Coordinate Systems

A review of the physical basis that governs mathematical representations

Location is relative

- *The position of a thing is stated relative to another thing (real or virtual)*
- **Reference object** must be defined
- **Distance** must be known
- **Direction** must be known

→ Vector

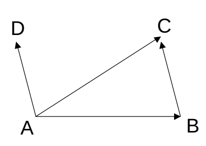
Vectors

- **Physical** - magnitude **and** direction - magnitude may be velocity, force, etc.
- **Spatial** - magnitude is a distance
- **Displacement -** spatial translation of single object. The displacement of an object can be given without respect to another object.
- ***Position* -** spatial relation (location) of an object with <u>respect to another</u> (reference object + spatial vector)
- **Translation -** Relates the changing position of an object. May be a displacement of the object or its reference.
- **Mathematical Representation-** Varied uses

Which of the following can be described with vectors??

- Temperature
- Electric Field
- Liquid Flow Rate
- Voltage
- Mass
- Position

Spatial Vector Addition



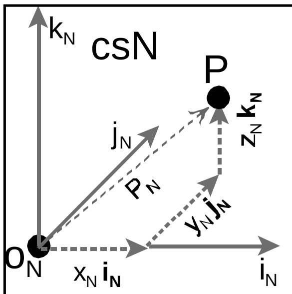
$\vec{AB} + \vec{BC} = \vec{AC}$, $\vec{AC} - \vec{BC} = \vec{AB}$

if $\vec{AD} = \vec{BC}$, then $\vec{AB} + \vec{AD} = \vec{AC}$

Spatial Reference Frame (Basis)

representation of length and direction

- Reference Directions
- Defined unit length (distance)
- Origin (reference point)
- Axes
  - through origin in reference directions
  - positive distinguished from negative
- **Spatial (Real Physical) Basis Vectors**
  - Unit displacement along axes
  - Linear sum of basis vectors describes position/direction



i_N is axis
$\mathbf{i_N}$ is basis vector
x_N is a scalar multiple

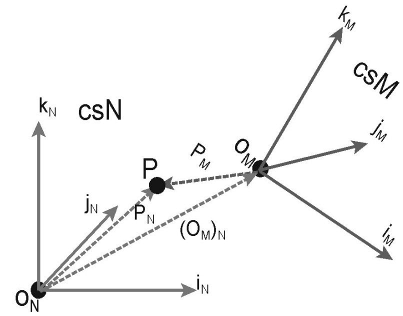
$$\vec{O_NP} = \mathbf{P_N} = x_N\mathbf{i_N} + y_N\mathbf{j_N} + z_N\mathbf{k_N}$$

$$= [\mathbf{i_N}\ \ \mathbf{j_N}\ \ \mathbf{k_N}]\begin{bmatrix} x_N \\ y_N \\ z_N \end{bmatrix}$$

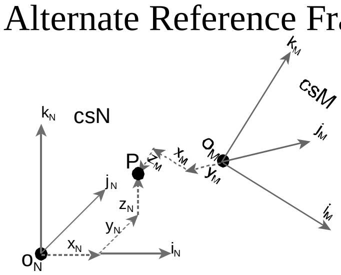
Coordinate representation of Position

$\mathbf{P_N}=[P]_N = \vec{P} = \vec{OP}$
$=(x_N, y_N, z_N)$
$=(x, y, z)_N$

Position Reference Relation Between Coordinate Frames

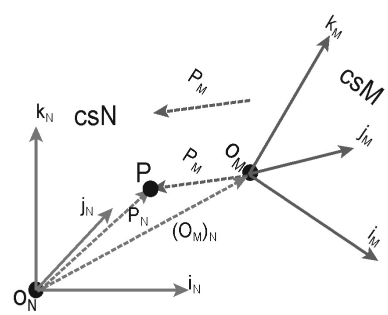


Alternate Reference Frames



- Position Coordinates
- $P_N=(x_N, y_N, z_N)= (x, y, z)_N$
- $P_M=(x_M, y_M, z_M)= (x, y, z)_M$

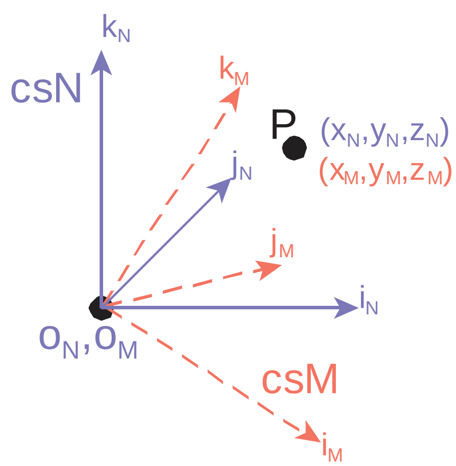
Vector Description Relation Between Coordinate Frames



So

- Position coordinates are dependent on the reference frame chosen
- When making measurements, the spatial reference frame must be understood or specified

Switching vector descriptions between non-aligned coordinate frames



$$(\vec{OP}) = [\mathbf{i}_N \;\; \mathbf{j}_N \;\; \mathbf{k}_N][\mathbf{P}]_N = [\mathbf{i}_M \;\; \mathbf{j}_M \;\; \mathbf{k}_M][\mathbf{P}]_M$$

$$= [\mathbf{i}_N \;\; \mathbf{j}_N \;\; \mathbf{k}_N]\begin{bmatrix} x_N \\ y_N \\ z_N \end{bmatrix}$$

and since $\begin{vmatrix} \mathbf{i}_N = a_1\mathbf{i}_M + a_2\mathbf{j}_M + a_3\mathbf{k}_M \\ \mathbf{j}_N = b_1\mathbf{i}_M + b_2\mathbf{j}_M + b_3\mathbf{k}_M \\ \mathbf{k}_N = c_1\mathbf{i}_M + c_2\mathbf{j}_M + c_3\mathbf{k}_M \end{vmatrix}$

it follows that

$$(\vec{OP}) = [\mathbf{i}_M \;\; \mathbf{j}_M \;\; \mathbf{k}_M]\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x_N \\ y_N \\ z_N \end{bmatrix}$$

$$= [\mathbf{i}_M \;\; \mathbf{j}_M \;\; \mathbf{k}_M][\mathbf{P}]_M$$

so

$$[\mathbf{P}]_M = \begin{bmatrix} [\mathbf{i}_N]_M & [\mathbf{j}_N]_M & [\mathbf{k}_N]_M \end{bmatrix}[\mathbf{P}]_N$$

Detail of previous transformation between rotated coordinate frames with common origin

$$(\vec{OP}) = [i_N \;\; j_N \;\; k_N][P]_N = [i_M \;\; j_M \;\; k_M][P]_M$$

$$= [i_N \;\; j_N \;\; k_N]\begin{bmatrix} x_N \\ y_N \\ z_N \end{bmatrix}, \quad \text{but} \begin{vmatrix} i_N = a_1 i_M + a_2 j_M + a_3 k_M \\ j_N = b_1 i_M + b_2 j_M + b_3 k_M \\ k_N = c_1 i_M + c_2 j_M + c_3 k_M \end{vmatrix}$$

$$= x_N(a_1 i_M + a_2 j_M + a_3 k_M) + y_N(b_1 i_M + b_2 j_M + b_3 k_M) + z_N(c_1 i_M + c_2 j_M + c_3 k_M)$$

$$= i_M(x_N a_1 + y_N b_1 + x_N c_1) + j_M(x_N a_2 + y_N b_2 + z_N c_2) + k_M(x_N a_3 + y_N b_3 + z_N c_3)$$

$$= [i_M \;\; j_M \;\; k_M]\begin{bmatrix} x_N a_1 + y_N b_1 + x_N c_1 \\ x_N a_2 + y_N b_2 + z_N c_2 \\ x_N a_3 + y_N b_3 + z_N c_3 \end{bmatrix}$$

$$= [i_M \;\; j_M \;\; k_M]\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x_N \\ y_N \\ z_N \end{bmatrix} = [i_M \;\; j_M \;\; k_M][P]_M$$

so

$$[P]_M = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{bmatrix} x_N \\ y_N \\ z_N \end{bmatrix} = \begin{bmatrix} [i_N]_M & [j_N]_M & [k_N]_M \end{bmatrix}[P]_N = R_{NM}[P]_N$$

Alternative representation of previous transformation between rotated coordinate frames with common origin

$$\overrightarrow{(OP)} = [P]_N \begin{bmatrix} i_N \\ j_N \\ k_N \end{bmatrix} = [P]_M \begin{bmatrix} i_M \\ j_M \\ k_M \end{bmatrix}$$

$$[P]_N \begin{bmatrix} i_N \\ j_N \\ k_N \end{bmatrix} = [x_N \quad y_N \quad z_N] \begin{bmatrix} i_N \\ j_N \\ k_N \end{bmatrix} = x_N i_N + y_N j_N + z_N k_N, \qquad \text{but} \begin{vmatrix} i_N = a_1 i_M + a_2 j_M + a_3 k_M \\ j_N = b_1 i_M + b_2 j_M + b_3 k_M \\ k_N = c_1 i_M + c_2 j_M + c_3 k_M \end{vmatrix}$$

$$= x_N(a_1 i_M + a_2 j_M + a_3 k_M) + y_N(b_1 i_M + b_2 j_M + b_3 k_M) + z_N(c_1 i_M + c_2 j_M + c_3 k_M)$$

$$= (x_N a_1 + y_N b_1 + x_N c_1) i_M + (x_N a_2 + y_N b_2 + z_N c_2) j_M + (x_N a_3 + y_N b_3 + z_N c_3) k_M$$

$$= [x_N a_1 + y_N b_1 + x_N c_1, \quad x_N a_2 + y_N b_2 + z_N c_2, \quad x_N a_3 + y_N b_3 + z_N c_3] \begin{bmatrix} i_M \\ j_M \\ k_M \end{bmatrix}$$

$$= [x_N \quad y_N \quad z_N] \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} i_M \\ j_M \\ k_M \end{bmatrix} = \overrightarrow{(OP)} = [P]_M \begin{bmatrix} i_M \\ j_M \\ k_M \end{bmatrix}$$

so

$$[P]_M = [x_N \quad y_N \quad z_N] \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = [P]_N \begin{bmatrix} (\; i_N \;)_M \\ (\; j_N \;)_M \\ (\; k_N \;)_M \end{bmatrix} = [P]_N R^T_{NM}$$

Machines and Metrology

- Basics of Model Building

Machine

- An assemblage of components designed to assist in a desired action being performed on an object.
- System which uses energy in any form to move or alter an object.
- *From the Greek '$M\epsilon\chi o\sigma$' - mechos meaning expedient*

Machine Tool

- Machine which uses any of a variety of tools to change the physical shape or other intrinsic state parameter of a workpiece.
- Common operations include cutting, turning, boring, drilling etc. Often incorporates probes, actuators, transducers, and stages to facilitate its operation.

Machine Design

- Frame - Basic structure of a machine which provides support to all elements.
- Dynamic Elements- *motors, ballscrews, carriages*
  Provide Actuation Forces
- Constraining Elements - *guideways, ballscrews*
  Limits Degrees of Freedom
- Metrology Elements *encoders, scales, readheads*
  Provide Positioning Reference

Structural Loop

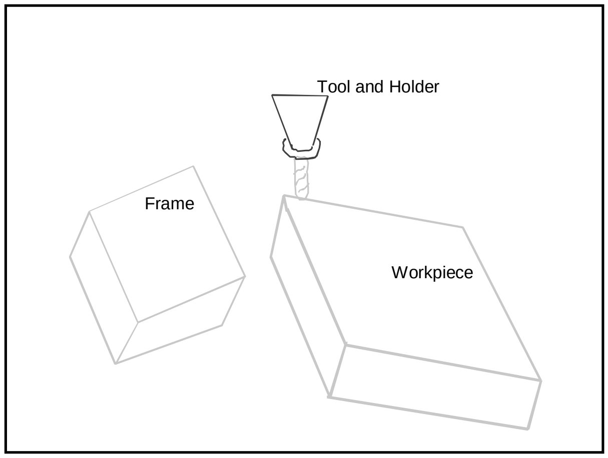
- The assembly chain from one machine element to another which provides the physical support and constraint for each element.
- Carriages, guideways (ways), clamping fixtures, toolholder, spindle, bearings
- From workpiece to tool

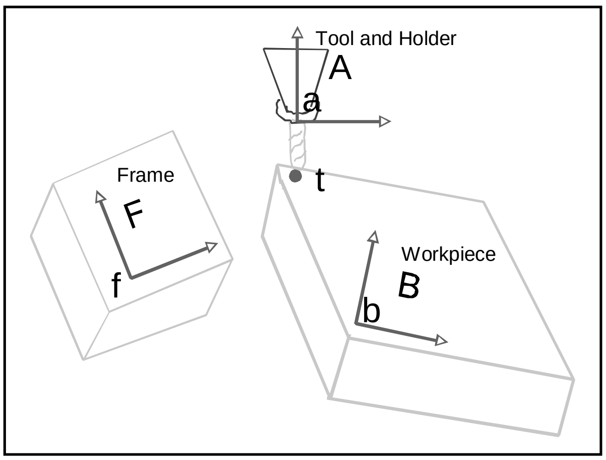
Metrology Loop/Links

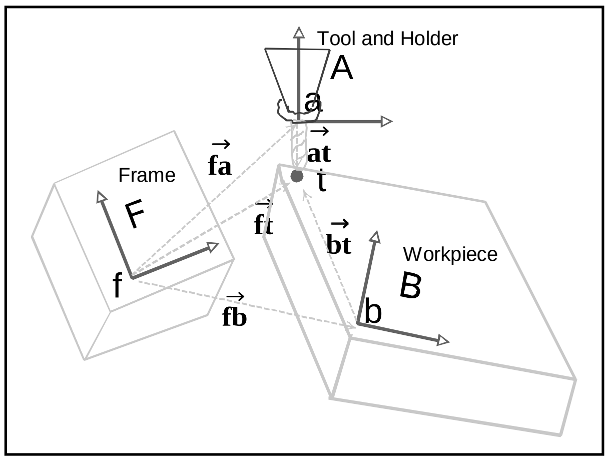
- **The path that weaves through the machine elements that provides/affects a machine's positioning reference**.
- **May interlace through the spindle, quill, table, carriages, scales, ballscrews, frame etc.**
- Can be separated to some extent from the structural and dynamic loops and thus not be subject to a distorted frame caused by actuation forces.
- Path may interlace through some supporting components and be affected by their thermal expansion, force distortions, and misalignments.

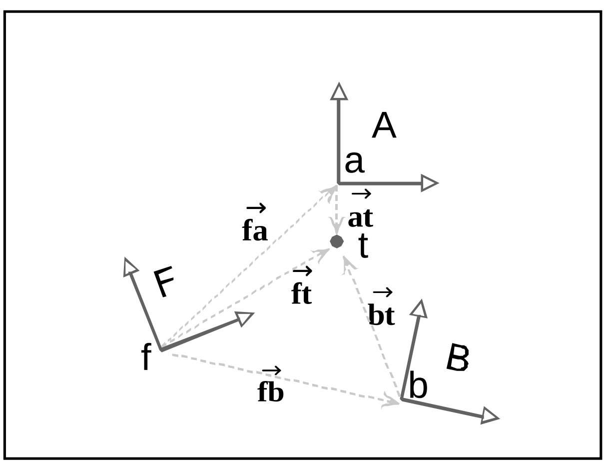
Dynamic Loops/Links

- The assembly chain from one machine element to another which directs the forces producing motion throughout the machine.
- For a machine tool, the dynamic loops may involve such things as the motors, actuators, spindle, carriages, frame, bearings, ballscrews, workpiece, tool, and clamping fixture.









Transformation Development on a Two Carriage Machine

$\vec{ft} = \vec{fa} + \vec{at} = \vec{fb} + \vec{bt}$ *(physical model)*

$\vec{ft}_F = \vec{fa}_F + \vec{at}_F = \vec{fb}_F + \vec{bt}_F$ *(coord representation)*

$\vec{ft}_F = \vec{fa}_F + R_{AF}\left(\vec{at}_A\right) = \vec{fb}_F + R_{BF}\left(\vec{bt}_B\right)$

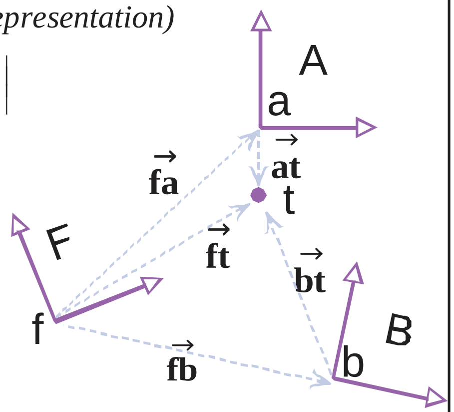
so that

$\vec{fa}_F + R_{AF}\left(\vec{at}_A\right) - \vec{fb}_F = R_{BF}\left(\vec{bt}_B\right)$

$R_{BF}^{-1}\left(\vec{fa}_F + R_{AF}\left(\vec{at}_A\right) - \vec{fb}_F\right) = \vec{bt}_B$

$R_{BF}^{-1}\left(\vec{a}_F + R_{AF}\left(\vec{t}_A\right) - \vec{b}_F\right) = \vec{t}_B$

$R_{BF}^{-1}\left(\vec{T}_{aF} + R_{AF}\left(\vec{t}_A\right) - \vec{T}_{bF}\right) = \vec{t}_B$



Measurements wrt frame

Transformation Development on a Two Carriage Machine

$\vec{bt} = \vec{bf} + \vec{ft} = \vec{bf} + \vec{fa} + \vec{at}$

$\vec{bt}_B = \vec{bf}_B + \vec{ft}_B$

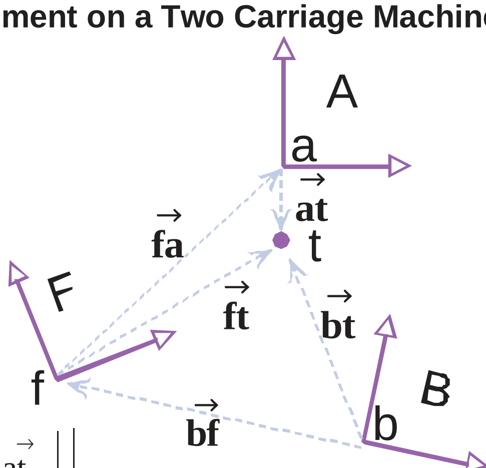
$\vec{bt}_B = \vec{bf}_B + R_{FB}\left(\vec{ft}_F\right)$

$\vec{bt}_B = \vec{bf}_B + R_{FB}\left(\vec{fa}_F + \vec{at}_F\right)$

$\vec{bt}_B = \vec{bf}_B + R_{FB}\left(\vec{fa}_F + R_{AF}\left(\vec{at}_A\right)\right)$

$\vec{t}_B = \vec{f}_B + R_{FB}\left(\vec{a}_F + R_{AF}\,\vec{t}_A\right)$

$\vec{t}_B = \vec{T}_{fB} + R_{FB}\left(\vec{T}_{aF} + R_{AF}\,\vec{t}_A\right)$



Measurement wrt B (table)