1}{\sqrt{2\pi bS}} e^{-\left[\ln(S/a) + b^2/2\right]^2/2b^2},
\]
a and \( b \) being positive numbers. Show that

a) \[
\int_0^c G(S) dS = N\left(\frac{\ln(c/a) + b^2/2}{b}\right);
\]

b) \[
\int_0^c S G(S) dS = a N\left(\frac{\ln(c/a) - b^2/2}{b}\right);
\]
c) for any real number \( n \)

\[
\int_0^c S^n G(S) dS = a^n e^{(n^2-n)b^2/2} N\left(\frac{\ln(c/a) + b^2/2}{b} - nb\right);
\]
d) for any real number \( n \)

\[
E [S^n] = a^n e^{(n^2-n)b^2/2};
\]
e) for any real number \( n \)

\[
\int_c^\infty S^n G(S) dS = a^n e^{(n^2-n)b^2/2} N\left(-\frac{\ln(c/a) + b^2/2}{b} + nb\right);
\]
f) \[
\int_0^c \ln S G(S) dS = -\frac{b}{\sqrt{2\pi}} e^{-\left[\ln(c/a) + b^2/2\right]^2/2b^2} + (\ln a - b^2/2) N\left(\frac{\ln(c/a) + b^2/2}{b}\right);
\]
g) \[
\int_c^\infty \ln S G(S) dS = \frac{b}{\sqrt{2\pi}} e^{-\left[\ln(c/a) + b^2/2\right]^2/2b^2} + (\ln a - b^2/2) N\left(-\frac{\ln(c/a) + b^2/2}{b}\right),
\]
where

\[
N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-\xi^2/2} d\xi.
\]
23. Using the results given in Problems 21 and 22, derive the Black-Scholes formula for a European put option.

24. As we know, the price of a call option on a forward/futures is the solution of the following problem:

\[
\begin{cases}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 V}{\partial F^2} - rV = 0, & 0 \leq F, \quad t \leq T, \\
V(F, T) = \max(F - K, 0), & 0 \leq F.
\end{cases}
\]

Using the results given in Problems 21 and 22, find a closed-form solution for this case.

25. Consider the following problem:

\[
\begin{cases}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV + k(t)Z = 0, & 0 \leq S, \quad 0 \leq t \leq T, \\
V(S, T) = \max(Z, nS), & 0 \leq S,
\end{cases}
\]

where \( \sigma, r, D_0, Z, n \) are constants and \( k(t) \) is a nonnegative function. Using the results given in Problems 21 and 22, find a closed-form solution for this case. (If \( D_0 = 0 \), this solution gives the price of a one-factor convertible bond paying coupon.) (Hint: Define \( V(S, t) = V(S, t) - b_0(t) \), where \( b_0(t) \) is the solution of the following problem:

\[
\begin{cases}
\frac{db_0}{dt} - rb_0 + k(t)Z = 0, & 0 \leq t \leq T, \\
b_0(T) = 0.
\end{cases}
\]

Find \( b_0(t) \) and a closed-form solution of \( V(S, t) \) first, then putting them together, we have \( V(S, t) \).

26. Using the Black–Scholes formula for a put option and the result in Problem 18 part c), find the formula for the price of a put option with time-dependent parameters.

27. As we know,

\[
e(S, t) = e^{-r(T-t)} \int_0^\infty \max(S' - E, 0) G(S', T; S, t) dS'
\]

and

\[
p(S, t) = e^{-r(T-t)} \int_0^\infty \max(E - S', 0) G(S', T; S, t) dS',
\]

where

\[
G(S', T; S, t) = \frac{1}{\sigma \sqrt{2\pi(T-t)S'}} e^{-\left[\ln(S'/S) - (r - D_0 - \sigma^2/2)(T-t)\right]^2/2\sigma^2(T-t)}.
\]
a) Using the expression above for \( c(S,t) \), show that if \( D_0 = 0 \), then
\[ c(S,t) \geq \max(S - E, 0) \],
which means that for this case the value of an American call option is the same as the value of a European call option.

b) Using the expression above for \( p(S,t) \), show that if \( r = 0 \), then
\[ p(S,t) \geq \max(E - S, 0) \]
which means that for this case the value of an American put option is the same as the value of a European put option.

28. Consider the problem
\[
\begin{align*}
\frac{\partial B_c}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 B_c}{\partial S^2} + (r - D_0)S \frac{\partial B_c}{\partial S} - rB_c &= 0, \\
B_c(S,T) &= \max(Z,nS),
\end{align*}
\]
where \( \sigma, r, D_0, Z, \) and \( n \) are constants. Show that if \( D_0 \leq 0 \), then
\[ B_c(S,t) \geq \max \left( Z e^{-(T-t)}, nS \right) \quad \text{for} \quad 0 \leq t \leq T. \]

29. Show
\[
Se^{-D_0(T-t)-d_1^2/2} = Ze^{-r(T-t)-d_2^2/2},
\]
i.e., \( Se^{-D_0(T-t)} N'(d_1) = Ze^{-r(T-t)} N'(d_2) \), where
\[
d_1 = \left[ \ln \left( \frac{Se^{(r-D_0)(T-t)}}{E} \right) + \frac{1}{2} \sigma^2(T-t) \right] \left( \sigma \sqrt{T-t} \right),
\]
\[d_2 = d_1 - \sigma \sqrt{T-t}.\]

30. Verify that the Black–Scholes formula for a put option is the solution of the following problem:
\[
\begin{align*}
\frac{\partial p}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + (r - D_0)S \frac{\partial p}{\partial S} - rp &= 0, \\
p(S,T) &= \max(E - S, 0), \\
0 &\leq S, \quad 0 \leq t \leq T.
\end{align*}
\]
(Hint: Use the following identity \( Ze^{-r(T-t)-d_2^2/2} = Se^{-D_0(T-t)-d_1^2/2} \).)

31. Find the expressions of \( \lim_{S \to 0} c(S,t) \) and \( \lim_{S \to 0} p(S,t) \).

32. Derive the expressions for derivatives of \( c(S,t) \) and \( p(S,t) \) with respect to \( r, D_0, \sigma, E \), and show that \( \frac{\partial c}{\partial r}, \frac{\partial c}{\partial \sigma}, \frac{\partial p}{\partial r}, \frac{\partial p}{\partial \sigma}, \frac{\partial p}{\partial E} \) are nonnegative, and others are nonpositive.

33. Let \( c(\xi, \tau) = c(S,t)/(S + P_m) \) and \( p(\xi, \tau) = p(S,t)/(S + P_m) \), where \( \xi = S/(S + P_m) \) and \( \tau = T - t \). Derive the expressions of \( c(\xi, \tau) \) and \( p(\xi, \tau) \) and find the limits of \( c(\xi, \tau) \) and \( p(\xi, \tau) \) as \( \xi \) tends to 0 and 1. Also write down the formulae for the case \( P_m = E \).
34. Consider the following problem

\[
\begin{align*}
\frac{\partial c_b}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 c_b}{\partial S^2} + (r - D_0) S \frac{\partial c_b}{\partial S} - r c_b &= 0, \\
0 &\leq S < \infty, \ 0 < t < T, \\
c_b(S,T) &= \begin{cases} 
0, & \text{if } 0 \leq S < S^{**}, \\
f(S), & \text{if } S^{**} \leq S < S^*, \\
S - E, & \text{if } S^* \leq S < \infty,
\end{cases}
\end{align*}
\]

where \( f(S) = a_0 + a_1 S + \cdots + a_J S^J \).

Show that it has a solution in the following closed form:

\[
c_b(S,t) = \sum_{n=0}^{J} \left\{ a_n S^n e^{(n-1)r - nD_0 + (n-1)n\sigma^2/2)(T-t)} \times \left[ N\left(d^* - n\sigma\sqrt{T-t}\right) - N\left(d^{**} - n\sigma\sqrt{T-t}\right) \right] \right\}
+ S e^{-D_0(T-t)} \left[ 1 - N\left(d^* - \sigma\sqrt{T-t}\right) - E e^{-r(T-t)} [1 - N(d^*)] \right],
\]

where

\[
d^* = \left[ \ln(S^*/S) - \left( r - D_0 - \frac{1}{2} \sigma^2 \right) (T-t) \right] / \left( \sigma\sqrt{T-t} \right),
\]

\[
d^{**} = \left[ \ln(S^{**}/S) - \left( r - D_0 - \frac{1}{2} \sigma^2 \right) (T-t) \right] / \left( \sigma\sqrt{T-t} \right).
\]

35. Consider a European call option on a non-dividend-paying stock. Use the Black–Scholes formula to find the option price when the stock price is $63, the strike price is $60, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months.

36. Consider a European put option on a dividend-paying stock. Use the Black–Scholes formula to find the option price when the stock price is $55, the strike price is $60, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, the dividend yield is 3% per annum, and the time to maturity is six months.

37. Consider a European call option on a non-dividend-paying stock. The option price is $4.5, the stock price is $86, the exercise price is $92, the risk-free interest rate is 5% per annum, and the time to maturity is three months. Use the Black–Scholes formula for a call option to find what the corresponding volatility should be. (This volatility is usually referred to as the implied volatility associated with the given option price.)
38. Consider a European option on a non-dividend-paying stock. The stock price is $37, the exercise price is $34, the risk-free interest rate is 5% per annum, the volatility is 30% per annum, and the time to maturity is six months. Find the call and put option prices by using the Black–Scholes formulae and verify that the put–call parity holds.

39. Suppose that $c(S,t)$ and $p(S,t)$ are the prices of European call and put options with the same parameters, respectively. Show the put–call parity $c(S,t) - p(S,t) = S e^{-D_0(T-t)} - E e^{-r(T-t)}$ without using the Black–Scholes formulae.

40. By using the put–call parity relation of European options $c(S,t) - p(S,t) = S e^{-D_0(T-t)} - E e^{-r(T-t)}$, show that the following relations hold:

\[
\begin{align*}
\frac{\partial p}{\partial S} &= \frac{\partial c}{\partial S} - e^{-D_0(T-t)} \frac{\partial^2 c}{\partial S^2} \\
\frac{\partial^2 p}{\partial S \partial \sigma} &= \frac{\partial^2 c}{\partial S \partial \sigma} \\
\frac{\partial p}{\partial \sigma} &= \frac{\partial c}{\partial \sigma} \\
\frac{\partial^2 p}{\partial \sigma^2} &= \frac{\partial^2 c}{\partial \sigma^2}
\end{align*}
\]

41. As we know, the prices of European call and put options are solutions of the problem

\[
\begin{align*}
\frac{\partial c}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 c}{\partial S^2} + (r - D_0)S \frac{\partial c}{\partial S} - rc &= 0, \quad 0 \leq S, \quad t \leq T, \\
c(S,T) &= \max(S - E, 0), \quad 0 \leq S,
\end{align*}
\]

and the problem

\[
\begin{align*}
\frac{\partial p}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 p}{\partial S^2} + (r - D_0)S \frac{\partial p}{\partial S} - rp &= 0, \quad 0 \leq S, \quad t \leq T, \\
p(S,T) &= \max(E - S, 0), \quad 0 \leq S,
\end{align*}
\]

respectively.

a) Let $S_0^* = E e^{-r(T-t)}$, $S_1^* = S e^{-D_0(T-t)}$, $\xi_{10} = S_1^* / S_0^*$, and $\xi_{01} = S_0^*/S_1^*$. Define $V_0(\xi_{10}, t) = c(S, t)/S_0^*$ and $V_1(\xi_{01}, t) = p(S, t)/S_1^*$. Find the PDEs and final conditions for $V_0(\xi_{10}, t)$ and $V_1(\xi_{01}, t)$.

b) Based on the results in part a), show that when $S_1^*$ is replaced by $S_0^*$ and $S_0^*$ by $S_1^*$ at the same time, the expression for $c(S, t) = S_0^* V_0(S_1^*/S_0^*, t)$ becomes the expression for $p(S, t) = S_1^* V_1(S_0^*/S_1^*, t)$, and explain the financial meaning of such a result.

42. Suppose that $S$ is the price of a stock,

\[
dS = \mu S dt + \sigma S dX,
\]
and $V(S,t)$ is the value of an option on the stock. Define $S_0^* = E^e_r(T-t)$, 
$S_1^* = S_0^* e^{-D_0(T-t)}$, $\xi_{10} = S_0^*/S_1^*$, $\xi_{01} = S_0^*/E$, $\xi_{1} = S_0^*$, $\xi_{1} = S_0^*$, $S_0^* = S_0^* e^{e_r(T-t)}$. 

$V_0(\xi_{10},t) = V(S(\xi_{10},t),t)/S_0^*(t)$, and $V_1(\xi_{01},t) = V(S(\xi_{01},t),t)/S_1^*(\xi_{01},t)$, 
where $E$ and $T$ are constants, $r$ is the interest rate, and $D_0$ is the dividend yield of the stock. Assume that we already know that 

$$d\xi_{10} = (\mu - r + D_0)\xi_{10} dt + \sigma\xi_{10} dX.$$ 

a) Derive the PDE for $V_0(\xi_{10},t)$ by using Itô’s lemma. (Hint: Set $II = V - \Delta S = S_0^*(t)V_0(\xi_{10},t) - E e^{e_r(T-t)}\xi_{10}$.)

b) From the PDE for $V_0(\xi_{10},t)$ obtained in a), find the PDE for $V_1(\xi_{01},t)$.

c) Show that for a call, $V_1(\xi_{01},T) = \max(\xi_{01} - 1, 0)$ and for a put, $V_1(\xi_{01},T) = \max(\xi_{01} - 1, 0)$. Based on this fact, show further that there exists such a function $W(\xi,t)$ that $c(S,t) = S_0^* W(S_1^*/S_0^*,t)$ and $p(S,t) = S_1^* W(S_0^*/S_1^*,t)$. This means that when $S_1^*$ is replaced by $S_0^*$ and $S_0^*$ by $S_1^*$ at the same time, the expression for $c(S,t) = S_0^* V_0(S_1^*/S_0^*,t)$ becomes the expression for $p(S,t) = S_1^* V_1(S_0^*/S_1^*,t)$.

d) Consider the problem:

$$\begin{cases}
\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 \xi^2 \frac{\partial^2 W}{\partial \xi^2} = 0, & 0 \leq \xi, \quad t \leq T, \\
W(\xi,T) = \max(\xi,1), & 0 \leq \xi.
\end{cases}$$

As we know, the solution of this problem is 

$$W(\xi,t) = \int_0^\infty \max(\xi',1)G(\xi',T;\xi,t)d\xi'$$

$$= \int_0^1 G(\xi',T;\xi,t)d\xi' + \int_1^\infty \xi' G(\xi',T;\xi,t)d\xi',$$

where

$$G(\xi',T;\xi,t) = \frac{1}{\sqrt{2\pi\xi'}} e^{-\frac{[\ln(\xi'/\xi)+b^2/2]T}{2b^2}}, \quad b \text{ being } \sigma \sqrt{T-t}.$$ 

Let $V(S,t)$ be the price of the option with payoff $\max(S,E)$. In this case $V_0(\xi_{10},T) = \max(S,E)/E = \max(\xi_{10},1)$ and $V_1(\xi_{01},T) = \max(\xi_{01},1)$. Thus, for $V(S,t)$ we have two expressions: 

$$V(S,t) = S_0^* \int_0^1 G(\xi_{10},T;\xi_{10},t)d\xi_{10} + S_0^* \int_1^\infty \xi_{10} G(\xi_{10},T;\xi_{10},t)d\xi_{10},$$

and
Consider the following option problem:

\[ V(S, t) = S_1^* W(\xi_{01}, t) \]

\[ = S_1^* \int_0^1 G(\xi_{01}, T; \xi_{01}, t) d\xi_{01} + S_1^* \int_1^\infty \xi_{01} G(\xi_{01}, T; \xi_{01}, t) d\xi_{01}. \]

Because both \( \xi_{10} < 1 \) and \( \xi_{01} > 1 \) correspond to \( S < E \), both the first term in the first expression and the second term in the second expression represent the contribution to the value \( V(S, t) \) of the function \( \max(S, E) \) for \( S < E \). Consequently, the two terms should be equal. Similarly the second term in the first expression should be equal to the first term in the second expression. Verify this conclusion by direct calculation.

43. Consider the following option problem:

\[
\begin{cases}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - r V = 0, & 0 \leq S, \ t \leq T, \\
V(S, T) = \max(E, S), & 0 \leq S.
\end{cases}
\]

Suppose that the uniqueness of the solution has been proved.

a) Let \( S_0^* = E e^{-r(T-t)} \), \( S_1^* = e^{-D_0(T-t)} \), \( \xi_{10} = S_1^*/S_0^* \), and \( \xi_{01} = S_0^*/S_1^* \). Define \( V_0(\xi_{10}, t) = V(S, t)/S_0^* \) and \( V_1(\xi_{01}, t) = V(S, t)/S_1^* \). Find the PDEs and final conditions for \( V_0(\xi_{10}, t) \) and \( V_1(\xi_{01}, t) \).

b) Based on the results in part a), show that \( V(S, t) \) can be expressed as a function \( f(S_0^*, S_1^*, t) \) and this function is symmetric for \( S_0^* \) and \( S_1^* \), i.e., \( f(S_0^*, S_1^*, t) = f(S_1^*, S_0^*, t) \). This result indicates that in this option problem, the position of the cash and the position of the value of the stock are symmetric in some sense. Explain why this happens by financial terminology.

44. a) Define \( S_0^* = E e^{-r(T-t)} \) and \( S_1^* = e^{-D_0(T-t)} \). Show that there exists a function \( f(x_1, x_2, t; \sigma) \) such that the following is true:

\[ e^{-r(T-t)} \int_0^E \max(E, S') G(S', T; S, t) dS' = f(S_0^*, S_1^*, t; \sigma) \]

and

\[ e^{-r(T-t)} \int_E^\infty \max(E, S') G(S', T; S, t) dS' = f(S_1^*, S_0^*, t; \sigma), \]

where

\[ G(S', T; S, t) = \frac{1}{\sigma \sqrt{2\pi(T-t)S'}} e^{-\frac{\ln S' - \ln S + (r-D_0-\sigma^2/2)(T-t))^2}{2\sigma^2(T-t)}}. \]

b) Let \( V(S, t) \) be the solution of the problem
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\[
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV &= 0, \quad 0 \leq S, \quad t \leq T, \\
V(S, T) &= \max(E, S), \quad 0 \leq S.
\end{aligned}
\]

Based on the results in part a), show that in the expression for \( V(S, t) \), the positions of \( S^*_0 \) and \( S^*_1 \) are symmetric, i.e., exchanging \( S^*_0 \) and \( S^*_1 \) in the expression for \( V(S, t) \) will generate the same expression.

45. Let \( L_{S,t} \) be an operator in an option problem in the form:

\[
L_{S,t}^S = a(S, t) \frac{\partial^2}{\partial S^2} + b(S, t) \frac{\partial}{\partial S} + c(S, t)
\]

and \( G(S, t) \) be the constraint function for an American option. Furthermore we assume that \( \frac{\partial G}{\partial t} + L_{S,t} G \) exists. Suppose \( V(S, t^*) = G(S, t^*) \) on an open interval \((A, B)\) on the \( S \)-axis. Let \( t = t^* - \Delta t \), where \( \Delta t \) is a sufficiently small positive number. Show the following conclusions: If for any \( S \in (A, B) \),

\[
\frac{\partial G}{\partial t}(S, t^*) + L_{S,t} G(S, t^*) + d(S, t^*) \geq 0,
\]

then the value \( V(S, t) \) determined by the equation

\[
\frac{\partial V}{\partial t}(S, t) + L_{S,t} V(S, t) + d(S, t) = 0
\]

satisfies the condition \( V(S, t) - G(S, t) \geq 0 \) on \((A, B)\); and if for any \( S \in (A, B) \),

\[
\frac{\partial G}{\partial t}(S, t^*) + L_{S,t} G(S, t^*) + d(S, t^*) < 0,
\]

then the equation

\[
\frac{\partial V}{\partial t}(S, t) + L_{S,t} V(S, t) + d(S, t) = 0
\]

cannot give a solution satisfying the condition \( V(S, t) - G(S, t) \geq 0 \) for any \( S \in (A, B) \).

46. Suppose that for an American option, the constraint is \( G(S, t) \), its value at time \( t \) is \( V(S, t) \), and \( V(S, t) = G(S, t) \) on \((A, B)\). Assume that when \( V(S, t) \) is given as the value of a European option at \( t \), the value of the European option at \( t - \Delta t \) for a positive and very small \( \Delta t \) is \( v(S, t - \Delta t) \). Explain that if in an open interval containing \( S^* \in (A, B) \), \( v(S, t - \Delta t) < G(S, t - \Delta t) \), then for the American option a fair value at the point \((S^*, t - \Delta t)\) should be \( G(S^*, t - \Delta t) \).

47. *Show that an American option is always worth at least as much as a European option on the same asset with the same strike price and exercise date if \( r, D_0, \sigma \) are constant, and give an financial explanation of this fact.
48. Show that a Bermudan option is always worth at least as much as a European option on the same asset with the same strike price and exercise date if \( r, D_0, \sigma \) are constant, and give a financial explanation of this fact. (Hint: For a Bermudan option, the approximate relation between the price at \( t_n \) and the price at \( t_{n+1} \) is the same as for a European option if at \( t = t_n \) the option cannot be exercised, and the same as for an American option if at \( t = t_n \) the option can be exercised.)

49. a) Explain why an American option is always worth at least as much as its intrinsic value. What is the definition of the time value of an American option?

b) Let \( V(S,t) \) be the price of a vanilla American option. Show that \( V(S,t^*) \geq V(S,t^{**}) \) is always true, where \( t^* \leq t^{**} \). This means that the time value of a vanilla American option for a fixed \( S \) is decreasing as \( t \to T \), and give a financial explanation of this fact.

50. a) The price of a one-factor convertible bond paying no coupon is the solution of the following linear complementarity problem:

\[
\begin{align*}
\min \left( - \frac{\partial V}{\partial t} - L_S V, \ V(S,t) - nS \right) &= 0, \quad 0 \leq S, \ 0 \leq t \leq T, \\
V(S,T) &= \max(Z,nS) \geq nS, \quad 0 \leq S,
\end{align*}
\]

where

\[ L_S = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r \]

and \( n, Z, \sigma, r, \) and \( D_0 \) are positive constants. Show

\[ V(S,t^*) - Ze^{-r(T-t^*)} \geq V(S,t^{**}) - Ze^{-r(T-t^{**})} \quad \text{if} \quad t^* \leq t^{**}. \]

(Hint: Define \( V(S,t) = V(S,t) - Ze^{-r(T-t)} \) and show \( V(S,t^*) \geq V(S,t^{**}) \) if \( t^* \leq t^{**} \).)

b) Can you prove that \( V(S,t^*) \geq V(S,t^{**}) \) for \( t^* \leq t^{**} \) by using the method used in a)? If your answer is “Yes”, give a proof; otherwise explain why you cannot.

c) “A holder of a convertible bond at time \( t^* \) has “more rights” than a holder of a convertible bond at time \( t^{**} \) does if \( t^* \leq t^{**} \), so the premium at \( t^* \) should be higher than the premium at \( t^{**} \), i.e., the inequality \( V(S,t^*) \geq V(S,t^{**}) \) should hold for any \( t^* \leq t^{**} \).” Do you think that this statement is true and why?

51. The price of a one-factor convertible bond paying constant coupon is the solution of the following linear complementarity problem:

\[
\begin{align*}
\min \left( - \frac{\partial V}{\partial t} - L_S V + kZ, \ V(S,t) - nS \right) &= 0, \quad 0 \leq S, \ 0 \leq t \leq T, \\
V(S,T) &= \max(Z,nS) \geq nS, \quad 0 \leq S,
\end{align*}
\]

where
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\[ \mathbf{L}_s = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r \]

and \( k, Z, n, \sigma, r, \) and \( D_0 \) are positive constants. Study whether or not \( V(S, t^*) \geq V(S, t^{**}) \) for \( t^* \leq t^{**} \) holds in the cases \( r > k \) and \( r = k \), and if not, try to find a relation between \( V(S, t^*) \) and \( V(S, t^{**}) \).

52. A European option is the solution of the problem

\[
\begin{aligned}
\frac{\partial V}{\partial t} + \mathbf{L}_S V &= 0, \quad 0 \leq S, \quad t \leq T, \\
V(S, T) &= V_T(S), \quad 0 \leq S,
\end{aligned}
\]

where

\[ \mathbf{L}_S = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r. \]

For an American option, the constraint is that the inequality

\[ V(S, t) \geq G(S, t) \]

holds for any \( S \) and \( t \), where \( G(S, T) = V_T(S) \). Derive the linear complementarity problem for the American option.

53. The American call option is the solution of the following linear complementarity problem on a finite domain:

\[
\begin{aligned}
\min \left( \frac{\partial V}{\partial \tau} - \mathbf{L}_\xi V, \ V(\xi, \tau) - \max(2\xi - 1, 0) \right) &= 0, \quad 0 \leq \xi \leq 1, \quad 0 \leq \tau, \\
V(\xi, 0) &= \max(2\xi - 1, 0), \quad 0 \leq \xi \leq 1,
\end{aligned}
\]

where

\[ \mathbf{L}_\xi = \frac{1}{2} \sigma^2 (\xi^2 (1 - \xi)^2 \frac{\partial^2}{\partial \xi^2} + (r - D_0)\xi (1 - \xi) \frac{\partial}{\partial \xi} - [r(1 - \xi) + D_0\xi]. \]

Reformulate this problem as a free-boundary problem if \( D_0 > 0 \).

54. The American put option is the solution of the following linear complementarity problem:

\[
\begin{aligned}
\min \left( \frac{\partial u}{\partial \bar{\tau}} - \frac{\partial^2 u}{\partial x^2}, \ u(x, \bar{\tau}) - g_p(x, \bar{\tau}) \right) &= 0, \quad -\infty < x < \infty, \quad 0 \leq \bar{\tau}, \\
u(x, 0) &= g_p(x, 0), \quad -\infty < x < \infty,
\end{aligned}
\]

where

\[ g_p(x, \bar{\tau}) = \max \left( e^{2r\tau/\sigma^2} - e^{\tau+(2D_0/\sigma^2+1)r}, 0 \right). \]

Find the domain where a free boundary may appear and the domain where it is impossible for a free boundary to appear, show that there is only one free boundary at \( \bar{\tau} = 0 \), and give the starting location of this free boundary.
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55. The price of a one-factor convertible bond is the solution of the linear complementarity problem

\[
\begin{aligned}
\min \left( -\frac{\partial V}{\partial t} - L_s V - kZ, V(S, t) - nS \right) &= 0, \quad 0 \leq S, \quad 0 \leq t \leq T, \\
V(S, T) &= \max(Z, nS) \geq nS, \quad 0 \leq S,
\end{aligned}
\]

where

\[L_s = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0) S \frac{\partial}{\partial S} - r,
\]

and \(k, Z, n, \sigma, r\) and \(D_0\) are constants. Show that if \(D_0 > 0\), then the solution of a one-factor convertible bond must involve a free boundary and its location at \(t = T\) is \(S = \max\left(\frac{Z}{n}, kZ \frac{n}{D_0}\right)\). Also, derive the corresponding free-boundary problem if this problem has only one free boundary.

56. Consider the following LC problem:

\[
\begin{aligned}
\min \left( -\frac{\partial W}{\partial t} - L_{\alpha,t} W, W(\eta, t) - \max(\alpha - \eta, 0) \right) &= 0, \quad 0 \leq \eta < \infty, \quad t \leq T, \\
W(\eta, T) &= \max(\alpha - \eta, 0), \quad 0 \leq \eta < \infty,
\end{aligned}
\]

where the operator \(L_{\alpha,t}\) is defined by

\[L_{\alpha,t} = \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2}{\partial \eta^2} + \left[(D_0 - r) \eta + \frac{1 - \eta}{t}\right] \frac{\partial}{\partial \eta} - D_0.
\]

Suppose that there is only one free-boundary for this problem, reformulate this problem as a free-boundary problem.

57. Consider the following LC problem:

\[
\begin{aligned}
\min \left( -\frac{\partial W}{\partial t} - L_\eta W, W(\eta, t) - G_{isp}(\eta, t) \right) &= 0, \quad 0 \leq \eta \leq 1, \quad t \leq T, \\
W(\eta, T) &= G_{isp}(\eta, T), \quad 0 \leq \eta \leq 1, \\
\frac{\partial W}{\partial \eta}(1, t) &= 0, \quad t \leq T,
\end{aligned}
\]

where \(G_{isp}(\eta, t) = \max(\eta - \beta, 0)\) and

\[L_\eta = \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2}{\partial \eta^2} + (D_0 - r) \eta \frac{\partial}{\partial \eta} - D_0.
\]

Find the domain where it is impossible for a free boundary to appear and the domain where a free boundary may appear.
58. As we know, when the LC problem of an American call option is formulated as a free-boundary problem, on the free boundary \( S = S_f(t) \geq \max(E, rE/D_0) \), we need to require \( C(S_f(t),t) = \max(S_f(t) - E, 0) = S_f(t) - E \) and \( \frac{\partial C(S_f(t),t)}{\partial S} = 1 \), where \( C(S,t) \) and \( \max(S - E, 0) \) are the solution of the free-boundary problem and the constraint. Show that if \( C(S,t) \geq 0 \) and \( \frac{\partial^2 C(S,t)}{\partial S^2} \geq 0 \) for \( S < S_f(t) \), then the solution of the free-boundary problem satisfies the LC condition

\[
\min \left( -\frac{\partial C}{\partial t} - L_s C, C - \max(S - E, 0) \right) = 0,
\]

where

\[
L_s = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S \frac{\partial}{\partial S} - r,
\]

that is, for \( S \in [0, S_f(t)] \), \( C(S,t) \) truly is a solution of the LC problem.

59. Suppose \( r, D_0 \), and \( \sigma \) are constant.

a) Derive the put–call symmetry relations.

b) Explain the financial meaning of the symmetry relation.

c) Explain how to use these relations when we write codes if a code for put options is quite different from a code for call options.

60. a) Suppose \( \sigma = \sigma(S,t) \), \( r = r(t) \), and \( D_0 = D_0(S,t) \). Show that the problem of pricing a put option can always be converted into a problem of pricing a call option. Also explain how to use this conclusion when we write codes if a code for put options is quite a different from a code for call options.

b) Let the exercise price be \( E \). Suppose that \( r, D_0 \) are constants and \( \sigma = \sigma(S) \).

Show

\[
P(S,t;b, a, \sigma(S)) = C \left( \frac{E^2}{S},t;a, b, \sigma(S) \right) S/E,
\]

\[
C(S,t;a, b, \sigma(S)) = P \left( \frac{E^2}{S},t;b, a, \sigma(S) \right) S/E
\]

and

\[
S_{cf}(t;a, b, \sigma(S)) \times S_{pf}(t;b, a, \sigma(E^2/S)) = E^2.
\]

Here, the first, second, and third parameters after the semicolon in \( P, C, S_{pf}, \) and \( S_{cf} \) are the interest rate, the dividend yield and the volatility function, respectively.

c) For Bermudan options the symmetry relation is still true.

61. Suppose that \( \sigma, r, D_0 \) are constants. In this case we have the following symmetry relation for European options

\[
p(S,t;b, a) = c \left( \frac{E^2}{S},t;a, b \right) S/E,
\]

where the first and second arguments after the semicolon in \( p \) and \( c \) are the values of the interest rate and the dividend yield, respectively. For a European call option, the price is
Basic Options

\[ c(S, t) = S e^{-D_0(T-t)} N(d_1) - E e^{-r(T-t)} N(d_2), \]

where

\[ d_1 = \frac{\ln(S e^{-D_0(T-t)} / E e^{-r(T-t)}) + \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}}, \]

\[ d_2 = \frac{\ln(S e^{-D_0(T-t)} / E e^{-r(T-t)}) - \frac{1}{2} \sigma^2(T-t)}{\sigma \sqrt{T-t}}. \]

Find the price of a European put option by using the symmetry relation.

62. Derive the formulation of the problem for \( \frac{\partial P}{\partial r} \) and write down the formulation of the problems for \( \frac{\partial P}{\partial \sigma} \) and \( \frac{\partial P}{\partial D_0} \), where \( P \) is the price of an American put option.

63. Define \( \alpha_\pm = \frac{1}{\sigma^2} \left[ -\left( r - D_0 - \frac{1}{2} \sigma^2 \right) \pm \sqrt{\left( r - D_0 - \frac{1}{2} \sigma^2 \right)^2 + 2\sigma^2} \right] \), where \( r \geq 0 \) and \( D_0 \geq 0 \).

a) Show that \( \alpha_+ \geq 1, \alpha_- \leq 0 \), and \( -(r - D_0)\alpha_\pm + r \geq 0 \).

b) Based on the results in a), show that \( 1/(1-1/\alpha_+) \geq \max(1, r/D_0) \) and \( 1/(1-1/\alpha_-) \leq \min(1, r/D_0) \).

64. a) Find the solution of the following free-boundary problem:

\[
\begin{align*}
\frac{1}{2} \sigma^2 S^2 \frac{d^2 P_\infty}{dS^2} + (r - D_0)S \frac{dP_\infty}{dS} - r P_\infty &= 0, \quad S_f \leq S, \\
P_\infty(S_f) &= E - S_f, \\
\frac{dP_\infty(S_f)}{dS} &= -1.
\end{align*}
\]

b) Define \( P_\infty(S) = \begin{cases} E - S, & 0 \leq S < S_f, \\ \text{the solution of the free-boundary problem}, & S_f \leq S. \end{cases} \)

Show that \( P_\infty(S) \) satisfies

\[
\min \left( -\frac{1}{2} \sigma^2 S^2 \frac{d^2 P_\infty}{dS^2} + (r - D_0)S \frac{dP_\infty}{dS} - r P_\infty \right) = 0,
\]

that is, \( P_\infty(S) \) is a solution of the perpetual American put option.
65. a) Find the solution of the following free-boundary problem:

\[
\begin{cases}
\frac{1}{2} \sigma^2 \eta^2 \frac{d^2 W_\infty}{d\eta^2} + (D_0 - r) \eta \frac{dW_\infty}{d\eta} - D_0 W_\infty = 0, & 1 \leq \eta \leq \eta_f, \\
\frac{dW_\infty(1)}{d\eta} = 0, \\
W_\infty (\eta_f) = \eta_f, \\
\frac{dW_\infty (\eta_f)}{d\eta} = 1,
\end{cases}
\]

where \( \eta_f \) is a number representing the location of this free boundary.

b) Define

\[
W_\infty (\eta) = \begin{cases}
\text{the solution of the free-boundary problem,} & 1 \leq \eta \leq \eta_f, \\
\eta, & \eta_f < \eta.
\end{cases}
\]

Show that \( W_\infty (\eta) \) is a solution of the following LC problem

\[
\min \left( -\frac{\sigma^2 \eta^2}{2} \frac{d^2 W_\infty}{d\eta^2} - (D_0 - r) \eta \frac{dW_\infty}{d\eta} + D_0 W_\infty, W_\infty - \eta \right) = 0, \quad 1 \leq \eta,
\]

\[
\frac{dW_\infty(1)}{d\eta} = 0.
\]

(This problem is related to the Russian option.)

66. Find the solution of the problem:

\[
\begin{cases}
\frac{1}{2} \sigma^2 \xi^2 \frac{d^2 W_\infty}{d\xi^2} + (D_{02} - D_{01}) \xi \frac{dW_\infty}{d\xi} - D_{02} W_\infty = 0, & \xi_{f1} \leq \xi \leq \xi_{f2}, \\
W_\infty (\xi_{f1}) = 1, \\
\frac{dW_\infty}{d\xi} (\xi_{f1}) = 0, \\
W_\infty (\xi_{f2}) = \xi_{f2}, \\
\frac{dW_\infty}{d\xi} (\xi_{f2}) = 1,
\end{cases}
\]

where \( \xi_{f1} < \xi_{f2} \). (This problem is related to the perpetual American better-of option.)

67. *Describe the reversion conditions of a stochastic process, and give the intuitive meaning of the conditions.

68. *Describe and derive the generalized Itô’s lemma.
69. Suppose that $S_1, S_2, \ldots, S_n$ are $n$ lognormal random variables satisfying the following stochastic differential equations:

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i, \quad i = 1, 2, \ldots, n,$$

where $\mu_i, \sigma_i, i = 1, 2, \ldots, n$, are constants and $dX_i, i = 1, 2, \ldots, n$, are $n$ Wiener processes, i.e., $dX_i = \phi_i \sqrt{dt}$, $\phi_i$ being distinct standardized normal random variables, $i = 1, 2, \ldots, n$. $\phi_i$ and $\phi_j$ could be correlated and

$$\text{E}[\phi_i \phi_j] = \rho_{ij}, \quad i, j = 1, 2, \ldots, n,$$

where $-1 \leq \rho_{ij} \leq 1$. Define

$$\xi_{ij} = S_i / S_j, \quad i \neq j.$$

a) Show that $\xi_{ij}$ satisfies the following stochastic differential equation

$$d\xi_{ij} = (\mu_i - \mu_j + \sigma_j^2 - \rho_{ij} \sigma_i \sigma_j) \xi_{ij} dt + \sigma_{ij} \xi_{ij} dX_{ij},$$

where

$$\sigma_{ij} = \sqrt{\sigma_i^2 - 2 \rho_{ij} \sigma_i \sigma_j + \sigma_j^2}$$

and $dX_{ij}$ is a Wiener process defined by

$$dX_{ij} = \frac{\sigma_i dX_i - \sigma_j dX_j}{\sigma_{ij}}.$$

That is, $\xi_{ij} = S_i / S_j$ is also a lognormal variable and its volatility is $\sigma_{ij}$.

b) Let $S_0$ be a function of $t$, satisfying

$$dS_0 = \mu_0 S_0 dt.$$

It is clear that if we think $S_0$ to be a random variable and let its volatility be $\sigma_0$, then $\sigma_0 = 0$. Show that if $S_i$ is $S_0$, then $\sigma_{0j} = \sigma_j$ and $dX_{0j} = -dX_j$; if $S_j$ is $S_0$, then $\sigma_{0i} = \sigma_i$ and $dX_{0i} = dX_i$.

c) Define

$$\rho_{ijk} = \frac{\rho_{ij} \sigma_i \sigma_j - \rho_{ik} \sigma_i \sigma_k - \rho_{jk} \sigma_j \sigma_k + \sigma_k^2}{\sigma_{ik} \sigma_{jk}}.$$

Show

$$\text{E}[dX_{ik} dX_{jk}] = \rho_{ijk} dt,$$

i.e., $\rho_{ijk}$ is the correlation coefficient between the Wiener processes related to $\xi_{ik}$ and $\xi_{jk}$.

d) Show that if $S_i = S_0$, then

$$\text{E}[dX_{0k} dX_{jk}] = \rho_{0jk} dt = \frac{\sigma_k - \rho_{jk} \sigma_j}{\sigma_{jk}} dt.$$
70. Suppose that $\xi_1$ and $\xi_2$ satisfy the system of stochastic differential equations:

$$d\xi_i = \mu_i(\xi_1, \xi_2, t)dt + \sigma_i(\xi_1, \xi_2, t)dX_i, \quad i = 1, 2,$$

where $dX_i$ are the Wiener processes and $E[dX_idX_j] = \rho_{ij}dt$ with $-1 \leq \rho_{ij} \leq 1$. Define

$$\begin{align*}
Z_1(\xi_1) &= Z_{1,t} + \xi_1 (1 - Z_{1,t}), \\
Z_2(\xi_1, \xi_2) &= Z_{2,t} + \xi_2 [Z_1(\xi_1) - Z_{2,t}] \\
&= Z_{2,t} + \xi_2 [Z_{1,t} + \xi_1 (1 - Z_{1,t}) - Z_{2,t}].
\end{align*}$$

Assume that $Z_1(\xi_1)$ and $Z_2(\xi_1, \xi_2)$ represent prices of two securities. Let $V(\xi_1, \xi_2, t)$ be the value of a derivative security. Setting a portfolio $\Pi = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2)$ and using Itô's lemma, show that $V(\xi_1, \xi_2, t)$ satisfies the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{rZ_1}{1 - Z_{1,t}} \frac{\partial V}{\partial \xi_1} + \left[ \frac{r(Z_2 - Z_1 \xi_2)}{Z_1 - Z_{2,t}} - \frac{\sigma_1 \sigma_2 \rho_{1,2} (1 - Z_{1,t})}{Z_1 - Z_{2,t}} \right] \frac{\partial V}{\partial \xi_2} - rV = 0.$$

71. Describe and derive the general equations for derivative securities.

72. Suppose that $S$ is the price of a dividend-paying stock and satisfies

$$dS = \mu(S, t)Sdt + \sigma SdX_1, \quad 0 \leq S < \infty,$$

where $dX_1$ is a Wiener process and $\sigma$ is another random variable. Let the dividend paid during the time period $[t, t + dt]$ be $D(S, t)dt$. Assume that for $\sigma$, the stochastic equation

$$d\sigma = p(\sigma, t)dt + q(\sigma, t)dX_2, \quad \sigma_1 \leq \sigma \leq \sigma_u$$

holds. Here, $p(\sigma, t)$ and $q(\sigma, t)$ are differentiable functions and satisfy the reversion conditions. $dX_2$ is another Wiener process correlated with $dX_1$, and the correlation coefficient between them is $\rho dt$. For options on such a stock, derive directly the partial differential equation that contains only the unknown market price of risk for the volatility. Here “Directly” means "without using the general PDE for derivatives." (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1 + \Delta_2 V_2 + S$, where $V_1$ and $V_2$ are two different options.)

73. Consider a two-factor convertible bond paying coupons with a rate $k(t)$. For such a convertible bond, derive directly the partial differential equation that contains only the unknown market price of risk for the spot interest rate. “Directly” means "without using the general PDE for derivatives". (Hint: Take a portfolio in the form $\Pi = \Delta_1 V_1 + \Delta_2 V_2 + S$, where $V_1$ and $V_2$ are two different convertible bonds.)
74. Suppose that $\xi_1, \xi_2$ and $\xi_3$ satisfy the system of stochastic differential equations:

$$d\xi_i = \tilde{\mu}_i(\xi_1, \xi_2, \xi_3, t)dt + \tilde{\sigma}_i(\xi_1, \xi_2, \xi_3, t)d\tilde{X}_i, \quad i = 1, 2, 3,$$

where $d\tilde{X}_i$ are the Wiener processes and $E[d\tilde{X}_i d\tilde{X}_j] = \tilde{\rho}_{ij} dt$ with $-1 \leq \tilde{\rho}_{ij} \leq 1$. Define

$$\begin{align*}
Z_1(\xi_1) &= Z_{1,t} + \xi_1 (1 - Z_{1,t}), \\
Z_2(\xi_1, \xi_2) &= Z_{2,t} + \xi_2 [Z_1(\xi_1) - Z_{2,t}] \\
&= Z_{2,t} + \xi_2 [Z_{1,t} + \xi_1 (1 - Z_{1,t}) - Z_{2,t}], \\
Z_3(\xi_1, \xi_2, \xi_3) &= Z_{3,t} + \xi_3 \{Z_2(\xi_1, \xi_2) - Z_{3,t} \}
&= Z_{3,t} + \xi_3 \{Z_{2,t} + \xi_2 [Z_{1,t} + \xi_1 (1 - Z_{1,t}) - Z_{2,t}] - Z_{3,t} \}.
\end{align*}$$

Assume that $Z_1(\xi_1), Z_2(\xi_1, \xi_2)$, and $Z_3(\xi_1, \xi_2, \xi_3)$ represent prices of three securities. Let $V(\xi_1, \xi_2, \xi_3, t)$ be the value of a derivative security. Setting a portfolio $H = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2) - \Delta_3 Z_3(\xi_1, \xi_2, \xi_3)$ and using Itô's lemma, show that $V(\xi_1, \xi_2, \xi_3, t)$ satisfies the following PDE:

$$\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \tilde{\sigma}_i \tilde{\sigma}_j \tilde{\rho}_{ij} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{r Z_1}{1 - Z_{1,t}} \frac{\partial V}{\partial \xi_1} \\
+ \left[ \frac{r (Z_2 - Z_{1,t})}{Z_1 - Z_{2,t}} - \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\rho}_{1,2} (1 - Z_{1,t}) \right] \frac{\partial V}{\partial \xi_2} \\
+ \left[ \frac{r (Z_3 - Z_{2,t})}{Z_2 - Z_{3,t}} - \tilde{\sigma}_1 \tilde{\sigma}_3 \tilde{\rho}_{1,3} (1 - Z_{1,t}) + \tilde{\sigma}_2 \tilde{\sigma}_3 \tilde{\rho}_{2,3} (Z_1 - Z_{2,t}) \right] \frac{\partial V}{\partial \xi_3} \\
- rV &= 0.
\end{align*}$$

75. Suppose $f_1(r, t) \geq 0$ and $f_2(r, t) = \frac{\partial f_1(r_1, t)}{\partial r} = f_1(r_u, t) = \frac{\partial f_1(r_u, t)}{\partial r} = 0$, and $f_2(r_1, t) > 0, f_2(r_2, t) > 0$. Explain why problem A

$$\begin{align*}
\frac{\partial V}{\partial t} &= f_1 \frac{\partial^2 V}{\partial r^2} + f_2 \frac{\partial V}{\partial r} + f_3 V, \quad r_1 \leq r \leq r_u, \quad 0 \leq t, \\
V(r, 0) &= V_0(r), \quad r_1 \leq r \leq r_u, \\
V(r_1, t) &= f_1(t), \quad 0 \leq t, \\
V(r_u, t) &= f_u(t), \quad 0 \leq t
\end{align*}$$

and problem B

$$\begin{align*}
\frac{\partial V}{\partial t} &= -f_1 \frac{\partial^2 V}{\partial r^2} + f_2 \frac{\partial V}{\partial r} + f_3 V, \quad r_1 \leq r \leq r_u, \quad t \leq T, \\
V(r, T) &= V_T(r), \quad r_1 \leq r \leq r_u
\end{align*}$$

have unique solutions.
76. a) Consider a linear hyperbolic partial differential equation
\[ \frac{\partial u}{\partial t} + f(x,t) \frac{\partial u}{\partial x} = 0. \]
Let \( x = x(t) \) be the curve \( C \) which is determined by the following ordinary differential equation
\[ \frac{dx(t)}{dt} = f(x,t) \]
with \( x(0) = \xi \). Show that \( u \) is a constant along the curve \( C \):
\[ u(x(t^*),t^*) = u(x(t^{**}),t^{**}), \]
where \( t^* \) and \( t^{**} \) are any two times, and that if
\[ f(x,t) = F(x,t)\delta(t-t_i), \]
where \( \delta(t-t_i) \) is the Dirac delta function, then
\[ u(x(t_i^-),t_i^-) = u(x(t_i^-) + F(x(t_i^-),t_i^-), t_i^+), \]
where \( t_i^- \) and \( t_i^+ \) denote the time just before and after \( t_i \), respectively.
b) Derive the jump condition for options on stocks with discrete dividends and explain its financial meaning.
c) Find the corresponding jump condition for the following PDE
\[ \frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2\eta^2 \frac{\partial^2 W}{\partial \eta^2} + \left( D_0 - r \right) \eta + \sum_{i=1}^{K} \delta(t-t_i) \frac{\partial W}{\partial \eta} - D_0W = 0. \]
d) Find the corresponding jump condition for the following PDE
\[ \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0)S \frac{\partial V}{\partial S} + \sum_{i=1}^{K} \left[ \max(S(t),H(t^-)) - H(t^-) \right] \delta(t-t_i) \frac{\partial V}{\partial H} - rV = 0. \]
77. Suppose that \( V(S,t) \) is the solution of the following PDE:
\[ \frac{\partial V}{\partial t} + a(S,t) \frac{\partial^2 V}{\partial S^2} + b(S,t) \frac{\partial V}{\partial S} + c(S,t)V + d(S,t)\delta(t-t_i) = 0. \]
Find the relation between \( V(S,t_i^+) \) and \( V(S,t_i^-) \), and describe the financial meaning of this relation.
78. *Use arbitrage arguments to show the put–call parity of European options for the following two cases.
a) When the dividend is paid continuously, the put–call parity is
\[ c(S,t) - p(S,t) = Se^{-D_0(T-t)} - Ee^{-r(T-t)}; \]
b) when the dividend is paid discretely, the put–call parity is
\[ c(S,t) - p(S,t) = S - D(t) - Ee^{-r(T-t)}, \]
where \( D(t) \) is the value of “will-be-paid” dividends at time \( t \).

79. Use arbitrage arguments to show the inequalities of American options for the following two cases.
a) When the dividend is paid continuously, there is the inequality
\[ Se^{-D_0(T-t)} - E \leq C(S,t) - P(S,t) \leq S - Ee^{-r(T-t)} \]
between American put option \( P(S,t) \) and American call option \( C(S,t) \) with the same parameters.
b) When the dividend is paid discretely, there is the inequality
\[ S - D(t) - E \leq C(S,t) - P(S,t) \leq S - Ee^{-r(T-t)} \]
between American put option \( P(S,t) \) and American call option \( C(S,t) \) with the same parameters.

80. Consider a European call option with \( T = 6 \) months and \( E = $80 \) on a dividend-paying stock. The dividend is paid continuously with a dividend yield \( D_0 = 0.05 \). Today, \( t = 0 \), \( r = 0.1 \) and \( S = $82 \).
a) Find the lower bound of the call option.
b) What are the least profits we could make at time \( T \) by arbitrage if the call option price today is $0.10 less than the lower bound and why?

81. Consider a European put option with \( T = 3 \) months and \( E = $60 \) on a dividend-paying stock. Today \( t = 0 \), \( r = 0.05 \), and \( S = $55 \). The dividends are paid discretely, and the total present value of them is \( D(0) = $0.30 \).
a) Find the lower bound of the put option.
b) What are the least profits we could make at time \( T \) by arbitrage if the put option price today is $0.20 less than the lower bound and why?

82. Suppose that there are an American call option and an American put option on the same stock that pays dividends discretely. For both of them, \( E = $90 \) and \( T = 3 \) months. At time \( t = 0 \), the stock price is $93 and the present value of dividend payments during the period \([0, T]\) is \( D(0) = $0.50 \). Assume that \( r = 0.10 \) and \( P(S,0) = $2.50 \).
a) Find the upper and lower bounds of the price of the American call option.
b) What are the risk-free profits we could make today by arbitrage if the price of the call option today is $0.10 greater than the calculated upper bound and why?
83. Suppose that $c_1(S,t)$ and $c_2(S,t)$ are the prices of European call options with strikes $E_1$ and $E_2$, respectively, where $E_1 < E_2$. Also assume that the two options have the same maturity $T$ and that the interest rate $r$ is a constant. Show

$$0 \leq c_1(S,t) - c_2(S,t) \leq (E_2 - E_1)e^{-r(T-t)}.$$ 

84. Suppose that $p_1$, $p_2$, and $p_3$ are the prices of European put options with strike prices $E_1$, $E_2$, and $E_3$ respectively, where $E_2 = \frac{1}{2}(E_1 + E_3)$. All the options have the same maturity. Show

$$p_2 \leq \frac{1}{2}(p_1 + p_3).$$
Exotic Options

Problems

1. Consider the following problem:

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV &= 0, \quad 0 \leq S, \ t \leq T, \\
V(S,T) &= \begin{cases} 
\varphi_1(S), & 0 \leq S \leq B, \\
\varphi_2(S), & B < S,
\end{cases}
\end{align*}
\]

where \( \varphi_1(S) \) and \( \varphi_2(S) \) are continuous functions and \( \varphi_1(B) = \varphi_2(B) \) may not hold.

a) Try to find such a relation between \( \varphi_1(S) \) and \( \varphi_2(S) \) that \( V(B,t) = 0 \).

b) Based on the result in a), show that for the problem

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV &= 0, \quad B_t \leq S, \ t \leq T, \\
V(S,T) &= V_T(S), \quad B_t \leq S,
\end{align*}
\]

the solution is

\[
V(S,t) = e^{-r(T-t)} \int_{B_t}^{\infty} V_T(S') G_1(S',T;S,t,B_t) dS',
\]

where

\[
G_1(S',T;S,t,B_t) = G(S',T;S,t) - (B_t/S)^{2(r-D_0-\sigma^2/2)/\sigma^2} G(S',T;B_t^2/S,t).
\]

Here
3 Exotic Options

\[ G(S', T; S, t) = \frac{1}{S' \sigma \sqrt{2\pi (T-t)}} e^{-\left[\ln(S'/S) - (r-D_0-\sigma^2/2)(T-t)\right]^2/2\sigma^2(T-t)}. \]

c) The value of a European down-and-out call option is the solution of the problem:

\[
\begin{aligned}
\frac{\partial c_o}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 c_o}{\partial S^2} + (r-D_0) S \frac{\partial c_o}{\partial S} - r c_o &= 0, & B_l \leq S, & t \leq T, \\
\end{aligned}
\]

\[
\begin{aligned}
c_o(S, T) = \max(S - E, 0), & & B_l \leq S, \\
\end{aligned}
\]

\[
\begin{aligned}
c_o(B_l, t) = 0, & & t \leq T.
\end{aligned}
\]

Based on the result in part b), show that for the case \(B_l \leq E\), the expression of \(c_o\) is

\[
c_o(S, t) = c(S, t) - \left(\frac{B_l}{S}\right)^{2(r-D_0-\sigma^2/2)/\sigma^2} c \left(\frac{B_l^2}{S^2}, t\right);
\]

and for the case \(B_l \geq E\), its expression is

\[
c_o(S, t) = S e^{-D_0(T-t)} e^{-r(T-t)} N \left(\tilde{d}_1(B_l)\right) - E e^{-r(T-t)} N \left(\tilde{d}_1(B_l) - \sigma \sqrt{T-t}\right)
\]

\[
- (B_l/S)^{2(r-D_0-\sigma^2/2)/\sigma^2} \left[\frac{B_l^2}{S} e^{-D_0(T-t)} N \left(\tilde{d}_1(B_l)\right) \right.
\]

\[
\left. - E e^{-r(T-t)} N \left(\tilde{d}_1(B_l) - \sigma \sqrt{T-t}\right) \right],
\]

where

\[
\tilde{d}_1(B_l) = \ln \left[\frac{S e^{(r-D_0)(T-t)}}{B_l} + \frac{1}{2} \sigma^2(T-t)\right] / \left(\sigma \sqrt{T-t}\right).
\]

\[
\tilde{d}_1(B_l) = \ln \left[\frac{B_l e^{(r-D_0)(T-t)}}{S} + \frac{1}{2} \sigma^2(T-t)\right] / \left(\sigma \sqrt{T-t}\right).
\]

d) Based on the result in a), show that for the problem

\[
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-D_0) S \frac{\partial V}{\partial S} - r V &= 0, & 0 \leq S \leq B_u, & t \leq T, \\
V(S, T) &= V_T(S), & 0 \leq S \leq B_u, \\
V(B_u, t) &= 0, & t \leq T.
\end{aligned}
\]

the solution is
\[ V(S,t) = e^{-r(T-t)} \int_0^{B_u} V_T(S') G_1(S',T;S,t,B_u) \, dS', \]

where

\[ G_1(S',T;S,t,B_u) = G(S',T;S,t) - (B_u/S)^{2(r-D_0-\sigma^2/2)/\sigma^2} G(S',T;B_u^2/S,t). \]

Here

\[ G(S',T;S,t) = \frac{1}{S'\sigma\sqrt{2\pi(T-t)}} e^{-\left[\ln(S'/S) - (r-D_0-\sigma^2/2)(T-t)\right]^2/2\sigma^2(T-t)}. \]

c) Based on the result in part d), find the closed-form solution of a European up-and-out put option for both the case \( B_u \geq E \) and the case \( B_u \leq E \).

2. Show the following results which are related to the down-and-out call options:

a) If \( S \geq B_l \) and \( S' \geq B_l \), then

\[ G_1(S',T;S,t,B_l) = G(S',T;S,t) - (B_l/S)^{2(r-D_0-\sigma^2/2)/\sigma^2} G(S',T;B_l^2/S,t) \geq 0, \]

where

\[ G(S',T;S,t) = \frac{1}{S'\sigma\sqrt{2\pi(T-t)}} e^{-\left[\ln(S'/S) - (r-D_0-\sigma^2/2)(T-t)\right]^2/2\sigma^2(T-t)}. \]

(Hint: First it should be shown that this inequality is equivalent to the following inequalities:

\[ \ln G(S',T;S,t) \geq \ln \left[ (B_l/S)^{2(r-D_0-\sigma^2/2)/\sigma^2} G(S',T;B_l^2/S,t) \right] \]

and

\[ \left( \ln \frac{S'}{B_l} + \ln \frac{S}{B_l} \right)^2 \geq \left( \ln \frac{S'}{B_l} - \ln \frac{S}{B_l} \right)^2. \]

b) Let \( c_o(S,t;B_l) \) be the price of the European down-and-out call option, where \( B_l \) is a parameter. For \( S \geq B_l \),

\[ \frac{\partial c_o(S,t;B_l)}{\partial B_l} \leq 0. \]

(Hint: Show \( \frac{\partial G_1}{\partial B_l} \leq 0 \) first.)
c) Let $c_o(S, t)$ and $C_o(S, t)$ be the prices of the European and American down-and-out call options, respectively. Between them the following is true:

$$C_o(S, t) \geq c_o(S, t) \text{ for any } t.$$ 

d) For $C_o(S, t)$ the following is true:

$$C_o(S, t^*) \geq C_o(S, t^{**}) \text{ if } t^* \leq t^{**}.$$ 

e) Let $C_o(S, t; B_l)$ be the price of the American down-and-out call option, where $B_l$ is a parameter. For $C_o(S, t; B_l)$ the following is true:

$$C_o(S, t; B_l^*) \geq C_o(S, t; B_l^{**}) \text{ if } 0 \leq B_l^* \leq B_l^{**}.$$ 

3. Show that a European up-and-out put option with $B_u > E$ plus a European up-and-in put option with the same parameters is equal to a vanilla European put option.

4. Find the solution of the European down-and-out call option

$$\begin{cases}
\frac{\partial c_o}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 c_o}{\partial S^2} + (r - D_0) S \frac{\partial c_o}{\partial S} - rc_o = 0, \\
bE_k(t) \leq S, \quad t \leq T, \\
c_o(S, T) = \max(S - E, 0), \\
c_o(B_l(t), t) = 0, \\
bE_k(t) \leq S, \quad t \leq T,
\end{cases}$$

where $bE_k(t) = bE e^{-\alpha(T-t)}$ with $b \in [0, 1]$ and $\alpha \geq 0$. (Hint: Let $\eta = S e^{\alpha(T-t)}$, the moving barrier becomes a fixed barrier in the $(\eta, t)$-plane. Then, solve a barrier option problem with a fixed barrier.)

5. Let $P_o(S, t)$ denote the price of an American up-and-out put option. Show that under the following transformation

$$\begin{cases}
\zeta = \frac{E^2}{S}, \\
C_o(\zeta, t) = \frac{EP_o(S, t)}{S},
\end{cases}$$

the new function $C_o(\zeta, t)$ represents the price of an American down-and-out call option. Based on this result, derive the symmetry relations between American down-and-out call and up-and-out put options.

6. Consider an average strike option with discrete arithmetic averaging. Assume that the stock pays dividends and that during the time step $[t, t+dt]$, the dividend payment is $D(S, t)dt$. Take $S$ and $I = \frac{1}{K} \int_0^t S(\tau) f(\tau) d\tau$ as state variables, where

$$f(t) = \sum_{i=1}^K \delta(t - t_i).$$
a) Derive the equation for such an option directly by using a portfolio
\[ H = V - \Delta S. \]
b) Find the jump condition at \( t = t_i, \ i = 1, 2, \cdots, K \) if at \( t = t_i \) no
discrete dividend is paid.
c) Finally under the assumption \( D(S, t) = D_0 S \), reduce an average strike
option problem to a problem with only two independent variables and
the payoff to a function with only one independent variable.

7. Let \( V(S, A, t) \) be the price of a European Asian option with continuous
arithmetic averaging, where \( A \) is the average of the price during the time
period \([0, t]\). As we know, the equation for European Asian option with
continuous arithmetic averaging is
\[
\frac{\partial W(\eta, t)}{\partial t} + L_{a,t} W(\eta, t) = 0,
\]
where \( W = V(S, A, t)/S, \eta = A/S \) and \( L_{a,t} \) is the time-dependent opera-
tor related to Asian options and given by
\[
L_{a,t} = \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2}{\partial \eta^2} + \left( (D_0 - r)\eta + \frac{1 - \eta}{t} \right) \frac{\partial}{\partial \eta} = D_0.
\]

a) Write down the LC problem for an American Asian put option with
a continuous arithmetic average strike price.

b) Determine where the PDE can always be used and a free bounda-
ry cannot appear and where a free boundary may appear.

c) Derive the free-boundary problem for this case. (Assume that there
exists at most one free boundary.)

8. Define
\[
L_{S,A} = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (r - D_0)S \frac{\partial}{\partial S} + \frac{S - A}{t} \frac{\partial}{\partial A} - r.
\]

a) Find the function of location of the free boundary at \( t = T, S = S_f(A, T) \), for the LC problem:
\[
\min \left( - \frac{\partial V}{\partial t} - L_{S,A} V, V - \max(\alpha S - A, 0) \right) = 0, \quad 0 \leq S, 0 \leq A,
\]
\[
t \leq T,
\]
\[
V(S, A, T) = \max(\alpha S - A, 0), \quad 0 \leq S, 0 \leq A.
\]

b) Find the function of location of the free boundary at \( t = T, A = A_f(S, T) \), for the LC problem:
\[
\min \left( - \frac{\partial V}{\partial t} - L_{S,A} V, V - \max(A - E, 0) \right) = 0, \quad 0 \leq S, 0 \leq A,
\]
\[
t \leq T,
\]
\[
V(S, A, T) = \max(A - E, 0), \quad 0 \leq S, 0 \leq A.
\]
9. Suppose that sampling is done discretely at \( t = t_1, t_2, \ldots, t_K \), where \( 0 \leq t_1 < t_2 < \cdots < t_K \leq T \). Let \( H(t) = \max\left(S(t_1), \ldots, S(t^*_i(t))\right) \), where \( i^*_i(t) \) is the number of samplings before time \( t \). Assume \( dS = \mu S dt + \sigma S dX \) and the dividends are paid continuously with dividend yield \( D_0 \). Let \( V(S, H, t) \) be the value of a lookback option with discrete sampling. Derive the PDE and the jump condition for such a lookback option by using a portfolio \( \Pi = V(S, H, t) - \Delta S \) (without using the general PDE for derivative securities).

10. Consider a lookback option \( V(S, H, t) \) with continuous sampling. Describe how to get the PDE and the boundary condition for such an option from the PDE and the jump condition for an identical lookback option with discrete sampling and reduce the PDE to a PDE involving only two independent variables and the boundary condition to a boundary condition involving only one independent variable.

11. Consider the following problem:

\[
\begin{aligned}
\frac{\partial W}{\partial t} + \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2 W}{\partial \eta^2} + (D_0 - r) \eta \frac{\partial W}{\partial \eta} - D_0 W &= 0, \quad 0 \leq \eta, \quad t \leq T, \\
W(\eta, T) &= \begin{cases} 
\varphi_1(\eta), & 0 \leq \eta \leq 1, \\
\varphi_2(\eta), & 1 \leq \eta,
\end{cases}
\end{aligned}
\]

where \( \varphi_1(\eta) \) and \( \varphi_2(\eta) \) are continuous functions, and \( \varphi_1(1) = \varphi_2(1) \) may not hold. Show that if

\[
\frac{d\varphi_1(\eta)}{d\eta} = \frac{\eta^2 (r-D_0+\sigma^2/2)/\sigma^2}{\eta} d\varphi_2(1/\eta)
\]

or

\[
\frac{d\varphi_2(\eta)}{d\eta} = \frac{\eta^2 (r-D_0+\sigma^2/2)/\sigma^2}{\eta} d\varphi_1(1/\eta)
\]

then \( \frac{\partial W(1,t)}{\partial \eta} = 0 \).

12. Suppose that the payoff of a lookback strike put option is

\[
\max(H - \beta S, 0),
\]

where \( \beta \geq 1 \). Show that if \( r \neq D_0 \), its solution is

\[
p_{\text{ls}}(S, H, t) = e^{-r(T-t)} S \left[ \frac{H}{S} N \left( \frac{\ln \frac{H}{\beta S} - \mu(T-t)}{\sigma \sqrt{T-t}} \right) \right]
\]
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\[ -\beta e^{(r-D_0)(T-t)} N \left( \ln \frac{H}{S} - (\mu + \sigma^2) (T-t) \right) \]

\[ - \frac{\sigma^2}{2(r-D_0)} \left( \frac{H}{S} \right)^{2(r-D_0)/\sigma^2} \frac{N \left( \ln \frac{S}{\beta H} - \mu (T-t) \right)}{\sigma \sqrt{T-t}} \]

\[ + \frac{\sigma^2 \beta^{-2(r-D_0)/\sigma^2} e^{(r-D_0)(T-t)}}{2(r-D_0)} \frac{N \left( \ln \frac{S}{\beta H} + (\mu + \sigma^2) (T-t) \right)}{\sigma \sqrt{T-t}} \]

where

\[ \mu = r - D_0 - \frac{\sigma^2}{2}, \]

and if \( r = D_0 \), the solution is

\[ p_{ls}(S, H, t) = e^{-r(T-t)} S \left[ \frac{H}{S} N \left( \frac{\ln \frac{H}{S} - \mu (T-t)}{\sigma \sqrt{T-t}} \right) \right. \]

\[ -\beta N \left( \ln \frac{H}{S} - (\mu + \sigma^2) (T-t) \right) \]

\[ + \left\{ \left[ 1 + \frac{\sigma^2}{2(r-D_0)} \right] S e^{(r-D_0)(T-t)} N \left( \ln \frac{S}{\beta H} + (\mu + \sigma^2) (T-t) \right) \right\} \frac{1}{\sigma \sqrt{T-t}} \]

\[ + \frac{\sigma \sqrt{T-t}}{\sqrt{2\pi}} e^{-\left[ \ln(S/\beta H) + \sigma^2(T-t)/2 \right] / 2\sigma^2(T-t)} \]

13. Suppose the payoff of a lookback price call option is

\[ \max(H - E, 0). \]

Show that if \( H > E \), the price is

\[ c_{lp}(S, H, t) = e^{-r(T-t)} \left\{ \frac{H N \left( \ln \frac{H}{S} - \mu (T-t) \right)}{\sigma \sqrt{T-t}} - E \right\} \]

\[ + \left[ 1 + \frac{\sigma^2}{2(r-D_0)} \right] S e^{(r-D_0)(T-t)} N \left( \ln \frac{S}{\beta H} + (\mu + \sigma^2) (T-t) \right) \frac{1}{\sigma \sqrt{T-t}} \]

\[ - \frac{\sigma^2 S}{2(r-D_0)} \left( \frac{H}{S} \right)^{2(r-D_0)/\sigma^2} \frac{N \left( \ln \frac{S}{\beta H} - \mu (T-t) \right)}{\sigma \sqrt{T-t}} \]

and that if \( H \leq E \), the price is

\[ c_{lp}(S, H, t) \]
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\[ e^{-r(T-t)} \left\{ -EN \left( \frac{\ln \frac{S}{E} + \mu (T-t)}{\sigma \sqrt{T-t}} \right) \right\} \]

\[ + \left[ 1 + \frac{\sigma^2}{2(r-D_0)} \right] S e^{(r-D_0)(T-t)} N \left( \frac{\ln \frac{S}{E} + (\mu + \sigma^2) (T-t)}{\sigma \sqrt{T-t}} \right) \]

\[ - \frac{\sigma^2 S}{2(r-D_0)} \left( \frac{E}{S} \right)^{2(r-D_0)/\sigma^2} N \left( \frac{\ln \frac{S}{E} - \mu (T-t)}{\sigma \sqrt{T-t}} \right) \],

where

\[ \mu = r - D_0 - \sigma^2 / 2. \]

14. Suppose the payoff of a lookback price put option is

\[ \max(E - L, 0). \]

Show that for the case \( E > L \), the price is

\[ p_{lp} (S, L, t) = e^{-r(T-t)} \left\{ E - LN \left( \frac{\ln \frac{S}{E} + \mu (T-t)}{\sigma \sqrt{T-t}} \right) \right\} \]

\[ - \left[ 1 + \frac{\sigma^2}{2(r-D_0)} \right] S e^{(r-D_0)(T-t)} N \left( \frac{\ln \frac{S}{E} - (\mu + \sigma^2) (T-t)}{\sigma \sqrt{T-t}} \right) \]

\[ + \frac{\sigma^2 S}{2(r-D_0)} \left( \frac{L}{S} \right)^{2(r-D_0)/\sigma^2} N \left( \frac{\ln \frac{S}{E} + \mu (T-t)}{\sigma \sqrt{T-t}} \right) \],

and that for \( E \leq L \), the price is

\[ p_{lp} (S, L, t) = e^{-r(T-t)} \left\{ EN \left( \frac{\ln \frac{E}{S} - \mu (T-t)}{\sigma \sqrt{T-t}} \right) \right\} \]

\[ - \left[ 1 + \frac{\sigma^2}{2(r-D_0)} \right] S e^{(r-D_0)(T-t)} N \left( \frac{\ln \frac{E}{S} - (\mu + \sigma^2) (T-t)}{\sigma \sqrt{T-t}} \right) \]

\[ + \frac{\sigma^2 S}{2(r-D_0)} \left( \frac{E}{S} \right)^{2(r-D_0)/\sigma^2} N \left( \frac{\ln \frac{E}{S} + \mu (T-t)}{\sigma \sqrt{T-t}} \right) \],

where

\[ \mu = r - D_0 - \sigma^2 / 2. \]

15. Show that for lookback options depending on \( S, L, t \), the Green's function is
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\[
g\left(S', L_t'; S, t' - t\right) = \frac{\partial}{\partial L_t'} \left[ \frac{1}{S'\sigma \sqrt{2\pi \tau'}} \left(\frac{L_t'}{S}\right)^{2\mu/\sigma^2} e^{-\left[\ln\left(S'/S/(L_t')^2\right) - \mu \tau'\right]^2/2\sigma^2\tau'} \right],
\]

where \(\tau' = t' - t\) and \(\mu = r - D_0 - \sigma^2/2\).

16. Let \(g\left(S', L_t'; S, t' - t\right)\) be Green’s function for lookback options depending on \(S, L, t\), and let \(c_{ls}(S, L, t)\) and \(C_{ls}(S, L, t)\) be the prices of the European and American lookback strike call options with continuous sampling, respectively.

a) As we know,

\[
g\left(S', L_t'; S, t' - t\right) = \frac{\partial}{\partial L_t'} \left[ \frac{1}{S'\sigma \sqrt{2\pi \tau'}} \left(\frac{L_t'}{S}\right)^{2\mu/\sigma^2} e^{-\left[\ln\left(S'/S/(L_t')^2\right) - \mu \tau'\right]^2/2\sigma^2\tau'} \right],
\]

for any \(L_t' \leq \min(S, S')\).

b) Show

\(C_{ls}(S, L, t) \geq c_{ls}(S, L, t)\)

always holds.

c) Show \(C_{ls}(S, L, t^*) \geq C_{ls}(S, L, t^{**})\) if \(t^* < t^{**}\).

17. As we know, for a European lookback strike call option with continuous sampling, the corresponding one-dimensional problem is

\[
\begin{cases}
\frac{\partial W}{\partial t} + L_{\eta} W = 0, & 0 \leq \eta \leq 1, \quad t \leq T, \\
W(\eta, T) = \max(\alpha - \eta, 0), & 0 \leq \eta \leq 1, \\
\frac{\partial W}{\partial \eta}(1, t) = 0, & t \leq T,
\end{cases}
\]

where

\[
\eta = L/S, \quad W(\eta, t) = V(S, L, t)/S,
\]

\[
L_{\eta} = \frac{1}{2} \sigma^2 \eta^2 \frac{\partial^2}{\partial \eta^2} + (D_0 - r)\eta \frac{\partial}{\partial \eta} - D_0,
\]

and \(0 < \alpha \leq 1\).
a) Let $V(S,L,t)$ denote the price of the American lookback strike call option with continuous sampling and define $W = V/S$. Derive the linear complementarity problem for $W$.

b) Assume that we have proven $V(S,L,t^*) \geq V(S,L,t^{**})$ for any $t^* \leq t^{**}$. Derive the free-boundary problem for $W$.

18. a) The price of a European better-of option is the solution of the following problem:

$$\begin{align*}
\frac{\partial V}{\partial t} + LV &= 0, \\
V(S_1,S_2,T) &= \max(S_1,S_2),
\end{align*}$$

where

$$L = \frac{1}{2} \sigma_1^2 S_1 \frac{\partial^2}{\partial S_1^2} + \rho \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2 \frac{\partial^2}{\partial S_2^2} + (r - D_{01}) S_1 \frac{\partial}{\partial S_1} + (r - D_{02}) S_2 \frac{\partial}{\partial S_2} - r.$$

Let $\xi = S_1 / S_2$, $W = V / S_2$, and $\tau = T - t$. Show that $W$ is the solution of the problem:

$$\begin{align*}
\frac{\partial W}{\partial \tau} &= L_\xi W, \\
W(\xi,0) &= \max(\xi,1),
\end{align*}$$

where

$$L_\xi = \frac{1}{2} \left[ \sigma_1^2 - 2 \rho \sigma_1 \sigma_2 + \sigma_2^2 \right] \xi^2 \frac{\partial^2}{\partial \xi^2} + (D_{02} - D_{01}) \xi \frac{\partial}{\partial \xi} - D_{02}.$$

b) For $W$, the corresponding American-style problem is

$$\begin{align*}
\min \left( \frac{\partial W}{\partial \tau} - L_\xi W, \ W(\xi,\tau) - \max(\xi,1) \right) &= 0, \\
W(\xi,0) &= \max(\xi,1)
\end{align*}$$

Based on this formulation, show that the corresponding free-boundary problem can be written as
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\[ \frac{\partial W}{\partial \tau} = L_\xi W, \quad \xi_{f_1}(\tau) \leq \xi \leq \xi_{f_2}(\tau), \quad \tau \geq 0, \]
\[ W(\xi, 0) = \max(\xi, 1), \quad \xi_{f_1}(0) \leq \xi \leq \xi_{f_2}(0), \]
\[ W(\xi_{f_1}(\tau), \tau) = 1, \quad \tau \geq 0, \]
\[ \frac{\partial W}{\partial \xi} (\xi_{f_1}(\tau), \tau) = 0, \quad \tau \geq 0, \]
\[ W(\xi_{f_2}(\tau), \tau) = \xi_{f_2}(\tau), \quad \tau \geq 0, \]
\[ \frac{\partial W}{\partial \xi} (\xi_{f_2}(\tau), \tau) = 1, \quad \tau \geq 0, \]
\[ \xi_{f_1}(0) = 1, \]
\[ \xi_{f_2}(0) = 1. \]

19. Suppose that the value \( V \) of an option depends on \( S, H, \) and \( t \), i.e., \( V = V(S, H, t) \). As we know, for such any European option, \( V \) satisfies the equation

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV = 0, \quad 0 \leq S \leq H, \quad t \leq T, \]

and the condition

\[ \frac{\partial V}{\partial H} (S, S, t) = 0, \quad 0 \leq S. \]

For such a perpetual American option with the constraint \( V \geq H \), which is called the Russian option, \( V = V(S, H) \) and \( V \) is the solution of the following LC problem

\[ \min \left( - \left[ \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D_0) S \frac{\partial V}{\partial S} - rV \right] , V(S, H) - H \right) = 0, \quad 0 \leq S \leq H, \]
\[ \frac{\partial V}{\partial H} (S, S) = 0, \quad 0 \leq S. \]

We can find a solution of this problem in the following way:

a) Find the solution of the following free-boundary problem:

\[ \frac{1}{2} \sigma^2 \eta^2 \frac{d^2 W_\infty}{d\eta^2} + (D_0 - r) \eta \frac{dW_\infty}{d\eta} - D_0 W_\infty = 0, \quad 1 \leq \eta \leq \eta_f, \]
\[ \frac{dW_\infty(1)}{d\eta} = 0, \]
\[ W_\infty(\eta_f) = \eta_f, \]
\[ \frac{dW_\infty(\eta_f)}{d\eta} = 1, \]
where \( \eta_f \) is a number representing the location of this free boundary.

b) Define

\[
W_\infty(\eta) = \begin{cases} 
\text{the solution of the free-boundary problem, } 1 \leq \eta \leq \eta_f, \\
\eta, \quad \eta_f < \eta.
\end{cases}
\]

Show that \( W_\infty(\eta) \) is a solution of the following LC problem

\[
\begin{align*}
\min \left( -\frac{\sigma^2}{2} \eta^2 \frac{d^2 W_\infty}{d\eta^2} - (D_0 - r) \eta \frac{dW_\infty}{d\eta} + D_0 W_\infty, \ W_\infty - \eta \right) &= 0, \\
1 \leq \eta,
\end{align*}
\]

\[
\frac{dW_\infty}{d\eta}(1) = 0.
\]

c) Show that the function \( SW_\infty(H/S) \) is a solution of the LC problem given at the beginning.

20. Find the solution of the problem:

\[
\begin{align*}
\frac{1}{2} \sigma^2 \xi^2 \frac{d^2 W_\infty}{d\xi^2} + (D_{02} - D_{01}) \xi \frac{dW_\infty}{d\xi} - D_{02} W_\infty &= 0, \quad \xi_{f_1} \leq \xi \leq \xi_{f_2}, \\
W_\infty(\xi_{f_1}) &= 1, \\
\frac{dW_\infty}{d\xi}(\xi_{f_1}) &= 0, \\
W_\infty(\xi_{f_2}) &= \xi_{f_2}, \\
\frac{dW_\infty}{d\xi}(\xi_{f_2}) &= 1,
\end{align*}
\]

where \( \xi_{f_1} < \xi_{f_2} \). (This problem is related to the perpetual American better-of option.)

21. Consider the following problem

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n} (r - D_{0i}) S_i \frac{\partial V}{\partial S_i} - r V &= 0, \\
0 \leq S_i, \quad 0 \leq t \leq T, \\
V(S, T) &= V_T(S_1, S_2, \ldots, S_n), \quad 0 \leq S.
\end{align*}
\]

a) Let \( V(S, t) = e^{-r(T-t)} \mathcal{V}(S, t) \). Show that \( \mathcal{V}(S, t) \) is the solution of the problem

\[
\begin{align*}
\frac{\partial \mathcal{V}}{\partial t} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 \mathcal{V}}{\partial S_i \partial S_j} + \sum_{i=1}^{n} (r - D_{0i}) S_i \frac{\partial \mathcal{V}}{\partial S_i} &= 0, \\
0 \leq S_i, \quad 0 \leq t \leq T, \\
\mathcal{V}(S, T) &= V_T(S_1, S_2, \ldots, S_n), \quad 0 \leq S.
\end{align*}
\]
b) Let
\[
\begin{aligned}
y_i &= a_i \ln S_i + b_i (T - t), \\
\tau &= T - t,
\end{aligned}
\]
and \( \nabla_1(y, \tau) = \nabla(S, t) \), \( y \) standing for \((y_1, y_2, \ldots, y_n)^T\). Find \( a_i \) and \( b_i \) such that \( \nabla_1(y, \tau) \) satisfies
\[
\frac{\partial \nabla_1(y, \tau)}{\partial \tau} = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \frac{\partial^2 \nabla_1(y, \tau)}{\partial y_i \partial y_j}, \quad -\infty < y < \infty, \quad 0 \leq \tau \leq T,
\]
where \( V_1(y, \tau) \equiv V(S, t) \). Find the equation and initial condition for \( \nabla_2(x, \tau) \).

c) Let
\[
x = Ry
\]
and
\[
\nabla_2(x, \tau) = \nabla_1(y, \tau),
\]
where \( R \) is a constant matrix:
\[
R = \begin{bmatrix}
r_{11} & r_{12} & \cdots & r_{1n} \\
r_{21} & r_{22} & \cdots & r_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
r_{n1} & r_{n2} & \cdots & r_{nn}
\end{bmatrix}
\]
Find the equation and initial condition for \( \nabla_2(x, \tau) \).

d) Find \( R \) such that \( \nabla_2(x, \tau) \) satisfies
\[
\frac{\partial \nabla_2(x, \tau)}{\partial \tau} = \sum_{l=1}^{n} \frac{\partial^2 \nabla_2(x, \tau)}{\partial x_l^2}, \quad -\infty < x < \infty, \quad 0 \leq \tau \leq T,
\]
where \( V_2(x, \tau) \equiv V_1(R^{-1}x) \).

22. a) Show that
\[
\phi(x_0; x, \tau) = \frac{1}{(4\pi \tau)^{n/2}} e^{-\sum_{i=1}^{n} (x_i - x_{i0})^2/(4\tau)}
\]
is a solution to
\[
\frac{\partial \phi}{\partial \tau} = \sum_{i=1}^{n} \frac{\partial^2 \phi}{\partial x_i^2}, \quad -\infty < x < \infty, \quad 0 \leq \tau,
\]
where
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\[
x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x_0 = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{n0} \end{pmatrix}
\]

and \(-\infty < x < \infty\) means \(-\infty < x_i < \infty, \quad i = 1, 2, \ldots, n\).

b) Show that the function \(\phi(x_0; x, \tau)\) satisfies the conditions

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \phi(x_0; x, \tau) dx_{10} dx_{20} \cdots dx_{n0} = 1
\]

and

\[
\lim_{\tau \to 0} \phi(x_0; x, \tau) = \begin{cases} 
\infty, & \text{at } x = x_0, \\
0, & \text{otherwise},
\end{cases}
\]

that is,

\[
\lim_{\tau \to 0} \phi(x_0; x, \tau) = \delta(x - x_0).
\]

c) Show that

\[
V(x, \tau) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} V_0(x_0) \phi(x_0; x, \tau) dx_{10} dx_{20} \cdots dx_{n0}
\]

is the solution of the problem

\[
\begin{align*}
\frac{\partial V}{\partial \tau} &= \sum_{i=1}^{n} \frac{\partial^2 V}{\partial x_i^2}, \quad -\infty < x < \infty, \quad 0 \leq \tau, \\
V(x, 0) &= V_0(x), \quad -\infty < x < \infty.
\end{align*}
\]

23. Let \(P\) be a positive definite matrix. As we know, in this case there exist a matrix \(Q\) and a diagonal matrix \(\Lambda\) such that \(P = \Lambda Q^T\), where all the components of \(\Lambda\) are positive and \(Q\) satisfies the conditions \(QQ^T = I\) and \(\det Q > 0\). Let \(y\) and \(y_0\) be two vectors and define \(R = \Lambda^{-1/2}Q^T\), \(x = Ry\), \(x_0 = Ry_0\), and \(\eta = \frac{y_0 - y}{\sqrt{2\tau}}\). Show

a) \(\det R = \frac{1}{\sqrt{\det P}}\);

b) \(\frac{(x_0 - x)^T (x_0 - x)}{4\tau} = \eta^T P^{-1} \eta\).

24. a) Suppose that \(V(S_1, S_2, t)\) satisfies
b) As we know, if let $\psi = S_0$, then

\[ \frac{\partial V}{\partial t} + \frac{1}{2} \sigma_1^2 S_1 \frac{\partial^2 V}{\partial S_1^2} + \rho_{12} \sigma_1 \sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2} \sigma_2^2 S_2 \frac{\partial^2 V}{\partial S_2^2} + (r - D_{01}) S_1 \frac{\partial V}{\partial S_1} + (r - D_{02}) S_2 \frac{\partial V}{\partial S_2} - rV = 0, \]

\[ V(S_1, S_2, T) = \max(S_0, S_1, S_2), \quad 0 \leq S_1, S_2, 0 \leq t \leq T, \]

we have the solution of the following problem:

Define $S_0^* = S_0 e^{-(T-t)}$, $S_i^* = S_i e^{-(T-t)}$, $i = 1, 2$. Let $\xi_{12} = S_1^*/S_2^*$ and $\xi_{10} = S_1^*/S_0^*$ as independent variables and $V_0(\xi_{10}, \xi_{20}, t) = V(S_1, S_2, t)/S_0^*$, then $V_0(\xi_{10}, \xi_{20}, t)$ is the solution of the following problem:

\[ \frac{\partial V_0}{\partial t} + \frac{1}{2} \sigma_{10}^2 \xi_{10} \frac{\partial^2 V_0}{\partial \xi_{10}^2} + \rho_{120} \sigma_{10} \sigma_{20} \xi_{10} \xi_{20} \frac{\partial^2 V_0}{\partial \xi_{10} \partial \xi_{20}} \]

\[ + \frac{1}{2} \sigma_{20}^2 \xi_{20} \frac{\partial^2 V_0}{\partial \xi_{20}^2} = 0, \quad 0 \leq \xi_{10}, \ 0 \leq \xi_{20}, \ 0 \leq t \leq T, \]

\[ V_0(\xi_{10}, \xi_{20}, T) = \max(1, \xi_{10}, \xi_{20}), \quad 0 \leq \xi_{10}, \ 0 \leq \xi_{20}. \]

Suppose that we know

\[ V(S_1, S_2, t) = S_0^* \int_{\max(\xi_{10}, 1, \xi_{20}, T)} \psi d\xi_{10} d\xi_{20}, \]

\[ + S_0^* \int_{\max(1, \xi_{20}, T)} \xi_{10} \psi d\xi_{10} d\xi_{20}, \]

\[ + S_0^* \int_{\max(1, \xi_{10}, T)} \xi_{20} \psi d\xi_{10} d\xi_{20}, \]

the solution is equal to

\[ S_0^* N_2 \left( \frac{\ln \frac{S_1^*}{S_1} + \frac{\sigma_{10}^2 \tau}{2}}{\sigma_{10} \sqrt{\tau}}, \frac{\ln \frac{S_2^*}{S_2} + \frac{\sigma_{20}^2 \tau}{2}}{\sigma_{20} \sqrt{\tau}}; \rho_{120} \right), \]

where $N_2(x, y, \rho)$ is a function of $x, y$, and $\rho$. Using the result in part a), show that the third term should be equal to

\[ S_2^* N_2 \left( \frac{\ln \frac{S_1^*}{S_1} + \frac{\sigma_{12}^2 \tau}{2}}{\sigma_{12} \sqrt{\tau}}, \frac{\ln \frac{S_2^*}{S_2} + \frac{\sigma_{12}^2 \tau}{2}}{\sigma_{12} \sqrt{\tau}}; \rho_{012} \right). \]
25. Show

\[
\begin{aligned}
e^{-r\tau} \int_{S_0}^{S_T} S_1 \psi(S_{1T}, S_{2T}; S_1, S_2, t)dS_{2T}dS_{1T}
&= S_1^* N_2 \left( \frac{\ln S_1^* + \frac{\sigma_1^2}{2} \tau}{\sigma_1 \sqrt{\tau}}, \frac{\ln S_1^* + \frac{\sigma_2^2}{2} \tau}{\sigma_2 \sqrt{\tau}}; \frac{\sigma_1 - \rho_1 \sigma_2}{\sigma_2} \right)
\end{aligned}
\]

by direct calculation, i.e., without using solutions of PDEs. Here

\[
\begin{aligned}
&\tau = T - t, \\
&\psi(S_{1T}, S_{2T}; S_1, S_2, t) = \frac{1}{2\pi \tau \sqrt{\det \mathbf{P}}} e^{-\eta^T \mathbf{P}^{-1} \eta / 2}, \\
&S_0^* = S_0 e^{-r\tau}, \quad S_1^* = S_1 e^{-D_{01} \tau}, \quad S_2^* = S_2 e^{-D_{02} \tau}, \\
&\sigma_{12} = \sqrt{\sigma_1^2 - 2\rho_1 \sigma_1 \sigma_2 + \sigma_2^2}, \\
&N_2(x_1, x_2; \rho) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} e^{-\frac{1}{2}(\eta_1^2 - 2\rho \eta_1 \eta_2 + \eta_2^2)/(1 - \rho^2)} d\eta_1 d\eta_2,
\end{aligned}
\]

where

\[
\mathbf{P} = \begin{bmatrix} 1 & \rho_{12} \\ \rho_{12} & 1 \end{bmatrix},
\]

the \(i\)-th component of \(\eta\) in \(\psi(S_{1T}, S_{2T}; S_1, S_2, t)\) is given by

\[
\eta_i(S_{iT}) = \frac{\ln S_{iT} - \ln S_i + (r - D_{0i} - \sigma_i^2/2)\tau}{\sigma_i \sqrt{\tau}}, \quad i = 1, 2,
\]

and \(r, D_{01}, D_{02}, \sigma_1, \sigma_2, \rho_{12}, \tau, S_0, S_1, S_2, t\) are parameters.

26. Suppose \(S_1\) and \(S_2\) are the prices of two assets \(A\) and \(B\), respectively. The random variables \(S_1\) and \(S_2\) satisfy

\[
\begin{aligned}
dS_1 &= \mu_1 S_1 dt + \sigma_1 S_1 dX_1, \\
dS_2 &= \mu_2 S_2 dt + \sigma_2 S_2 dX_2,
\end{aligned}
\]

where \(\mu_1, \mu_2, \sigma_1,\) and \(\sigma_2\) are constants, and \(dX_1\) and \(dX_2\) are two Wiener processes with

\[
\mathbb{E}[dX_1 dX_2] = \rho_{12} dt.
\]

Also, suppose that the two assets pay dividends continuously and that the dividend yields of the assets \(A\) and \(B\) are \(D_{01}\) and \(D_{02}\), respectively. Consider a European option on the minimum of \(S_1, S_2,\) and \(S_0\), i.e., its payoff function is

\[
\min(S_0, S_1, S_2),
\]
where \( S_0 \) is a constant. Let \( V_{\min}(S_1, S_2, t) \) be the price of the option. Show that

\[
V_{\min}(S_1, S_2, t) = S_0^* N_2 \left( \frac{\ln S_1^*}{\sigma_1 \sqrt{\tau}} - \frac{\sigma_1^2 \tau}{2}, \frac{\ln S_2^*}{\sigma_2 \sqrt{\tau}} - \frac{\sigma_2^2 \tau}{2}, \rho_{12} \right) \\
+ S_1^* N_2 \left( \frac{\ln S_2^*}{\sigma_2 \sqrt{\tau}} - \frac{\sigma_2^2 \tau}{2}, \frac{\ln S_1^*}{\sigma_1 \sqrt{\tau}} - \frac{\sigma_1^2 \tau}{2}, \rho_{12} - \frac{\sigma_1 \sigma_2}{\sigma_2 \sqrt{\tau}} \right) \\
+ S_2^* N_2 \left( \frac{\ln S_1^*}{\sigma_1 \sqrt{\tau}} - \frac{\sigma_1^2 \tau}{2}, \frac{\ln S_2^*}{\sigma_2 \sqrt{\tau}} - \frac{\sigma_2^2 \tau}{2}, \rho_{12} - \frac{\sigma_1 \sigma_2}{\sigma_1 \sqrt{\tau}} \right),
\]

where \( S_0^* = S_0 e^{-r \tau}, S_1^* = S_1 e^{-D_{01} \tau} \) and \( S_2^* = S_2 e^{-D_{02} \tau} \), \( \tau \) denoting \( T - t \).

27. \( \mathbf{S}, \mathbf{S}_T, \xi, \) and \( \eta \) are \( n \)-dimensional vectors. The \( i \)-th components of \( \mathbf{S} \) and \( \mathbf{S}_T \) are \( S_i \) and \( S_i T \) respectively. \( \tau, \sigma, D_{0_i}, \) and \( \sigma_i, i = 1, 2, \cdots, n, \) are numbers. We further define

\[
S_0^* = S_0 e^{-r \tau}, \quad S_i^* = S_i e^{-D_{0_i} \tau}, \quad i = 1, 2, \cdots, n.
\]

The \( i \)-th components of \( \xi \) and \( \eta \) are given by

\[
\xi_i(S_{i\tau}) = \frac{\ln S_{i\tau} - \ln S_0}{\sigma_i \sqrt{\tau}}, \\
\eta_i(S_{i\tau}) = \frac{\ln S_{i\tau} - \{\ln S_i + (r - D_{0i} - \sigma_i^2/2)\tau\}}{\sigma_i \sqrt{\tau}} \\
= \frac{\ln S_{i\tau} - \ln S_0 - [\ln S_i^* - \ln S_0^*] + \sigma_i^2 \tau / 2}{\sigma_i \sqrt{\tau}} \\
= \xi_i + \frac{\ln S_0^* - \ln S_i^*}{\sigma_i \sqrt{\tau}} + \sigma_i \sqrt{\tau} / 2.
\]

Define

\[
\mathbf{P} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix} \quad \text{with} \quad \rho_{ij} = \rho_{ji},
\]

\[
\psi(\mathbf{S}_T; \mathbf{S}, t) = \frac{1}{(2\pi \tau)^{n/2} \sqrt{\det \mathbf{P}}} \prod_{i=1}^{n} \left( \sigma_i S_{i\tau} \right)^{-1} e^{-\frac{1}{2} \eta^T \mathbf{P}^{-1} \eta}.
\]
and
\[ \sigma_{ij} = \sqrt{\sigma_i^2 - 2 \rho_{ij} \sigma_i \sigma_j + \sigma_j^2}, \quad \text{for any } i \text{ and } j. \]
a) Show
\[ e^{-r\tau} S_1 T e^{-\frac{1}{2}\eta^T P^{-1} \eta} = S_1^* e^{-\frac{1}{2}(\eta - \sigma \sqrt{\tau} Pe_1)^T P^{-1}(\eta - \sigma \sqrt{\tau} Pe_1)}, \]
where \( e_i \) is the vector, the \( i \)-th component of which is one and the other component of which is zero.
b) \( R \) is a given matrix. Define
\[ z = R \xi, \quad b = R a, \quad \zeta = z + b, \quad \text{and} \quad Q = R P R^T. \]
Show
\[ (\xi + a)^T P^{-1} (\xi + a) = (z + b)^T Q^{-1} (z + b) = \zeta^T Q^{-1} \zeta. \]
c) Show
\[ \frac{1}{\tau^{n/2} \prod_{i=1}^n (\sigma_i S_{1r}) \sqrt{\det P}} ds_1 T ds_2 T \cdots ds_n T = \frac{1}{\sqrt{\det Q}} dz_1 dz_2 \cdots dz_n. \]
d) Suppose that the domain \( \Omega \) in the \( (S_1 T, S_2 T, \cdots, S_n T) \)-space is equivalent to the domain \( \Omega^* \) : \(-\infty < z_i \leq 0, \quad i = 1, 2, \cdots, n \), in the \( (z_1, z_2, \cdots, z_n) \)-space. Show
\[ e^{-r\tau} \int \cdots \int_\Omega S_1 T \psi(S_1 T, S_2 T; S, t) ds_2 T \cdots ds_n T = S_1^* N_n(b; Q), \]
where
\[ N_n(b; Q) = \frac{1}{(2\pi)^{n/2} \sqrt{\det Q}} \int_{-\infty}^{b_n} \int_{-\infty}^{b_n-1} \cdots \int_{-\infty}^{b_1} e^{-\frac{1}{2} \zeta^T Q^{-1} \zeta} d\zeta_1 d\zeta_2 \cdots d\zeta_n, \]
b_1 being the \( i \)-th component of \( b \) and
\[ b = R (\eta - \xi - \sigma \sqrt{\tau} Pe_1). \]
e) Define \( \Omega \) is the domain : \( \max(S_0, S_2) \leq S_1 T \). Show
\[ e^{-r\tau} \int \int_\Omega S_1 T \psi(S_1 T, S_2 T; S_1, S_2, t) ds_2 T ds_1 T = S_1^* N_2 \left( \ln \frac{S_1^*}{S_0^*} + \frac{\sigma_1^2 \tau}{2}, \ln \frac{S_1^*}{S_0^*} + \frac{\sigma_2^2 \tau}{2}, \frac{\sigma_1 - \rho_{12} \sigma_2}{\sigma_{12} \sqrt{\tau}} \right) \]
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where

\[ N_2(x_1, x_2; \rho) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} e^{-\frac{1}{2}(\eta_1^2 - 2\rho\eta_1\eta_2 + \eta_2^2)/(1 - \rho^2)} d\eta_1 d\eta_2. \]

f) Define \( \Omega \) is the domain : \( \max(S_0, S_{1T}, S_{2T}) \leq S_{3T} \). Show

\[
e^{-r\tau} \int \int \int_{\Omega} S_{3T}\psi(S_{1T}, S_{2T}, S_{3T}; S_{1}, S_{2}, S_{3}, t)dS_{1T}dS_{2T}dS_{3T} = S_3^* \quad N_3 \left( \begin{array}{c}
\ln \frac{S_3^*}{S_0} + \frac{\sigma_3^2 T}{2} \\
\ln \frac{S_3^*}{S_1} + \frac{\sigma_1^2 T}{2} \\
\ln \frac{S_3^*}{S_2} + \frac{\sigma_2^2 T}{2} \\
\rho_{13}, \rho_{23}, \rho_{12} \end{array} \right),
\]

where

\[
N_3(x_1, x_2, x_3; \rho_{12}, \rho_{13}, \rho_{23}) = \frac{1}{(2\pi)^{3/2} \det P} \int_{-\infty}^{x_3} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} e^{-\frac{1}{2} \eta^T P^{-1} \eta} d\eta_1 d\eta_2 d\eta_3,
\]

P being

\[
\begin{bmatrix}
1 & \rho_{12} & \rho_{13} \\
\rho_{12} & 1 & \rho_{23} \\
\rho_{13} & \rho_{23} & 1
\end{bmatrix}
\]

and \( \eta \) being

\[
\begin{bmatrix}
\eta_1 \\
\eta_2 \\
\eta_3
\end{bmatrix}
\]

g) Suppose that \( \Omega \) is the domain : \( \max(S_0, S_{1T}, \ldots, S_{i-1T}, S_{i+1T}, \ldots, S_{nT}) \leq S_{iT} \). Show that in this case if let

\[
R = \begin{bmatrix}
0 & \cdots & 0 & -\sigma_i & \cdots & \sigma_{i+1} & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & -\sigma_i & \cdots & \sigma_n & \cdots & 0 \\
0 & \cdots & 0 & -1 & \cdots & \sigma_n & \cdots & 0 \\
\sigma_{i-1} & \cdots & 0 & -\sigma_i & \cdots & \sigma_{i+1} & \cdots & 0 \\
\sigma_i & \cdots & 0 & \sigma_{i-1} & \ddots & \vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & \sigma_{i-1} & -\sigma_i & \cdots & \sigma_{i-1} & 0 & \cdots & 0
\end{bmatrix},
\]

then the domain \( \Omega \) is equivalent to the domain \( \Omega^* \) in d) and for b we have
28. Suppose that $S_1, S_2, S_3$ are the prices of three assets satisfying

$$dS_i = \mu_i S_i dt + \sigma_i S_i dX_i, \quad i = 1, 2, 3,$$

where $\mu_i, \sigma_i, i = 1, 2, 3$ are constants and $dX_i, i = 1, 2, 3$ are the Wiener processes with

$$E[dX_i dX_j] = \rho_{ij} dt, \quad i, j = 1, 2, 3.$$

Also, suppose that the three assets pay dividends continuously and that the dividend yields of the three assets are $D_i$, $i = 1, 2, 3$.

a) Consider a European option on the maximum of $S_1, S_2, S_3$, and $S_0$, i.e., its payoff is

$$\text{max}(S_0, S_1, S_2, S_3),$$

where $S_0$ is a certain amount of cash. If we understand $S_0$ as a random variable, then its volatility, $\sigma_0$, is equal to zero. Let $V_{max}(S_1, S_2, S_3, t)$ be the price of such a $T$-year option. Show that

$$V_{max}(S_1, S_2, S_3, t) = S_0^* N_3 (A_{10}, A_{20}, A_{30}; \rho_{120}, \rho_{130}, \rho_{230})$$

$$+ S_1^* N_3 (A_{21}, A_{31}, A_{01}; \rho_{231}, \rho_{201}, \rho_{301})$$

$$+ S_2^* N_3 (A_{32}, A_{02}, A_{12}; \rho_{302}, \rho_{312}, \rho_{012})$$

$$+ S_3^* N_3 (A_{03}, A_{13}, A_{23}; \rho_{013}, \rho_{023}, \rho_{123}),$$

where

$$S_0^* = S_0 e^{-r(T-t)},$$

$$S_i^* = S_i e^{-D_i(T-t)}, \quad i = 1, 2, 3,$$
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\[ A_{ij} = \ln \frac{S_i^*}{S_i} + \frac{\sigma_i^2 (T - t)}{\sigma_{ij} \sqrt{T - t}}, \quad i, j = 0, 1, 2, 3 \text{ but } i \neq j, \]

\[ \sigma_{ij} = \sqrt{\sigma_i^2 - 2\rho_{ij} \sigma_i \sigma_j + \sigma_j^2}, \]

\[ \rho_{ijk} = \frac{\rho_{ij} \sigma_i \sigma_j - \rho_{ik} \sigma_i \sigma_k - \rho_{jk} \sigma_j \sigma_k + \sigma_k^2}{\sigma_{ik} \sigma_{jk}}, \]

\[ i, j, k = 0, 1, 2, 3 \text{ but } i, j, k \text{ being distinct}, \]

and \( N_3(x_1, x_2, x_3; \rho_{12}, \rho_{13}, \rho_{23}) \) is the trivariate cumulative distribution function:

\[ N_3(x_1, x_2, x_3; \rho_{12}, \rho_{13}, \rho_{23}) = \frac{1}{(2\pi)^{3/2} \det F} \int_{-\infty}^{x_3} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} e^{-\frac{1}{2} \eta^T \rho^{-1} \eta} d\eta_1 d\eta_2 d\eta_3, \]

\[ \rho \text{ being } \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} \text{ and } \eta \text{ being } \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{bmatrix}. \]

b) Consider a European option on the minimum of \( S_1, S_2, S_3, \) and \( S_0, \)

i.e., its payoff is

\[ \min(S_0, S_1, S_2, S_3), \]

where \( S_0 \) is a certain amount of cash. Find the expression of the price for this option in terms of the trivariate cumulative distribution function.

29. Suppose that \( S_1, S_2, \ldots, S_n \) are the prices of \( n \) assets and that each asset pays a dividend continuously, the dividend yield for \( S_i \) being \( D_{0i}, i = 1, 2, \ldots, n. \) Each price \( S_i \) satisfies the stochastic equation

\[ dS_i = \mu_i S_i dt + \sigma_i S_i dX_i, \]

where \( \mu_i, \sigma_i \) are constants and \( dX_i \) is a Wiener process, and

\[ E[dX_i dX_j] = \rho_{ij} dt, \quad i, j = 1, 2, \ldots, n. \]

a) Guess the expression of price of the European option on the maximum of \( S_1, S_2, \ldots, S_n \) and \( S_0 \) according to the result given in part a) of Problem 28, where \( S_0 \) is a certain amount of cash.

b) Guess the expression of price of the European option on the minimum of \( S_1, S_2, \ldots, S_n \) and \( S_0 \) according to the result given in part b) of Problem 28.

30. Suppose that \( c_{\text{max}}(S_1, S_2, t), c_{\text{min}}(S_1, S_2, t), c(S_1, t) \) and \( c(S_2, t) \) are the prices of four call options with payoff functions

\[ \max(\max(S_1, S_2) - E, 0), \quad \max(\min(S_1, S_2) - E, 0), \quad \max(S_1 - E, 0), \]
and
\[ \max(S_2 - E, 0), \]
respectively. Show
\[ c_{\text{max}}(S_1, S_2, t) + c_{\text{min}}(S_1, S_2, t) = c(S_1, t) + c(S_2, t). \]
(Hint: Show that the total payoff of the two options on the left-hand side is equal to the total payoff of the two options on the right-hand side.)

31. Let \( p_{\text{max}}(S_1, S_2, t) \) and \( p_{\text{min}}(S_1, S_2, t) \) be the prices of two European put options with payoff functions
\[ \max(E - \max(S_1, S_2), 0) \]
\[ \max(E - \min(S_1, S_2), 0), \]
respectively. Suppose that \( c_{\text{max}}(S_1, S_2, t) \) and \( c_{\text{min}}(S_1, S_2, t) \) are the prices of two European call options with payoff functions
\[ \max(\max(S_1, S_2) - E, 0) \]
\[ \min(\min(S_1, S_2) - E, 0), \]
respectively. \( \bar{c}_{\text{max}}(S_1, S_2, t) \) and \( \bar{c}_{\text{min}}(S_1, S_2, t) \) denote the prices of European options with payoff functions
\[ \max(S_1, S_2), \]
\[ \min(S_1, S_2). \]
Show
a) \( p_{\text{max}}(S_1, S_2, t) = E e^{-r(T-t)} - \bar{c}_{\text{max}}(S_1, S_2, t) + c_{\text{max}}(S_1, S_2, t); \)
b) \( p_{\text{min}}(S_1, S_2, t) = E e^{-r(T-t)} - \bar{c}_{\text{min}}(S_1, S_2, t) + c_{\text{min}}(S_1, S_2, t). \)
(Hint: Show that the payoff of the option on the left-hand side is equal to the total value of the three terms at \( t = T \) on the right-hand side.)

32. Show that the closed-form solutions of cash-or-nothing puts, asset-or-nothing calls, and asset-or-nothing puts are
\[ Be^{-r(T-t)} N(-d_2), \]
\[ Se^{-D_0(T-t)} N(d_1), \]
\[ Se^{-D_0(T-t)} N(-d_1), \]
respectively. Here
\[ d_1 = \frac{\ln(Se^{(r-D_0)(T-t)}/E) + \sigma^2(T-t)/2}{\sigma \sqrt{T-t}}, \]
\[ d_2 = \frac{\ln(Se^{(r-D_0)(T-t)}/E) - \sigma^2(T-t)/2}{\sigma \sqrt{T-t}}. \]

33. Show that the value of a forward start American put option with exercise price \( E = \alpha S_{T_1} \) at time \( t_0 < T_1 \) is
\[ \alpha Se^{-D_0(T_1-t_0)} P^* \left( \frac{1}{\alpha}, T_1 \right), \]
where \( P^* \left( \frac{1}{\alpha}, T_1 \right) \) is the value of a standard American put option.
34. Consider compound options and assume that both options are European. Let \( c_1(S,t; c_2) \), \( c_1(S,t; p_2) \), and \( p_1(S,t; p_2) \) denote the prices of a call on a call, a call on a put, and a put on a put, respectively. Show that their closed-form solutions are

\[
\begin{align*}
  c_1(S,t;c_2) &= S e^{-D_0(T_2-t)} N_2(d_{11},d_{12};\rho) - E e^{-r(T_2-t)} N_2(d_{21},d_{22};\rho) \\
  c_1(S,t;p_2) &= E e^{-r(T_2-t)} N_2(-d_{23},-d_{22};\rho) - S e^{-D_0(T_2-t)} N_2(-d_{13},-d_{12};\rho) \\
  p_1(S,t;p_2) &= E e^{-r(T_2-t)} N(-d_{23}) - E e^{-r(T_2-t)} N_2(d_{23},-d_{22};\rho) \\
    &+ S e^{-D_0(T_2-t)} N_2(d_{13},-d_{12};\rho),
\end{align*}
\]

where

\[
\begin{align*}
  d_{11} &= \frac{\ln(S/S^*) + (r - D_0 + \sigma^2/2)(T_1 - t)}{\sigma \sqrt{T_1 - t}}, \\
  d_{21} &= \frac{\ln(S/S^*) + (r - D_0 - \sigma^2/2)(T_1 - t)}{\sigma \sqrt{T_1 - t}}, \\
  d_{12} &= \frac{\ln(S/E_2) + (r - D_0 + \sigma^2/2)(T_2 - t)}{\sigma \sqrt{T_2 - t}}, \\
  d_{22} &= \frac{\ln(S/E_2) + (r - D_0 - \sigma^2/2)(T_2 - t)}{\sigma \sqrt{T_2 - t}}, \\
  d_{13} &= \frac{\ln(S/S^{**}) + (r - D_0 + \sigma^2/2)(T_1 - t)}{\sigma \sqrt{T_1 - t}}, \\
  d_{23} &= \frac{\ln(S/S^{**}) + (r - D_0 - \sigma^2/2)(T_1 - t)}{\sigma \sqrt{T_1 - t}}, \\
  \rho &= \sqrt{\frac{T_1 - t}{T_2 - t}}.
\end{align*}
\]

Here, \( S^* \) and \( S^{**} \) are the solutions of the following equations:

\[
c_2(S^*,T_1) = E_1
\]

and

\[
p_2(S^{**},T_1) = E_1.
\]

35. How do we determine the price of a European put option on an American put option?

36. Show

\[
\int_{0}^{\infty} G(S'',T_2;S',T_1) G(S',T_1;S,t) dS' = G(S'',T_2;S,t),
\]
where

\[
G(S', T_1; S, t) = \frac{1}{\sigma \sqrt{2\pi (T_1 - t)}} e^{-\left[\ln S' - \ln S - \left(r - D_0 - \sigma^2/2\right)(T_1 - t)\right]^2/2\sigma^2(T_1 - t)}
\]

and \(G(S'', T_2; S', T_1)\) and \(G(S'', T_2; S, t)\) are defined in the same way.

37. The payoff of a standard chooser option is

\[
V(S, T_1) = \max\left(c(S, T_1), p(S, T_1)\right),
\]

where \(c(S, T_1)\) and \(p(S, T_1)\) are the prices of European call and put options with the same exercise price \(E_2\) and the same expiration date \(T_2\). Find its closed-form solution. (Hint: Use the put–call parity relation and the result of Problem 36.)
4

Interest Rate Derivative Securities

Problems

1. a) Suppose the spot interest rate is a known function \( r(t) \). Consider a bond with a face value \( Z \) and assume that it pays a coupon with a coupon rate \( k(t) \), that is, during a time interval \((t, t + dt]\), the coupon payment is \( Zk(t)dt \). Show that the value of the bond is

\[
V(t) = Ze^{-\int_t^T r(\tau) d\tau} \left[ 1 + \int_t^T k(\tau) e^{\int_t^\tau r(\tau) d\tau} d\tau \right].
\]

b) Suppose that \( r(t) \) and \( k(t) \) are equal to constants \( r \) and \( k \), respectively. Show that in this case,

\[
V(t) = Ze^{-rt} \left[ 1 + k(e^{r(T-t)} - 1)/r \right].
\]

c) Suppose that the bond pays coupon payments at two specified dates \( T_1 \) and \( T_2 \) before the maturity date \( T \) and the payments are \( Zk_1 \) and \( Zk_2 \), respectively. According to the formula given in a), and assuming \( T_1 < T_2 \), find the values of the bond for \( t \in [0, T_1) \), \( t \in (T_1, T_2) \), and \( t \in (T_2, T) \), respectively, and give a financial interpretation of these expressions.

2. Suppose that the spot interest rate \( r \) satisfies

\[
\dot{r} = u dt + \sigma dX,
\]

where \( dX \) is a Wiener process. Assume that during the time period \([0, t^*]\) the interest rate is equal to the spot rate \( r \). Thus the price of a zero-coupon bond at \( t = 0 \) with face value one and maturity date \( t^* \) is \( e^{-rt^*} \). Because the zero-coupon bond can be traded on the market, we can take \( \Pi = V(r, t) - \Delta e^{-rt^*} \) as the portfolio in order to derive the PDE for \( V(r, t) \), the price of an interest rate derivative. Derive the PDE for \( V(r, t) \) in this way.
3. Suppose that the spot interest rate $r$ satisfies

$$dr = udt + w dB$$

where $dX$ is a Wiener process.

a) Find the stochastic equation for $B(r) = e^{-rt^*}$ by using Itô’s lemma, where $t^*$ is a constant.

b) As we know, $B(r)$ is the price of a zero-coupon bond at $t = 0$ with face value one and maturity date $t^*$ if during the time period $[0, t^*]$ the interest rate is a constant. $V(B, t)$ is any derivative on the zero-coupon bond. Derive the PDE for $V(B, t)$ by using Itô’s lemma directly.

c) As we know, any derivative on $r$, $V(r, t)$, satisfies the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV + kZ = 0,$$

where $kZ$ is the coupon of the derivative. Let $V(B, t) = V(-\ln B/t^*, t)$. By using transformation, derive the PDE for $V(B, t)$ from the PDE above. This equation should be the same equation derived in b). Based on this, determine $\lambda$ and write down the PDE for $V$ in which there is no $\lambda$.

4. Suppose that $a(r, t) = \sum_{i=0}^{\infty} a_i(t) r^i$ and $b(r, t) = \sum_{i=0}^{\infty} b_i(t) r^i$ and require that the problem

$$\begin{cases}
\frac{\partial V}{\partial t} + a(r, t) \frac{\partial^2 V}{\partial r^2} + b(r, t) \frac{\partial V}{\partial r} - rV = 0, & 0 \leq t \leq T; \\
V(r, T) = 1
\end{cases}$$

has a solution in the form

$$V(r, t) = e^{A(t) - tB(t)}.$$ 

Show that in order to fulfill this requirement, between $a_i$ and $b_i$, $i = 2, 3, \cdots$, there must exist the following relations:

$$a_i B - b_i = 0, \quad i = 2, 3, \cdots.$$ 

This means that in order to choose $a(r, t)$ and $b(r, t)$ independently and for the solution to be in the form $e^{A(t) - tB(t)}$, we have to assume $a(r, t) = a_0(t) + a_1(t) r$ and $b(r, t) = b_0(t) + b_1(t) r$.

5. Suppose that $a(r, t) = a_0(t) + a_1(t) r$ and $b(r, t) = b_0(t) + b_1(t) r$. Show that the problem

$$\begin{cases}
\frac{\partial V}{\partial t} + a(r, t) \frac{\partial^2 V}{\partial r^2} + b(r, t) \frac{\partial V}{\partial r} - rV = 0, & 0 \leq t \leq T; \\
V(r, T) = 1
\end{cases}$$

has a solution in the form

$$V(r, t) = e^{A(t) - tB(t)}.$$ 

Show that in order to fulfill this requirement, between $a_i$ and $b_i$, $i = 2, 3, \cdots$, there must exist the following relations:

$$a_i B - b_i = 0, \quad i = 2, 3, \cdots.$$ 

This means that in order to choose $a(r, t)$ and $b(r, t)$ independently and for the solution to be in the form $e^{A(t) - tB(t)}$, we have to assume $a(r, t) = a_0(t) + a_1(t) r$ and $b(r, t) = b_0(t) + b_1(t) r$. 

has a solution in the form

\[ V(r, t) = e^{A(t) - rB(t)} \]

with \( A(T) = B(T) = 0 \) and determine the system of ordinary differential equations the functions \( A(t) \) and \( B(t) \) should satisfy.

6. *In the Vasicek model, the spot interest rate is assumed to satisfy

\[ dr = (\bar{\mu} - \gamma r)dt + \sqrt{-\beta}dX, \quad \beta < 0, \quad \gamma > 0, \]

where \( \bar{\mu}, \gamma, \) and \( \beta \) are constants, and \( dX \) is a Wiener process. Let the market price of risk \( \lambda(r, t) = \lambda \sqrt{-\beta} \). Then, the price \( V(r, t; T) \) of a zero-coupon bond maturing at time \( T \) with a face value \( Z \) is the solution of the problem

\[
\begin{aligned}
\frac{\partial V}{\partial t} + \frac{1}{2}(-\beta)\frac{\partial^2 V}{\partial r^2} + (\mu - \gamma r)\frac{\partial V}{\partial r} - rV &= 0, \\
-\infty < r < \infty, \quad 0 \leq t \leq T, \\
V(r, T; T) &= Z, \quad -\infty < r < \infty, \quad 0 \leq t \leq T,
\end{aligned}
\]

where

\[ \mu = \bar{\mu} + \bar{\lambda}\beta. \]

a) Show that this problem has a solution in the form

\[ V(r, t; T) = Ze^{A(t, T) - rB(t, T)} \]

and \( A \) and \( B \) are the solution of the system of ordinary differential equations

\[
\begin{aligned}
\frac{dA}{dt} &= \frac{1}{2} \beta B^2 + \mu B, \\
\frac{dB}{dt} &= \gamma B - 1
\end{aligned}
\]

with the conditions

\[ A(T, T) = 0, \]

\[ B(T, T) = 0. \]

b) Show that the solution of the above problem of ordinary differential equations is

\[
\begin{aligned}
B &= \frac{1}{\gamma} \left( 1 - e^{-\gamma(T-t)} \right), \\
A &= - \left( \frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma} \right) (T-t) + \left( \frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma} \right) B + \frac{\beta}{4\gamma} B^2.
\end{aligned}
\]
7. Show

\[
\lim_{\alpha \to 0} \left\{ \frac{\beta}{\alpha} B + \left[ \frac{\beta (\gamma - \psi)}{\alpha (\gamma + \psi)} + \frac{\gamma - \psi}{\alpha \psi} \right] \ln \frac{(\gamma - \psi)/\alpha}{B + (\gamma - \psi)/\alpha} \right. \\
- \left[ \frac{\beta (\gamma + \psi)}{\alpha (\gamma - \psi)} + \frac{\gamma + \psi}{\alpha \psi} \right] \ln \frac{(\gamma + \psi)/\alpha}{B + (\gamma + \psi)/\alpha} \right. \\
= - \left( \frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma} \right) (T - t) + \left( \frac{\beta}{2\gamma^2} + \frac{\mu}{\gamma} \right) B + \frac{\beta B^2}{4\gamma},
\]

where

\[
B(t, T) = \frac{1}{\gamma} \left( 1 - e^{-\gamma(T-t)} \right) \quad \text{and} \quad \psi = \sqrt{\gamma^2 + 2\alpha}.
\]

(The two sides are two expressions for \( A \) associated with the Vasicek model obtained by different approaches. This confirms that the two different approaches give the same answer.)

8. *In the Cox–Ingersoll–Ross model, the spot interest rate is assumed to satisfy

\[
dr = (\mu - \bar{\gamma}r)dt + \sqrt{\alpha r}dX
\]

where \( \mu, \bar{\gamma}, \) and \( \alpha \) are constants, and \( dX \) is a Wiener process. Let the market price of risk \( \lambda(r, t) \) be \( \bar{\lambda}\sqrt{\alpha r} \). Then, the price \( V(r, t; T) \) of a zero-coupon bond maturing at time \( T \) with a face value \( Z \) is the solution of the problem

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \alpha r \frac{\partial^2 V}{\partial r^2} + (\mu - \gamma r) \frac{\partial V}{\partial r} - rV &= 0, \\
0 &\leq r, \quad 0 \leq t \leq T, \\
V(r, T; T) &= Z,
\end{align*}
\]

where \( \gamma = \bar{\gamma} + \bar{\lambda} \alpha \).

a) Show that this problem has a solution in the form

\[
V(r, t; T) = Z e^{A(t, T) - rB(t, T)}
\]

and \( A \) and \( B \) are the solutions of the system of ordinary differential equations

\[
\begin{align*}
\frac{dA}{dt} &= \mu B, \\
\frac{dB}{dt} &= \frac{1}{2} \alpha B^2 + \gamma B - 1
\end{align*}
\]

with the conditions

\( A(T, T) = 0 \)

and

\( B(T, T) = 0 \)
b) Show that the solution of the above problem of ordinary differential equations is

\[ B = \frac{2(e^{\psi(T-t)} - 1)}{(\gamma + \psi)e^{\psi(T-t)} - (\gamma - \psi)}, \]

\[ A = \ln \left( \frac{2\psi e^{(\gamma + \psi)(T-t)/2}}{(\gamma + \psi)e^{\psi(T-t)} - (\gamma - \psi)} \right)^{2\mu/\alpha}, \]

where

\[ \psi = \sqrt{\gamma^2 + 2\alpha}. \]

9. Show

\[ Z \left[ B + \frac{(\gamma - \psi)}{\alpha} \right]^{\mu(\psi-\gamma)/\alpha} \left[ B + \frac{(\gamma + \psi)}{\alpha} \right]^{\mu(\gamma+\psi)/\alpha} e^{-rB} \]

\[ = Z \left[ 2\psi e^{(\gamma+\psi)(T-t)/2} \right]^{2\mu/\alpha} e^{-rB}, \]

where

\[ B = \frac{2(e^{\psi(T-t)} - 1)}{(\gamma + \psi)e^{\psi(T-t)} - (\gamma - \psi)}. \]

(The two sides are two expressions for the zero-coupon bond price associated with the Cox–Ingersoll–Ross model obtained by different approaches. This confirms that the two different approaches give the same answer.)

10. *Describe a way to determine the market price of risk for the spot interest rate.

11. *Suppose that any European-style interest rate derivative with a continuous coupon satisfies the equation:

\[ \frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV + k = 0, \quad r_1 \leq r \leq r_u, \quad t \leq T, \]

where \( k \) is the coupon rate corresponding to the derivative, the coefficients \( u \) and \( w \) satisfy the reversion conditions on the boundaries \( r = r_1 \), \( r = r_u \), and \( \lambda \) is a given bounded function. Describe how to evaluate the price of a European call option on a bond with coupon by using this equation.

12. Show that the value of an \( N \)-year swap with swap rate \( r_s \) is

\[ V_s(T; r_s) = Q \left[ 1 - Z(T; T + N) - \frac{r_s}{2} \sum_{k=1}^{2N} Z(T; T + k/2) \right], \]

where \( T \) is the time the swap initiates.

13. Show that the solution of the problem

\[ \begin{cases} \frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV = 0, \\ r_1 \leq r \leq r_u, \quad t \leq T, \\ V(r, T; T) = 1, \quad r_1 \leq r \leq r_u \end{cases} \]
is the same as that of the problem
\[
\begin{cases}
\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV + \delta(t - T) = 0, \\
V(r, T^+; T) = 0,
\end{cases}
\]
for any \( t < T \).

14. Let \( V_{s1k}(r; T) \) denote the price of a \((k/2)\)-year zero-coupon bond, \( k = 1, 2, \ldots, 2N \), and we want to get \( \sum_{k=1}^{2N} V_{s1k}(r; T) \). Consider the following procedures. The first one is to solve the following problems
\[
\begin{cases}
\frac{\partial V_{s1k}}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_{s1k}}{\partial r^2} + (u - \lambda w) \frac{\partial V_{s1k}}{\partial r} - rV_{s1k} = 0, \\
V_{s1k}(r; T + k/2) = 1,
\end{cases}
\]
k = 1, 2, \ldots, 2N, and then obtain \( \sum_{k=1}^{2N} V_{s1k}(r; T) \) by adding \( V_{s1k}(r; T) \), \( k = 1, 2, \ldots, 2N \), together. The second one is to solve the problem:
\[
\begin{cases}
\frac{\partial V_s}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_s}{\partial r^2} + (u - \lambda w) \frac{\partial V_s}{\partial r} - rV_s + \sum_{k=1}^{2N} \delta(t - T - k/2) = 0, \\
V_s(r; T + N) = 0,
\end{cases}
\]
\( r_l \leq r \leq r_u \), \( T \leq t \leq T + N \).

a) Show \( V_s(r; T) = \sum_{k=1}^{2N} V_{s1k}(r; T) \) holds;

b) In order to get \( \sum_{k=1}^{2N} V_{s1k}(r; T) \), which procedure is better and why?

15. Suppose that any European-style interest rate derivative satisfies the equation:
\[
\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - rV + f(t) = 0, \quad r_l \leq r \leq r_u,
\]
where all the coefficients in the equation are known. The value of \( N \)-year swap at time \( T \) is given by
\[
V_s(T; r_s) = Q \left[ 1 - Z(T; T + N) - \frac{r_s}{2} \sum_{k=1}^{2N} Z(T; T + k/2) \right],
\]
17. Consider an $N$-year swap rate and $Z(T; T+k/2)$ is the value of zero-coupon bond with maturity $k/2$ at time $T$. Describe how to find the price of a swaption with exercise swap rate $r_{sc}$ and maturity $T$, including to find $Z(T; T + N)$ and $\sum_{k=1}^{2N} Z(T; T + k/2)$, by solving this equation from $T + N$ to $T$ twice and from $T$ to 0 once.

16. Suppose that the solution of

$$\frac{\partial V}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V}{\partial r^2} + (u - \lambda w) \frac{\partial V}{\partial r} - r V = 0, \quad r_l \leq r \leq r_u, \quad t \leq t_k,$$

is $V(r, t; t_k)$ and that $V(r^*, t^*; t_k) = Z(t^*; t_k)$. Also assume that $V_s(r; t; r_s)$ is the solution of

$$\begin{align*}
\frac{\partial V_s}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_s}{\partial r^2} + & (u - \lambda w) \frac{\partial V_s}{\partial r} - r V_s - \sum_{k=k^*+1}^{2N} \frac{\partial V_s}{\partial r} - \frac{1}{2} \delta(t - t_k) \\
+ & Q \left[ 1 + \frac{f(t_{k^*}, t_{k^*+1}) \delta(t - t_{k^*+1})}{2} \right], \quad r_l \leq r \leq r_u, \quad t^* \leq t \leq T + N,
\end{align*}$$

$$V_s(r, T + N; r_s) = -Q(1 + r_s/2), \quad r_l \leq r \leq r_u.$$

Here, $V_s(r; t; r_s)$ actually is the value of a swap, $Q$ and $r_s$ are the notional principal and the swap rate, respectively, $t^*$, $T$, and $N$ denote the time today, the time the swap is initiated, and the duration of the swap with the relation $T \leq t^* < T + N$. $k^*$ is the integer part of $(t^* - T)/2$, and $t_k = T + k/2$, $k = k^* + 1, k^* + 2, \ldots, 2N$. $f(t_{k^*}, t_{k^*+1})$ is the six-month LIBOR for the period $[t_{k^*}, t_{k^*+1}]$ determined at time $t_{k^*}$. Then

$$V_s(r^*, t^*; r_s) = Q Z(t^*; t_{k^*+1}) \left[ 1 + \frac{1}{2} f(t_{k^*}, t_{k^*+1}) \right]$$

$$- \frac{1}{2} \sum_{k=k^*+1}^{2N} \frac{r_s}{Z(t^*; t_k) + Z(t^*; T + N)}.$$

17. Consider an $N$-year floor with a floor rate $r_f$. Suppose that the money will be paid quarterly at time $t_k = t^* + k/4$, $k = 2, 3, \ldots, 4N$, and the floating rate is the three-month LIBOR. Suppose that $V_{bk}(r; t)$ is the solution of the problem

$$\begin{align*}
\frac{\partial V_{bk}}{\partial t} + \frac{1}{2} w^2 \frac{\partial^2 V_{bk}}{\partial r^2} + (u - \lambda w) \frac{\partial V_{bk}}{\partial r} - r V_{bk} &= 0, \quad r_l \leq r \leq r_u, \quad t_{k-1} \leq t \leq t_k, \\
V_{bk}(r, t_k) &= Q \left( 1 + \frac{r_f}{4} \right), \quad r_l \leq r \leq r_u,
\end{align*}$$

where $Q$ is the notional principal, $r_s$ is the $N$-year swap rate and $Z(T; T+k/2)$ is the value of zero-coupon bond with maturity $k/2$ at time $T$.
where \( k = 2, 3, \ldots, 4N \) and \( V_f(r, t) \) is the solution of the problem

\[
\begin{align*}
\frac{\partial V_f}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 V_f}{\partial r^2} + (\mu - \lambda \sigma) \frac{\partial V_f}{\partial r} - r V_f \\
+ \sum_{k=2}^{4N} \max(V_{bk}(r, t_{k-1}) - Q, 0) \delta(t - t_{k-1}) = 0,
\end{align*}
\]

\( r_l \leq r \leq r_u, \quad t_4 N \leq t \leq t_{4N-1}, \)

\( V_f(r, t_{4N-1}) = 0, \quad r_l \leq r \leq r_u. \)

Show that the premium of the floor should be

\( V_f(r^*, t^*), \)

where \( r^* \) is the spot interest rate at time \( t^* \).

18. a) \( S \) is a random vector and its covariance matrix is \( B \). Let \( \bar{S} = A S \), \( A \) being a constant matrix, and its covariance matrix be \( C \). Find the relation among \( A, B, \) and \( C \).

b) How do we choose \( A \) so that \( C \) will be a diagonal matrix?

c) Suppose that \( S_1, S_2, \ldots, S_n \) are variables and \( \bar{S}_{K+1}, \bar{S}_{K+2}, \ldots, \bar{S}_N \) are fixed numbers. Find the dependence of \( S_{K+1}, S_{K+2}, \ldots, S_N \) on \( S_1, S_2, \ldots, S_K \).

19. a) Suppose that there is a domain \( \Omega \) on the \((Z_1, Z_2)\)-plane, the boundary of \( \Omega \) is \( \Gamma \), and \((n_1, n_2)^T\) is the outer normal vector of the boundary \( \Gamma \). Assume that \( Z_1 \) and \( Z_2 \) are two stochastic processes and satisfy the system of stochastic differential equations:

\[
dZ_i = \mu_i(Z_1, Z_2, t) dt + \sigma_i(Z_1, Z_2, t) dX_i \quad \text{with} \quad \sigma_i \geq 0, \quad i = 1, 2,
\]

where \( dX_1, i = 1, 2 \), are the Wiener processes and \( E[dX_1 dX_2] = \rho_{12} dt \) with \( \rho_{12} \in [-1, 1] \). Suppose that at \( t = 0 \), \( (Z_1, Z_2) \in \Omega \). Show that in order to guarantee \( (Z_1, Z_2) \in \Omega \) for any time \( t \in [0, T] \), we need to require, for any \( t \in [0, T] \) and for any point on \( \Gamma \), the following condition to be held:

i. if \( n_1 \neq 0 \) and \( n_2 = 0 \), then

\[
\begin{align*}
n_1 \mu_1 &\leq 0, \\
\sigma_1 &\leq 0;
\end{align*}
\]

ii. if \( n_1 = 0 \) and \( n_2 \neq 0 \), then

\[
\begin{align*}
n_2 \mu_2 &\leq 0, \\
\sigma_2 &\leq 0;
\end{align*}
\]
iii. if \( n_1 \neq 0 \) and \( n_2 \neq 0 \), then
\[
\begin{align*}
& \begin{cases}
  n_1 \mu_1 + n_2 \mu_2 \leq 0, \\
  n_1 \sigma_1 - \text{sign}(n_1 n_2) n_2 \sigma_2 = 0, \quad \text{and} \quad \rho_{12} = -\text{sign}(n_1 n_2),
\end{cases}
\end{align*}
\]
where
\[
\text{sign}(n_1 n_2) = \begin{cases}
  1, & \text{if } n_1 n_2 > 0, \\
  -1, & \text{if } n_1 n_2 < 0.
\end{cases}
\]

If a point is a corner point, then there are two normals and we need to require this condition to be held for the two outer normal vectors.

b) Suppose that the domain \( \Omega \) is \( Z_{1t} \leq Z_1 \leq 1 \) and \( Z_{2i} \leq Z_2 \leq Z_1 \), where \( Z_{1t} \) and \( Z_{2i} \) are constants, and \( Z_{1t} \geq Z_{2i} \). Find the concrete condition corresponding to the condition given in a) on each segment of the boundary.

20. Assume that \( Z_1, Z_2, Z_3 \) are random variables and satisfy the system of stochastic differential equations:
\[
dZ_i = \mu_i(Z_1, Z_2, Z_3, t) \, dt + \sigma_i(Z_1, Z_2, Z_3, t) \, dX_i, \quad i = 1, 2, 3,
\]
where \( dX_i \) are the Wiener processes and \( E[dX_i dX_j] = \rho_{i,j} dt \) with \(-1 \leq \rho_{i,j} \leq 1\). In order to guarantee that if a point is in a domain \( \Omega \) at time \( t^* \), then the point is still in the domain \( \Omega \) at \( t = t^* + dt \) for a positive \( dt \), it is necessary to require that the condition
\[
n_1 dZ_1 + n_2 dZ_2 + n_3 dZ_3 \leq 0
\]
holds at any point on the boundary of the domain \( \Omega \), where \( n_1, n_2, \) and \( n_3 \) are the three components of the outer normal vector of the boundary at the point. Suppose that the domain \( \Omega \) is \( \{Z_{1t} \leq Z_1 \leq 1, \ Z_{2i} \leq Z_2 \leq Z_{1t}, \ Z_{3i} \leq Z_3 \leq Z_{2i}\} \). Show that on the surfaces \( Z_1 = 1, Z_2 = Z_{2i}, \) and \( Z_3 = Z_{2i} \), the condition is equivalent to \( \{\mu_1 \leq 0, \sigma_1 = 0\}, \{\mu_2 \geq 0, \sigma_2 = 0\}, \) and \( \{-\rho_{23} + \rho_{3i} \leq 0, \sigma_2 = \sigma_3, \rho_{23} = 1\} \), respectively.

21. Suppose that \( \sigma_1(Z_1, Z_2, Z_3, t), \sigma_2(Z_1, Z_2, Z_3, t), \) and \( \sigma_3(Z_1, Z_2, Z_3, t) \) are defined on \( \Omega : \{Z_{1t} \leq Z_1 \leq 1, \ Z_{2i} \leq Z_2 \leq Z_{1t}, \ Z_{3i} \leq Z_3 \leq Z_{2i}\} \). Assume that
\[
i) \ \sigma_1(Z_{1t}, Z_2, Z_3, t) = 0 \text{ on surface I: } \{Z_1 = Z_{1t}, Z_{2i} \leq Z_2 \leq Z_{1t}, \ Z_{3i} \leq Z_3 \leq Z_{2i}\};
\]
\[
ii) \ \sigma_1(1, Z_2, Z_3, t) = 0 \text{ on surface II: } \{Z_1 = 1, Z_{2i} \leq Z_2 \leq Z_{1t}, \ Z_{3i} \leq Z_3 \leq Z_{2i}\};
\]
\[
iii) \ \sigma_2(Z_1, Z_{2i}, Z_3, t) = 0 \text{ on surface III: } \{Z_{1t} \leq Z_1 \leq 1, \ Z_2 = Z_{2i}, \ Z_{3i} \leq Z_3 \leq Z_{2i}\};
\]
\[
iv) \ \sigma_3(Z_1, 1, Z_3, t) = \sigma_3(Z_1, Z_1, Z_3, t), \ \rho_{12}(Z_1, Z_1, Z_3, t) = 1 \text{ on surface IV: } \{Z_{1t} \leq Z_1 \leq 1, \ Z_2 = Z_1, \ Z_{3i} \leq Z_3 \leq Z_{2i}\};
\]
\[
v) \ \sigma_3(Z_1, Z_2, Z_{3i}, t) = 0 \text{ on surface V: } \{Z_{1t} \leq Z_1 \leq 1, \ Z_{2i} \leq Z_2 \leq Z_{1t}, \ Z_3 = Z_{3i}\};
\]
vi) \( \sigma_2(Z_1, Z_2, Z_2, t) = \sigma_3(Z_1, Z_2, Z_2, t), \quad \rho_{2,3}(Z_1, Z_2, Z_2, t) = 1 \) on surface VI: \( n \{ Z_{1,l} \leq Z_1 \leq 1, Z_{2,l} \leq Z_2 \leq Z_1, Z_3 = Z_2 \} \).

Define
\[
\begin{aligned}
\xi_1 &= \frac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\
\xi_2 &= \frac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}}, \\
\xi_3 &= \frac{Z_3 - Z_{3,l}}{Z_2 - Z_{3,l}},
\end{aligned}
\]

and
\[
\begin{aligned}
\tilde{\sigma}_1^2(\xi_1, \xi_2, \xi_3, t) &= \frac{\sigma_1^2}{(1 - Z_{1,l})^2}, \\
\tilde{\sigma}_2^2(\xi_1, \xi_2, \xi_3, t) &= \frac{\sigma_2^2 \xi_2^2 - 2\sigma_1 \sigma_2 \xi_2 \rho_{1,2} + \sigma_2^2}{(Z_1 - Z_{2,l})^2}, \\
\tilde{\sigma}_3^2(\xi_1, \xi_2, \xi_3, t) &= \frac{\sigma_3^2 \xi_3^2 - 2\sigma_2 \sigma_3 \xi_3 \rho_{2,3} + \sigma_3^2}{(Z_2 - Z_{3,l})^2}.
\end{aligned}
\]

Show that the assumption on \( \sigma_1, \sigma_2, \) and \( \sigma_3 \) is equivalent to
\[
\begin{aligned}
\tilde{\sigma}_1(0, \xi_2, \xi_3, t) &= \tilde{\sigma}_1(1, \xi_2, \xi_3, t) = 0, \quad 0 \leq \xi_2 \leq 1, 0 \leq \xi_3 \leq 1, \\
\tilde{\sigma}_2(\xi_1, 0, \xi_3, t) &= \tilde{\sigma}_2(\xi_1, 1, \xi_3, t) = 0, \quad 0 \leq \xi_1 \leq 1, 0 \leq \xi_3 \leq 1, \\
\tilde{\sigma}_3(\xi_1, \xi_2, 0, t) &= \tilde{\sigma}_3(\xi_1, \xi_2, 1, t) = 0, \quad 0 \leq \xi_1 \leq 1, 0 \leq \xi_2 \leq 1.
\end{aligned}
\]

22. a) Show that under the transformation
\[
\begin{aligned}
\xi_1 &= \frac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\
\xi_2 &= \frac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}},
\end{aligned}
\]
the partial differential equation
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial Z_i \partial Z_j} + r \sum_{i=1}^{2} Z_i \frac{\partial V}{\partial Z_i} - rV = 0
\]
becomes
\[
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \tilde{\sigma}_i \tilde{\sigma}_j \tilde{\rho}_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + r \sum_{i=1}^{2} \tilde{\theta}_i \frac{\partial V}{\partial \xi_i} - rV = 0,
\]
where
23. a) * Show that under the transformation

\[
\begin{align*}
    b_1 &= \frac{rZ_1}{1 - Z_{1,t}}, \\
    b_2 &= \frac{r(Z_2 - Z_1 \xi_2)}{Z_1 - Z_{2,t}} + \frac{\sigma_1 (\sigma_1 \xi_2 - \sigma_2 \rho_{1,2})}{(Z_1 - Z_{2,t})^2},
\end{align*}
\]

and \( \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\rho}_{1,2} \) are determined by

\[
\begin{align*}
    \frac{1}{2} \tilde{\sigma}_1^2 &= \frac{1}{2} \sigma_1^2 (1 - Z_{1,t})^2, \\
    \frac{1}{2} \tilde{\sigma}_2^2 &= \frac{1}{2} \left( \sigma_1^2 \xi_2^2 - 2\sigma_1 \sigma_2 \xi_2 \rho_{1,2} + \sigma_2^2 \right) (Z_1 - Z_{2,t})^2, \\
    \tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\rho}_{1,2} &= \frac{\sigma_1 (\sigma_2 \rho_{1,2} - \sigma_1 \xi_2)}{(1 - Z_{1,t})(Z_1 - Z_{2,t})}.
\end{align*}
\]

b) Show further that the expression of \( b_2 \) can be rewritten as

\[
b_2 = \frac{r(Z_2 - Z_1 \xi_2)}{Z_1 - Z_{2,t}} - \frac{\tilde{\sigma}_1 \tilde{\sigma}_2 \tilde{\rho}_{1,2} (1 - Z_{1,t})}{Z_1 - Z_{2,t}}.
\]

c) \( \tilde{\sigma}_i \) and \( b_i \) given above are functions of \( \xi_1, \xi_2, t \) and let \( \tilde{\sigma}_i(\xi_1, \xi_2, t) \) and \( b_i(\xi_1, \xi_2, t) \) denote these functions, \( i = 1 \) and \( 2 \). Show that if

\[
\begin{align*}
    \tilde{\sigma}_1(0, \xi_2, t) &= \tilde{\sigma}_1(1, \xi_2, t) = 0, \quad 0 \leq \xi_2 \leq 1, \\
    \tilde{\sigma}_2(\xi_1, 0, t) &= \tilde{\sigma}_2(\xi_1, 1, t) = 0, \quad 0 \leq \xi_1 \leq 1,
\end{align*}
\]

then

\[
\begin{align*}
    b_1(0, \xi_2, t) &\geq 0, \quad b_1(1, \xi_2, t) = 0, \quad 0 \leq \xi_2 \leq 1, \\
    b_2(\xi_1, 0, t) &\geq 0, \quad b_2(\xi_1, 1, t) = 0, \quad 0 \leq \xi_1 \leq 1.
\end{align*}
\]

(Hint: \( r(\xi_1, \xi_2, t)|_{\xi_1 = 1} = 0 \). This can be explained as follows, \( \xi_1 = 1 \) means \( Z_1 = 1 \), thus the zero-coupon bond curve must be flat near \( T = 0 \) and its derivative with respect to \( T \) at \( T = 0, r(\xi_1, \xi_2, t)|_{\xi_1 = 1} \) equals zero. When \( \tilde{\sigma}_i, b_i, i = 1, 2, \) satisfy these conditions here, it can be proved that the final value problem

\[
\begin{align*}
    \frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \tilde{\sigma}_i \tilde{\sigma}_j \tilde{\rho}_{ij} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{2} b_i \frac{\partial V}{\partial \xi_i} - rV &= 0 \\
    V(\xi_1, \xi_2, T) = V_r(\xi_1, \xi_2) \quad &\text{on } [0, 1] \times [0, 1] \times [0, T],
\end{align*}
\]

has a unique solution.)

23. a) * Show that under the transformation
24. Assume that $Z_1, Z_2, Z_3$ are random variables and satisfy the system of stochastic differential equations:

$$dZ_i = \mu_i (Z_1, Z_2, Z_3, t) \, dt + \sigma_i (Z_1, Z_2, Z_3, t) \, dX_i, \quad i = 1, 2, 3,$$

the partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{i,j} \rho_{i,j} \frac{\partial^2 V}{\partial Z_i \partial Z_j} + \sum_{i=1}^{3} Z_i \frac{\partial V}{\partial Z_i} - rV = 0,$$

becomes

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \tilde{\sigma}_{i,j} \tilde{\rho}_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{3} b_i \frac{\partial V}{\partial \xi_i} - rV = 0,$$

and find the expressions of $\tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \tilde{\rho}_{1,2}, \tilde{\rho}_{1,3}, \tilde{\rho}_{2,3}, b_1, b_2,$ and $b_3$. 

b) Show

$$\begin{cases}
\tilde{\sigma}_1(0, \xi_2, \xi_3, t) = \tilde{\sigma}_1(1, \xi_2, \xi_3, t) = 0, & 0 \leq \xi_2 \leq 1, \ 0 \leq \xi_3 \leq 1, \\
\tilde{\sigma}_2(\xi_1, 0, \xi_3, t) = \tilde{\sigma}_2(\xi_1, 1, \xi_3, t) = 0, & 0 \leq \xi_1 \leq 1, \ 0 \leq \xi_3 \leq 1, \\
\tilde{\sigma}_3(\xi_1, \xi_2, 0, t) = \tilde{\sigma}_3(\xi_1, \xi_2, 1, t) = 0, & 0 \leq \xi_1 \leq 1, \ 0 \leq \xi_2 \leq 1,
\end{cases}$$

and

$$\begin{cases}
b_1(0, \xi_2, \xi_3, t) \geq 0, & b_1(1, \xi_2, \xi_3, t) = 0, \ 0 \leq \xi_2 \leq 1, \ 0 \leq \xi_3 \leq 1, \\
b_2(\xi_1, 0, \xi_3, t) \geq 0, & b_2(\xi_1, 1, \xi_3, t) = 0, \ 0 \leq \xi_1 \leq 1, \ 0 \leq \xi_3 \leq 1, \\
b_3(\xi_1, \xi_2, 0, t) \geq 0, & b_3(\xi_1, \xi_2, 1, t) = 0, \ 0 \leq \xi_1 \leq 1, \ 0 \leq \xi_2 \leq 1.
\end{cases}$$

(When $\tilde{\sigma}_i, b_i, i = 1, 2, 3,$ satisfy these conditions here, it can be proved that the final value problem

$$\begin{cases}
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \tilde{\sigma}_{i,j} \tilde{\rho}_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{3} b_i \frac{\partial V}{\partial \xi_i} - rV = 0 \\
\text{on } [0, 1] \times [0, 1] \times [0, 1] \times [0, T], \\
V(\xi_1, \xi_2, \xi_3, T) = V_T(\xi_1, \xi_2, \xi_3) \quad \text{on } [0, 1] \times [0, 1] \times [0, 1],
\end{cases}$$

has a unique solution.)
where \(dX_i\) are the Wiener processes and \(E[dX_i dX_j] = \rho_{ij} dt\) with \(-1 \leq \rho_{ij} \leq 1\), and that \(\xi_1, \xi_2\) and \(\xi_3\) are governed by

\[
d\xi_i = \tilde{\mu}_i(\xi_1, \xi_2, \xi_3, t) dt + \tilde{\sigma}_i(\xi_1, \xi_2, \xi_3, t) d\tilde{X}_i, \quad i = 1, 2, 3,
\]

where \(d\tilde{X}_i\) are the Wiener processes and \(E[d\tilde{X}_i d\tilde{X}_j] = \tilde{\rho}_{ij} dt\) with \(-1 \leq \tilde{\rho}_{ij} \leq 1\). Furthermore, we suppose that \(\xi_1, \xi_2\) and \(\xi_3\) are defined by

\[
\begin{align*}
\xi_1 &= \frac{Z_1 - Z_{1,l}}{1 - Z_{1,l}}, \\
\xi_2 &= \frac{Z_2 - Z_{2,l}}{Z_1 - Z_{2,l}}, \\
\xi_3 &= \frac{Z_3 - Z_{3,l}}{Z_2 - Z_{3,l}},
\end{align*}
\]

where \(Z_{1,l}, Z_{2,l},\) and \(Z_{3,l}\) are constants. Find the expressions of \(\tilde{\sigma}_1, \tilde{\sigma}_2,\) \(\tilde{\sigma}_3, \tilde{\rho}_{12}, \tilde{\rho}_{13}, \tilde{\rho}_{23}\) as functions of \(\sigma_1, \sigma_2, \sigma_3, \rho_{1,2}, \rho_{1,3}, \rho_{2,3}, Z_1, Z_2,\) and \(Z_3\) by using Itô’s lemma.

25. Suppose that \(\xi_1, \xi_2\) and \(\xi_3\) satisfy the system of stochastic differential equations:

\[
d\xi_i = \mu_i(\xi_1, \xi_2, \xi_3, t) dt + \sigma_i(\xi_1, \xi_2, \xi_3, t) dX_i, \quad i = 1, 2, 3,
\]

where \(dX_i\) are the Wiener processes and \(E[dX_i dX_j] = \rho_{ij} dt\) with \(-1 \leq \rho_{ij} \leq 1\). Define

\[
\begin{align*}
Z_1(\xi_1) &= Z_{1,l} + \xi_1 (1 - Z_{1,l}), \\
Z_2(\xi_1, \xi_2) &= Z_{2,l} + \xi_2 [Z_1(\xi_1) - Z_{2,l}] \\
&= Z_{2,l} + \xi_2 [Z_{1,l} + \xi_1 (1 - Z_{1,l}) - Z_{2,l}], \\
Z_3(\xi_1, \xi_2, \xi_3) &= Z_{3,l} + \xi_3 \{Z_2(\xi_1, \xi_2) - Z_{3,l}\} \\
&= Z_{3,l} + \xi_3 \{Z_{2,l} + \xi_2 [Z_{1,l} + \xi_1 (1 - Z_{1,l}) - Z_{2,l}] - Z_{3,l}\}.
\end{align*}
\]

Assume that \(Z_1(\xi_1), Z_2(\xi_1, \xi_2),\) and \(Z_3(\xi_1, \xi_2, \xi_3)\) represent prices of three securities. Let \(V(\xi_1, \xi_2, \xi_3, t)\) be the value of a derivative security. Setting a portfolio \(H = V - \Delta_1 Z_1(\xi_1) - \Delta_2 Z_2(\xi_1, \xi_2) - \Delta_3 Z_3(\xi_1, \xi_2, \xi_3)\) and using Itô’s lemma, show that \(V(\xi_1, \xi_2, \xi_3, t)\) satisfies the following PDE:

\[
\begin{align*}
\frac{\partial V}{\partial t} + \frac{1}{2} \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_i \sigma_j \rho_{i,j} \frac{\partial^2 V}{\partial \xi_i \partial \xi_j} + \frac{r Z_1}{1 - Z_{1,l}} \frac{\partial V}{\partial \xi_1} \\
+ \frac{r (Z_2 - Z_{1,l})}{Z_1 - Z_{2,l}} \frac{\partial V}{\partial \xi_2} - \frac{\sigma_1 \sigma_2 \rho_{1,2} (1 - Z_{1,l})}{Z_1 - Z_{2,l}} \frac{\partial V}{\partial \xi_2} \\
+ \frac{r (Z_3 - Z_{2,l})}{Z_2 - Z_{3,l}} \frac{\partial V}{\partial \xi_3} - \frac{\sigma_1 \sigma_3 \rho_{1,3} (1 - Z_{1,l}) + \sigma_2 \sigma_3 \rho_{2,3} (Z_1 - Z_{2,l})}{Z_2 - Z_{3,l}} \frac{\partial V}{\partial \xi_3} \\
- r V &= 0.
\end{align*}
\]
26. Consider a two-factor convertible bond paying coupons with a rate $k$. For such a convertible bond, derive directly the partial differential equation that contains only the unknown market price of risk for the spot interest rate. “Directly” means “without using the general PDE for derivatives.” (Hint: Take a portfolio in the form

$$II = \Delta_1 V_1 + \Delta_2 V_2 + S,$$

where $V_1$ and $V_2$ are two different convertible bonds.)

27. *Formulate the two-factor convertible coupon-paying bond problem as a linear complementarity problem.

   a) Show that if $D_0 \leq 0$, then there is no free boundary; if $D_0 > 0$, then there exists at least one free boundary.
   b) *Formulate a two-factor convertible coupon-paying bond problem as a free-boundary problem if $D_0 > 0$. (Suppose it is known that on the free boundary, the price of the convertible bond and its derivative are continuous, and assume that there exists only one free boundary.)

29. Consider the problem

$$\begin{cases} 
\frac{\partial B_c}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 B_c}{\partial S^2} + (r - D_0) S \frac{\partial B_c}{\partial S} - r B_c + kZ = 0, \\
B_c(S,T) = \max(Z,nS), \quad 0 \leq S, \quad 0 \leq t \leq T,
\end{cases}$$

where $\sigma, r, D_0, k, Z,$ and $n$ are constants. Show that if $D_0 \leq 0$, then

$$B_c(S,t) \geq nS \quad \text{for} \quad 0 \leq t \leq T.$$

(Hint: Define $\overline{B}_c(S,t) = B_c(S,t) - b_0(t)$, where $b_0(t)$ is the solution of the problem:

$$\begin{cases} 
\frac{db_0}{dt} - rb_0 + kZ = 0, \quad 0 \leq t \leq T, \\
b_0(T) = 0.
\end{cases}$$

Show $\overline{B}_c(S,t) \geq nS$ and $b_0(t) \geq 0$, and then show $B_c(S,t) \geq nS$.)

(Remark: If the solution of this problem fulfills the constraint condition

$$B_c(S,t) \geq nS \quad \text{for} \quad 0 \leq t \leq T,$$

then the solution of the problem above represents the price of a one-factor convertible bond. In this case, the solution of a one-factor convertible bond does not involve any free boundary. Therefore, no free boundary will be encountered when one prices a one-factor convertible bond with $D_0 \leq 0$.)
30. Consider the problem

\[
\begin{align*}
\frac{\partial B_c}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 B_c}{\partial S^2} + (r - D_0)S \frac{\partial B_c}{\partial S} - rB_c + kZ &= 0, \\
0 &\leq S, \ 0 \leq t \leq T,
\end{align*}
\]

\[
B_c(S, T) = \max(Z, nS) = n \max(S - Z/n, 0) + Z, \quad 0 \leq S,
\]

where \(\sigma, r, D_0, k, Z,\) and \(n\) are constants. Show that its solution is

\[
c(S, t; Z/n) + Ze^{-r(T-t)} \left[1 + k \left(e^{r(T-t)} - 1\right) / r\right],
\]

where \(c(S, t; Z/n)\) is the price of a European call option with an exercise price \(E = Z/n\). This means that the problem can be understood as a problem to determine the value of an investment consisting of \(n\) units of European call options with \(E = Z/n\) and a bond with face value \(Z\) and coupon rate \(k\) (see the result of Problem 1 part b)). According to the result of Problem 29, if \(D_0 \leq 0\), then it is the price of a convertible bond. Therefore when \(D_0 \leq 0\), the value of a one-factor convertible bond is equal to the price of \(n\) units of European call options with \(E = Z/n\) plus the price of a bond with face value \(Z\) and coupon rate \(k\).