Coloring Graphs with Intervals

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Scheduling for Large Scale Systems Workshop
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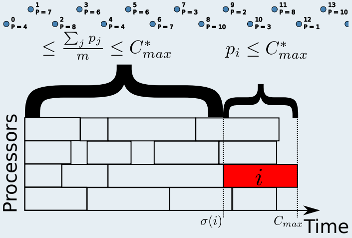
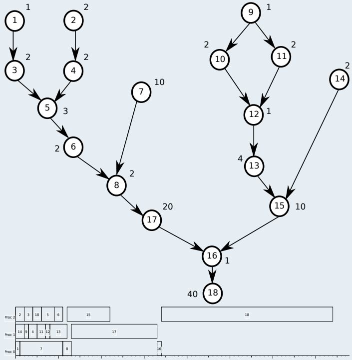
Scheduling, Traditionally

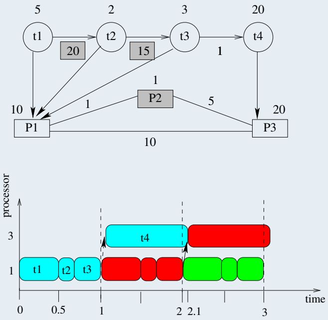
Independent Tasks



Parallel Task Graph



Pipelined Graphs



All these cases have precedence dependencies that come from the application

Conflict Graph Model

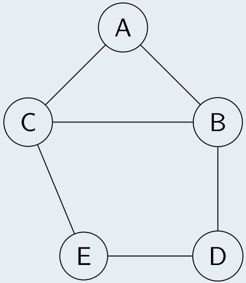
Applications

Process X can't run while Process Y runs.

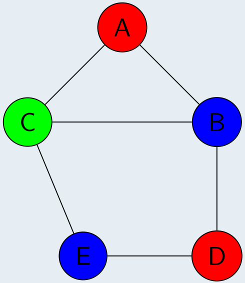
- Transactions in Databases
- Transpose sparse matrix operations
- Anytimes tasks share memory writes

Traditionally solved with mutex and atomics.

A Coloring Model



Solution



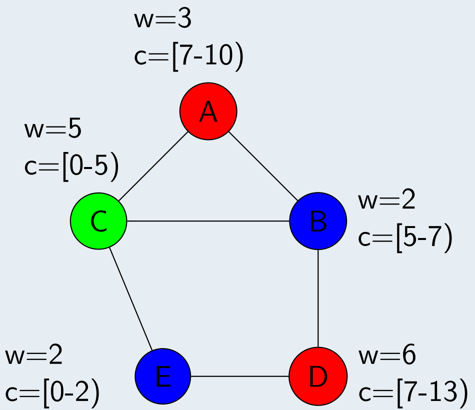
Schedule

In batches of colors with a hard synchronization between colors.

Conflict models do not account for runtime of tasks.

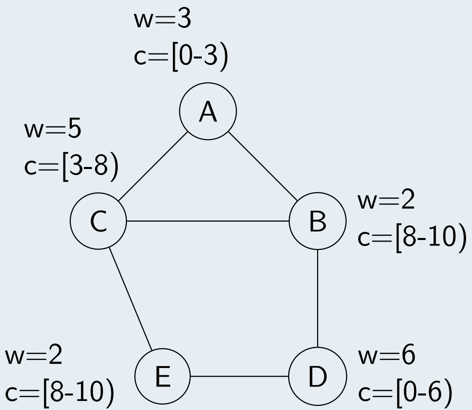
The runtime of tasks matters

Executing colors one at a time



Solution in 13

Optimal schedule



Solution in 10

Formal Definition of the Interval Vertex Coloring Problem

Interval Vertex Coloring Problem (IVC)

Let $G = (V, E)$ be an undirected graph and $w : V \to \mathbb{Z}^+$ be a weight function.
An interval coloring of the vertices of G is a function $start : V \to \mathbb{Z}^+$.
We say that vertex v is colored with the open interval $[start(v), start(v) + w(v))$.

Valid Interval Coloring

For the coloring to be valid, neighboring vertices must have disjoint color intervals:
$\forall (a, b) \in E, [start(a), start(a) + w(a)) \cap [start(b), start(b) + w(b)) = \emptyset$.

Optimal Interval Coloring

A particular coloring of vertices is said to use $maxcolor = \max_{v \in V} start(v) + w(v)$ colors.
The optimization problem is to find a coloring that minimizes $maxcolor$.
We denote the optimal value of $maxcolor$ as $maxcolor^*$.

Coloring vertices with interval is harder than regular graph coloring

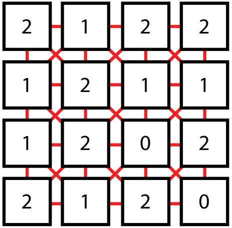
- NP Complete in general
- No general approximation algorithm

Outline

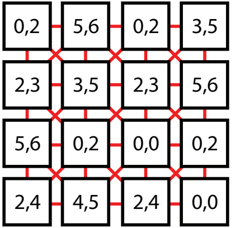
1. Introduction
2. Coloring 9-pt and 27-pt stencils with intervals [IPDPS22]
3. Optimizing Distributed Dataflow Algorithms [PDCO23]
4. Conclusion

The Problem of Coloring Stencils with Intervals

2D Example (4x4)

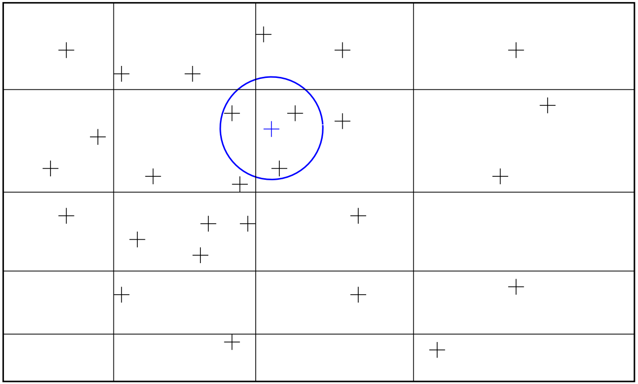


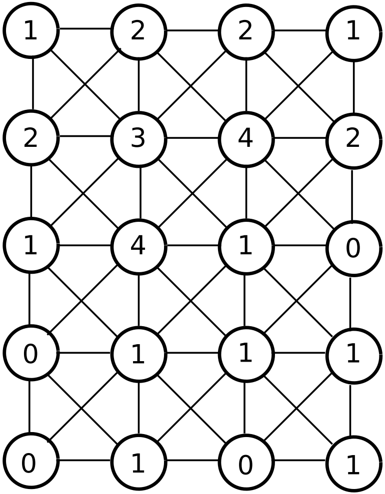
Example Solution



- A graph which is a
  - 2D 9-pt stencil
  - or 3D 27-pt stencil
- Each vertex has a weight $w(v)$
- Color each vertex with an interval larger than its weight
  - Intervals should have the form $[start(v), start(v) + w(v))$
- No adjacent vertices can have overlapping intervals
- *maxcolors* is the largest right endpoint in the set of intervals
- Objective is to minimize *maxcolor*

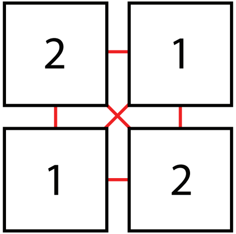
Spatial Applications often Parallelize as Stencils



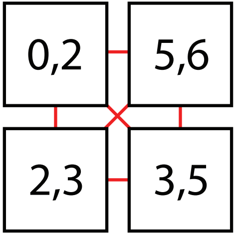


Cliques can be Colored in Linear Time

K4 Example



K4 Solution



Algorithm

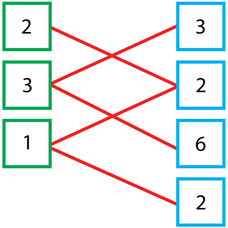
- No vertex can share any color with any other vertex in clique
- We must use at least $\sum_{v \in K} w(v)$ colors
- Greedily color the interval with the lowest available $start(v)$
- Complexity $\Theta(V)$

Implications

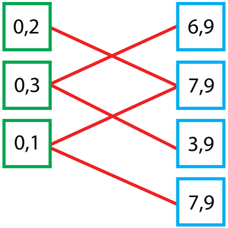
- Each square block of 4 vertices is a K_4
  - Sum of weights of K_4 is a lower bound of 2D 9-pt stencil
- Each square block of 8 vertices is a K_8
  - Sum of weights of K_8 is a lower bound of 3D 27-pt stencil

Bipartite Graphs can be Colored in Linear Time

Bipartite Example



Bipartite Solution



Algorithm

- Partition vertices into A, B, s.t, $(i, j) \in E \implies i \in A, j, \in B$
- Compute $maxcolor = \max_{(i,j)\in E} w(i) + w(j)$
- Color $i \in A$ starting at 0 with $[0; w(i))$
- Color $j \in B$ ending at $maxcolor$ with $[maxcolor - w(j); maxcolor)$
- Complexity $\Theta(E)$

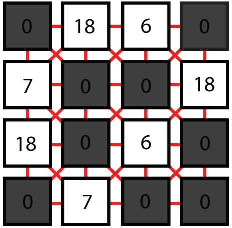
Implications

Many subgraphs of a stencil are bipartite and induce lower bounds:

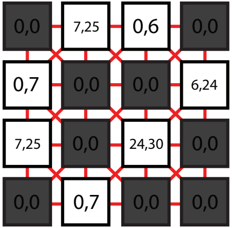
- Each edge in the graph
- 2D 5-pt stencils
- 3D 7-pt stencils
- Many cycles of even length

Odd Cycles can be Colored in Linear Time

Odd Cycle Example



Odd Cycle Solution



Algorithm

- Let *maxpair* be the largest sum of any 2 consecutive vertices
- Let *minchain*3 is the smallest sum of 3 consecutive vertices
- We have $maxcolor = max(maxpair, minchain3)$
- Identify the *minchain*3 triplet: $0, 1, 2$
  - Color 0 with $[0; w(0))$
  - Color 1 with $[w(0); w(0) + w(1))$
  - Color 2 with $[w(0) + w(1); w(2))$
  - Color the other alternatively with $[0; w(v))$
  - or $[maxcolor - w(v); maxcolor)$
- Complexity $\Theta(E)$

Implications

Many odd cycles in 2D 9-pt stencils and 3D 27-pt stencils

27pt-Stencil is NP-Complete by Reduction from NAE-3SAT

NAE-3SAT: Not-All-Equal 3-SAT

- n binary variables in m groups of 3 variables
- Assign true or false to each variable
- The instance is positive if every group has at least one variable that is true and at least one that is false

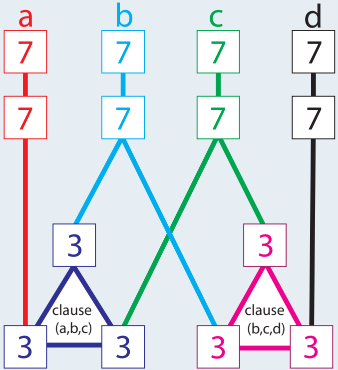
NAE-3SAT is known to be NP-Complete

Solving NAE-3SAT by Coloring a Simple Graph with 14 Colors

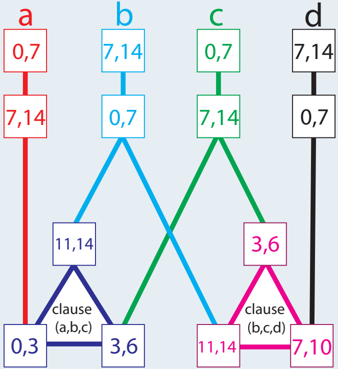
NAE-3SAT Instance

Variables: $\{a, b, c, d\}$; Clauses: $\{(a, b, c), (b, c, d)\}$

Constructed Graph Instance

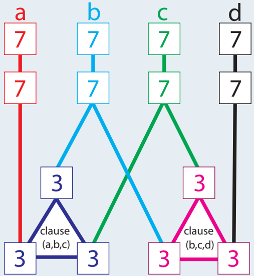


Solution in 14 Colors

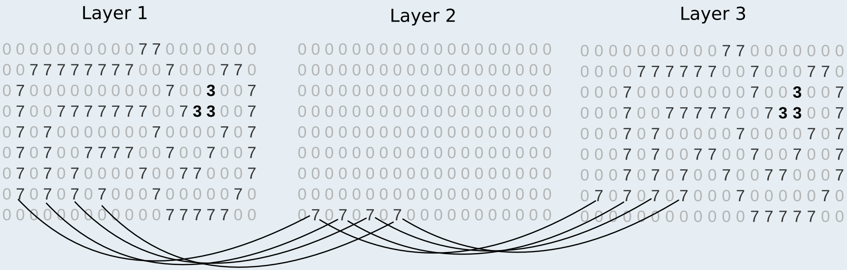


Embedding the Constructed Graph in a 3D 27-pt Stencil

From Graph ...



... To a 27-pt 3D stencil



Greedy Algorithms

Greedy Principles

Any greedy coloring will color vertex v with an interval that ends before: $\sum_{j \in \Gamma(v)} w(j) + (\Gamma(v) + 1)w(v) - \Gamma(v)$

By Vertex

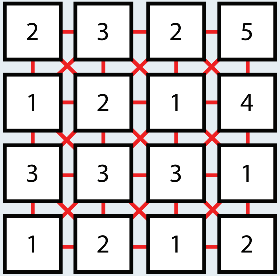
- Greedy Largest First
- Greedy Line by Line
- Greedy Z-Order

By Set

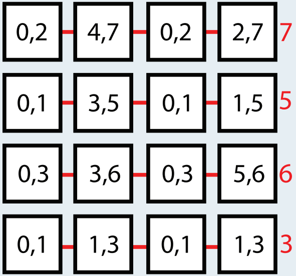
- Greedy Largest Clique First
  - Schedule vertices in the largest clique first; order within clique uses vertex id
- Smart Greedy Largest Clique First
  - Permute each clique and use the order with least *maxcolor*

Bipartite Decomposition is a 2-approx. in 2D (and 4-approx. in 3D)

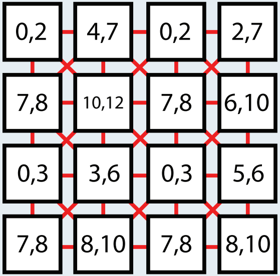
Example



Do Rows Independently



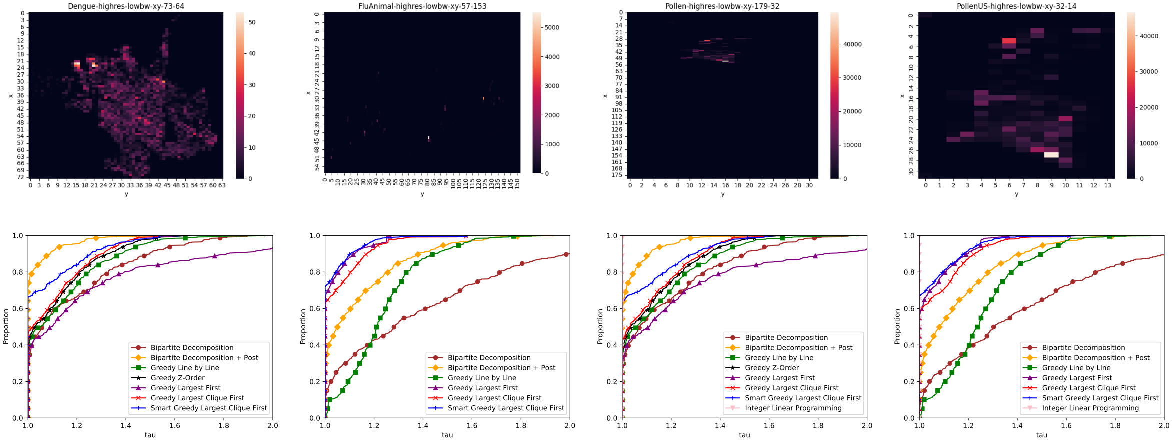
Shift Odd Rows by Max Color



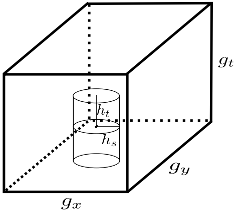
Bipartite Decomposition with Post Optimization

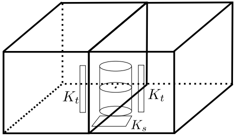
- Sort K_4 or K_8 (in 3D) by non-increasing order by the sum total of their weights
- Sort vertices within K_4 by increasing order of lowest value in their scheduled interval
- Recolor each vertex one at a time using a greedy principle

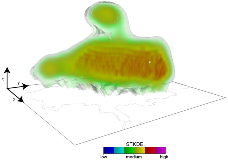
We Ran Simulations. These Methods Work.

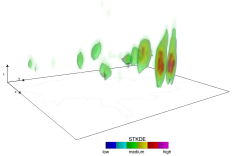


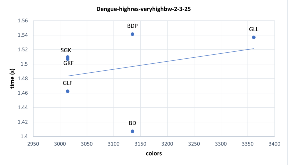
We Integrated that in a Real Application. These Methods Work.

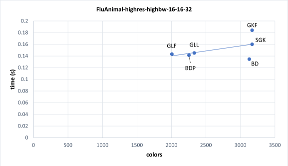


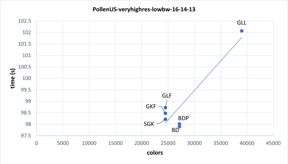


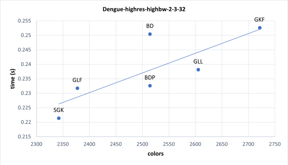


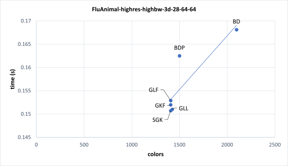


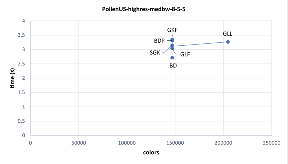












Outline

Luby's Algorithm is an Example of a Dataflow Algorithm

Luby's Algorithm for Maximal Independent Set

- Each vertex v picks unique random number $r(v)$ uniformly in $[0; 1)$
- v sends $r(v)$ to each of its neighbors u
- v notes which neighbors u have the property $r(u) > r(v)$
- v marks its own state as `unknown`
- v awaits a message from each of its neighbors u if $r(u) < r(v)$
- If the state of u is `marked`, the state of v is changed to `unmarked`
- After receiving messages from all neighbors u, if the state of v is `unknown`, the state of v is changed to `marked`
- v sends its state to all neighbors u, such that $r(u) > r(v)$
- All vertices in the `marked` state are a maximal independent set

Introduction to Dataflow Algorithms

Distributed Dataflow Algorithms

- Only use local information
- Processing order of vertices is generated randomly
- Once the order is picked the vertices are processed from low to high in each neighborhood
- Cost to determine other desirable properties is too high
- We are interested in these methods as a model for distributed graph algorithms

Examples

- Luby's Algorithm for Maximal Independent Set
- Jones-Plassmann Algorithm for Graph Coloring

Choice of random order matters to algorithm runtime

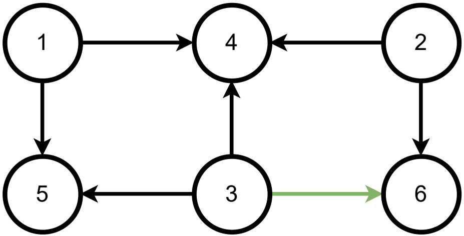


Figure 1: Lucky Draw

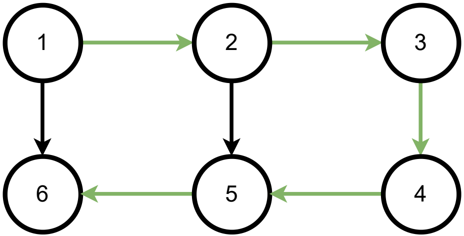


Figure 2: Unlucky Draw

Choice of random order matters to algorithm runtime

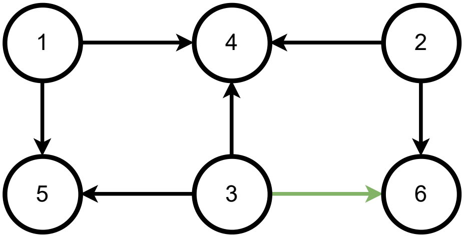


Figure 1: Lucky Draw

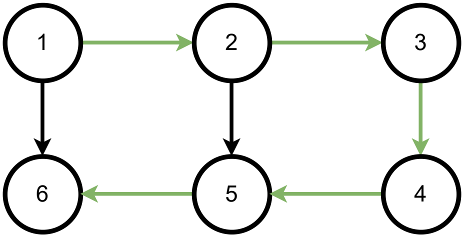


Figure 2: Unlucky Draw

The question

How can we avoid unlucky draws?

Developing a Dataflow Model for Distributed Graph Algorithms

Model

- Let $G = \{V, E\}$ be an undirected graph and $w : V \rightarrow \mathbb{Z}^+$ be a weight function
- Assign $r(v)$ to each vertex v with your algorithm of choice
- Construct directed graph $\bar{G}$ by orienting the existing edges from low $r(v)$ to high $r(v)$
- Calculate length of critical path of $\bar{G}$

Critical Path

Longest weighted path in $\bar{G}$

Objective

Minimizing the length of the critical path (which minimizes the algorithm execution time)

The Weight Function is Non-trivial

Special Case: $w(v) = 1$

- Execution time is dominated by latency.
- The critical path is the number of phases for the graph.
- Critical path is the same as longest path using euclidean distance

Special Case: $w(v) = \delta(v)$

- Bandwidth or the cost of algorithms on the vertices themselves dominates the total execution time of the algorithm.
- Each vertex sends and receives $\delta(v)$ messages
- Most dataflow algorithms have each vertex do $O(\delta(v))$ computations
- Largest Degree First Order closely resembles that of Largest Processing Time First

Deriving Better Partial Orders for Distributed Graph Algorithms

`Uniform` (aka draw in $[0; 1)$)

- Existing method of random number generation in dataflow algorithms

`Linear` (aka draw in $[0; \delta(v))$)

- v is guaranteed to be after all vertices u, such that $\delta(u) = \delta(v) - 1$ with probability $\frac{1}{\delta(v)}$
- Good approximation of Largest Degree First with vertices of dramatic difference in degrees
- Poor approximation when $\Delta(G)$ is large and G has many vertices of large degrees

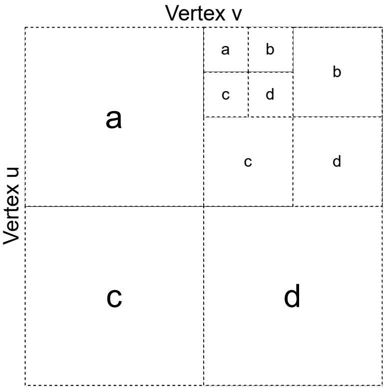
`Exponential` (aka draw in $[0; 2^{\delta(v)})$)

- v is guaranteed to be after all vertices u, such that $\delta(u) = \delta(v) - 1$ with probability $> \frac{1}{2}$
- Better approximation of Largest Degree First

- Communication and Computational cost is the same for each algorithm
- Sampling uniformly in those intervals despite naming conventions

Construction of RMAT Graphs

- 2^n nodes
- Recursively split square matrix into 4 quadrants: a, b, c, d
- Each quadrant has an associated probability that a given edge will fall into that quadrant: $a + b + c + d = 1$
- Edges are generated one at a time and placed in a quadrant recursively following those probabilities until the edge is placed in a 1×1 submatrix.
- $ef * 2^n$ edges



RMAT Graphs Study

Methodology

- Sampled RMAT parameter space with constant *ef*
- Computed critical path length for each algorithm
- Calculated 95% confidence intervals
- Computed pairwise ratios of critical paths
- Conducted Z-Test to validate statistical significance

Results

- `Exponential` path < `Linear` path < `Uniform` path
- `Exponential` was never worse than `Uniform`
- At best, `Exponential` was 50% better
- On average, `Exponential` was about 10% better

a	b	c	d	Uniform CI	Exponential CI	Linear CI	U/E	U/L	L/E
0.30	0.28	0.28	0.14	[2944; 2947]	[2623; 2625]	[2850; 2853]	1.123	1.033	1.087
0.40	0.24	0.24	0.12	[4950; 4953]	[4680; 4683]	[4859; 4862]	1.058	1.019	1.038
0.50	0.20	0.20	0.10	[6968; 6971]	[6643; 6647]	[6845; 6848]	1.049	1.018	1.030
0.60	0.16	0.16	0.08	[8353; 8357]	[7949; 7952]	[8175; 8178]	1.051	1.022	1.028
0.70	0.12	0.12	0.06	[9211; 9214]	[8738; 8740]	[8987; 8990]	1.054	1.025	1.029
0.30	0.28	0.17	0.25	[2111; 2113]	[2064; 2066]	[2103; 2105]	1.023	1.004	1.019
0.30	0.28	0.25	0.17	[2633; 2635]	[2437; 2439]	[2583; 2585]	1.080	1.019	1.060
0.30	0.28	0.34	0.08	[3782; 3785]	[3112; 3115]	[3510; 3513]	**1.215**	1.077	1.128
0.30	0.35	0.14	0.21	[2625; 2627]	[2047; 2049]	[2402; 2404]	**1.282**	1.093	**1.173**
0.30	0.35	0.21	0.14	[3064; 3067]	[2464; 2467]	[2825; 2827]	**1.243**	1.085	1.146
0.30	0.35	0.28	0.07	[3959; 3962]	[3221; 3225]	[3644; 3647]	**1.229**	1.086	1.131
0.30	0.42	0.11	0.17	[3632; 3635]	[2181; 2183]	[2880; 2883]	**1.665**	**1.261**	**1.321**
0.30	0.42	0.17	0.11	[3821; 3824]	[2433; 2436]	[3061; 3064]	**1.570**	**1.248**	**1.258**
0.30	0.42	0.22	0.06	[4343; 4346]	[3110; 3113]	[3685; 3688]	**1.396**	**1.178**	**1.185**
0.30	0.49	0.08	0.13	[4852; 4855]	[2245; 2249]	[3437; 3441]	**2.160**	**1.411**	**1.530**
0.30	0.49	0.13	0.08	[4775; 4778]	[2505; 2508]	[3325; 3330]	**1.906**	**1.435**	**1.328**
0.30	0.49	0.17	0.04	[5052; 5056]	[3151; 3155]	[3883; 3887]	**1.603**	**1.301**	**1.232**
0.40	0.24	0.14	0.22	[3246; 3249]	[3185; 3188]	[3242; 3245]	1.019	1.001	1.018
0.40	0.24	0.22	0.14	[4571; 4574]	[4377; 4380]	[4509; 4512]	1.044	1.014	1.030
0.40	0.24	0.29	0.07	[5946; 5950]	[5500; 5504]	[5759; 5763]	1.081	1.032	1.047
0.40	0.30	0.12	0.18	[4026; 4029]	[3457; 3460]	[3811; 3814]	**1.165**	1.056	1.102
0.40	0.30	0.18	0.12	[5003; 5006]	[4551; 4555]	[4821; 4825]	1.099	1.038	1.059
0.40	0.30	0.24	0.06	[6141; 6144]	[5652; 5656]	[5932; 5935]	1.086	1.035	1.049
0.40	0.36	0.10	0.14	[4903; 4906]	[3767; 3771]	[4333; 4336]	**1.301**	1.132	1.150
0.40	0.36	0.14	0.10	[5506; 5509]	[4637; 4641]	[5071; 5075]	**1.187**	1.086	1.094
0.40	0.36	0.19	0.05	[6383; 6387]	[5684; 5688]	[6045; 6049]	1.123	1.056	1.063
0.40	0.42	0.07	0.11	[5667; 5670]	[3901; 3906]	[4639; 4643]	**1.452**	**1.221**	**1.189**
0.40	0.42	0.11	0.07	[6196; 6199]	[4927; 4932]	[5478; 5483]	**1.257**	1.131	1.112
0.40	0.42	0.14	0.04	[6670; 6674]	[5681; 5685]	[6132; 6136]	**1.174**	1.088	1.079
0.50	0.20	0.12	0.18	[5009; 5012]	[4881; 4884]	[4981; 4984]	1.026	1.006	1.020
0.50	0.20	0.18	0.12	[6489; 6492]	[6213; 6216]	[6399; 6402]	1.044	1.014	1.030
0.50	0.20	0.24	0.06	[7813; 7816]	[7365; 7368]	[7616; 7620]	1.061	1.026	1.034
0.50	0.25	0.10	0.15	[5687; 5690]	[5230; 5234]	[5515; 5519]	1.087	1.031	1.054
0.50	0.25	0.15	0.10	[6920; 6924]	[6500; 6504]	[6748; 6751]	1.065	1.026	1.038
0.50	0.25	0.20	0.05	[7984; 7987]	[7503; 7507]	[7766; 7770]	1.064	1.028	1.035
0.50	0.30	0.08	0.12	[6294; 6297]	[5524; 5528]	[5943; 5946]	1.139	1.059	1.076
0.50	0.30	0.12	0.08	[7263; 7267]	[6644; 6648]	[6969; 6972]	1.093	1.042	1.049
0.50	0.30	0.16	0.04	[8073; 8076]	[7495; 7499]	[7786; 7789]	1.077	1.037	1.039
0.50	0.35	0.06	0.09	[6812; 6815]	[5778; 5781]	[6278; 6281]	**1.179**	1.085	1.087
0.50	0.35	0.09	0.06	[7537; 7540]	[6729; 6732]	[7108; 7111]	1.120	1.060	1.056
0.50	0.35	0.12	0.03	[8135; 8138]	[7420; 7423]	[7754; 7757]	1.096	1.049	1.045
0.60	0.16	0.10	0.14	[6491; 6495]	[6204; 6208]	[6406; 6409]	1.046	1.013	1.032
0.60	0.16	0.14	0.10	[7774; 7777]	[7418; 7422]	[7631; 7635]	1.048	1.019	1.029
0.60	0.16	0.19	0.05	[9077; 9080]	[8613; 8616]	[8843; 8846]	1.054	1.026	1.027
0.60	0.20	0.08	0.12	[6942; 6945]	[6427; 6431]	[6741; 6745]	1.080	1.030	1.049
0.60	0.20	0.12	0.08	[8257; 8260]	[7803; 7806]	[8044; 8048]	1.058	1.026	1.031
0.60	0.20	0.16	0.04	[9253; 9256]	[8754; 8757]	[8997; 9000]	1.057	1.028	1.028
0.60	0.24	0.06	0.10	[7261; 7264]	[6516; 6519]	[6926; 6929]	1.114	1.048	1.063
0.60	0.24	0.10	0.06	[8597; 8600]	[8041; 8044]	[8308; 8311]	1.069	1.035	1.033
0.60	0.24	0.13	0.03	[9283; 9286]	[8731; 8734]	[8987; 8990]	1.063	1.033	1.029
0.60	0.28	0.05	0.07	[7829; 7832]	[6958; 6962]	[7380; 7383]	1.125	1.061	1.061
0.60	0.28	0.07	0.05	[8537; 8540]	[7838; 7841]	[8157; 8160]	1.089	1.047	1.041
0.60	0.28	0.10	0.02	[9249; 9252]	[8631; 8634]	[8900; 8903]	1.072	1.039	1.031
0.70	0.12	0.07	0.11	[7229; 7232]	[6852; 6856]	[7101; 7105]	1.055	1.018	1.036
0.70	0.12	0.11	0.07	[8854; 8857]	[8424; 8427]	[8659; 8662]	1.051	1.023	1.028
0.70	0.12	0.14	0.04	[9813; 9816]	[9197; 9200]	[9514; 9517]	1.067	1.031	1.034
0.70	0.15	0.06	0.09	[7797; 7800]	[7252; 7255]	[7573; 7576]	1.075	1.030	1.044
0.70	0.15	0.09	0.06	[9093; 9096]	[8589; 8592]	[8850; 8853]	1.059	1.027	1.030
0.70	0.15	0.12	0.03	[10032; 10035]	[9347; 9350]	[9694; 9696]	1.073	1.035	1.037
0.70	0.18	0.05	0.07	[8267; 8270]	[7587; 7591]	[7941; 7945]	1.090	1.041	1.047
0.70	0.18	0.07	0.05	[9183; 9186]	[8599; 8602]	[8876; 8879]	1.068	1.035	1.032
0.70	0.18	0.10	0.02	[10074; 10076]	[9370; 9372]	[9700; 9702]	1.075	1.039	1.035
0.70	0.21	0.04	0.05	[8667; 8669]	[7899; 7902]	[8263; 8266]	1.097	1.049	1.046
0.70	0.21	0.05	0.04	[9168; 9171]	[8503; 8506]	[8798; 8800]	1.078	1.042	1.035
0.70	0.21	0.07	0.02	[9844; 9846]	[9198; 9200]	[9470; 9472]	1.070	1.039	1.030

Understanding the Results of the RMAT Graphs Study

Why is `Exponential` better on RMAT Graphs

- RMAT Graphs have the same properties of a social network
- Social networks are "Onion-like" - dense core, but outer layers become less dense
- `Exponential` is similar to Largest First

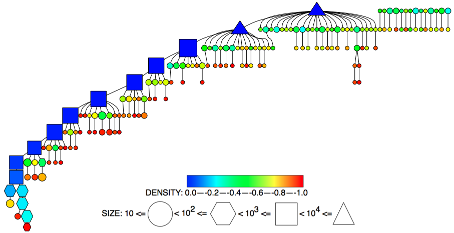


Figure 3: Hierarchy of Dense Subgraphs by Sariyuce et al. (2015)

Real World Application

- Conducted similar experiment on real world graphs from SNAP
- All graphs have small world properties, except roads of Pennsylvania
- `Exponential` was better except on ca-HepPh and roadNet-PA

Name	Vertices	Edges	Max Degree	Clustering Coefficient	Diameter	Uniform CI	Exponential CI	Linear CI	U/E	U/L	L/E
CA-HepPh	89,209	118,521	491	0.6115	13	[1030; 1036]	[1040; 1045]	[1032; 1037]	0.991	0.999	0.992
Email-Enron	36,692	183,831	1,383	0.4970	11	[43437; 43720]	[38836; 38982]	[40688; 41002]	1.120	1.067	1.050
p2p-Gnutella04	10,879	39,994	103	0.0062	9	[911; 925]	[568; 575]	[728; 740]	**1.606**	**1.251**	**1.284**
roadNet-PA	1,090,920	1,541,898	9	0.0465	786	[49; 49]	[49; 50]	[48; 49]	0.990	1.010	0.980
soc-Epinions1	75,888	405,740	3,044	0.1378	14	[94793; 95270]	[88297; 88593]	[89488; 90034]	1.074	1.059	1.015
soc-pokec-relationships	1,632,804	22,301,964	14,854	0.1094	11	[118924; 119528]	[96958; 97239]	[100836; 101775]	**1.228**	**1.177**	1.043
web-Google	916,428	4,322,051	6,332	0.5143	21	[80466; 81618]	[18166; 18192]	[20577; 21084]	**4.458**	**3.891**	1.146
WikiTalk	2,394,385	4,659,565	100,029	0.0526	9	[1352414; 1357165]	[1101248; 1103043]	[1145942; 1151894]	**1.229**	**1.179**	1.042

Results of the Real World Application

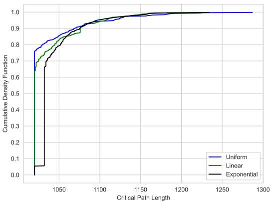


Figure 4: ca-HepPh

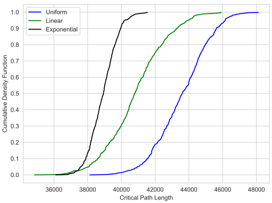


Figure 5: email-Enron

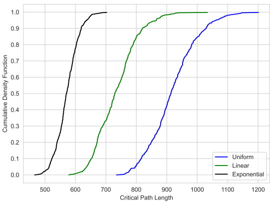


Figure 6: p2p-Gnutella04

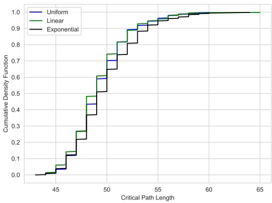


Figure 7: roadNet-PA

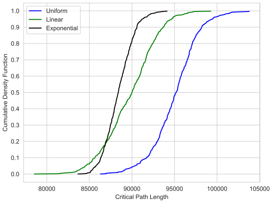


Figure 8: soc-Epinions1

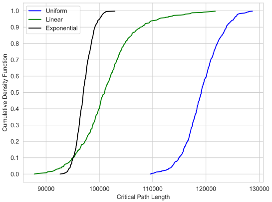


Figure 9: soc-pokec

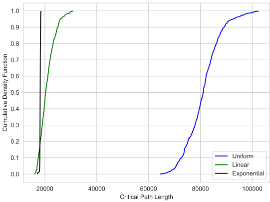


Figure 10: web-Google

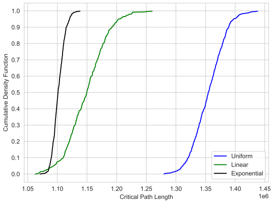


Figure 11: wiki-Talk

Outline

Tackled an understudied coloring problem

Structured graphs

- 27-pt 3D stencil is NP Complete
- Polynomiality of simple structures
- Approximation algorithms for 2D and 3D stencils
- Validated in simulation
- Validated in a real application

Distributed coloring with interval

- Recast graph dataflow algorithm optimization as interval coloring
- Suggested new algorithms for distributed interval coloring
- Statistically proved soundness on RMAT graphs
- Validated on some real world graphs

Future Works

- Complexity of coloring 2D 9pt stencil with intervals?
- Can we do better than 4-approximation for 3D 27-pt stencils?
- Are there other particular graphs it would make sense to consider?
- Can we prove that largest degree first lead to shorter path for some categories of graphs?
- Can we find more applications where coloring with intervals is a good model?

Thank you!

Papers

Dante Durrman and Erik Saule. Optimizing the critical path of distributed dataflow graph algorithms. In Proceedings of IPDPS Workshops (IPDPSW); PDCO, 2023.

Dante Durrman and Erik Saule. Coloring the vertices of 9-pt and 27-pt stencils with intervals. In Proc. of IPDPS, May 2022.

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