On Relating the Disconnectedness of a Contact Formation to the Geometric Properties of its Constituent Objects *

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Abstract

An object with certain geometric features, when it is in contact with another object, may cause disjoint regions of contact configurations under the same topological contact formation (CF). In this paper we analyze such geometric features and introduce the concept of Limiting Channel Element of CF-Compliance *(LIMITEL). We further develop a simple algorithm for finding potential LIMITELs of an arbitrary polygon that may cause disjoint regions of contact configurations, given a second, contacting polygon. The algorithm is purely geometric, and assumes exact knowledge of the geometries of the objects.*

1 Introduction

Knowledge of contact states is necessary for many robotic operations involving contact motion, such as assembly operations. A contact state between two contacting objects can be topologically characterized by a *contact formation*(CF), which, in turn, is made up of one or more contact primitives, called *principal contacts*(PCs)[3]. Geometrically, each CF characterizes a set of relative contact configurations which satisfy the contact constraints of the CF. A contact motion from one such contact configuration to another is called *CF-compliant* if it can be made without changing the CF (i.e., the motion is constrained only by the current CF, without adding a new PC or breaking a current PC).

There are times, however, when the set of the contact configurations satisfying a particular CF is made up of *disjoint* subsets. This arises when there is no CF-compliant motion between two or more contact configurations sharing the same CF. In such a case, each disjoint subset is itself a *CF-connected region* as there exists a CF-compliant motion between any two configurations within the subset, but there are multiple, mutually disjoint CF-connected regions under the same CF. Whether this phenomenon occurs for a topological contact formation depends on whether the contacting objects have certain geometrical features. Although examples were used in the past to demonstrate certain phenomena of disjoint regions [1], to date, there has been little or no work done on identifying the geometrical features of objects causing the disjoint regions in a systematic way.

In this paper we characterize these features that cause disjoint regions of contact configurations under the same contact formation in terms of *Limiting Channel Elements of CF-Compliance* (LIMITELs). We restrict our study to 2-D polygons here. We also present a simple algorithm for extracting the possible LIMITELs of a given arbitrary 2-D polygon relative to some polygon of given dimensions.

The paper is organized as follows: In Section 2, we review the notion of *contact formation* (CF) in terms of *principal contacts* (PC), and we revisit the idea of *CF-compliant motion.* We show an example of non-CF connected configurations under the same contact formation. In Section 3, we develop the ideas of *channel elements* and *LIMITEL*, which is a geometric structure limiting CF-compliant motion. We then present our algorithm for determining possible LIMITELs of an arbitrary polygon with respect to contacts with another polygon. In Section 4 we describe areas for future extensions and conclude the paper.

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2 Context

In this section we give related concepts, some of which were previously defined, to provide needed background.

2.1 Contact Formations

We denote the *boundary elements* of a face as the edges and vertices bounding it, and the boundary elements of an edge as the vertices bounding it[3]. With this notation, a *principal contact* (PC) can be defined as follows:

> A PC is the contact between a pair of surface elements which are not the boundary elements of other contacting surface elements.

Now a *contact formation*(CF) can be described as the unique set of PCs that are formed when arbitrary polyhedra are in contact. If two configurations share the same contact formation, they are *CF-equivalent* configurations. A *CF-compliant* motion is any motion that at all times maintains the same contact formation (i.e., the same set of PCs). Finally, two configurations are *CF-connected* if and only if there is at least one CF-compliant motion that leads from one to the other. We consider that a CF-connected region of configurations forms a unique contact state [2, 4]. If two CF-equivalent configurations are not CF-connected, they indicate disjoint regions in the contact configuration space (i.e., they produce different contact states).

It should be noted here that the above definitions and descriptions are general for 2-D polygons and 3-D polyhedra. For the rest of this paper, we concentrate on the 2-D case.

2.2 Connectivity of Contact Formations

Figure 1 shows an example where two feasible configurations with the same contact formation are not CF-connected. There is no motion that can change C1 to C2 without changing the contact formation.

For two polygons in contact, whether the CF has more than one CF-connected region depends on the geometries of the polygons and their geometric relationship.

In the example in Figure 1, this limiting relationship is the size of object B relative to the distance between edges e_1 and e_3 of object A. Edges e_3 and e_1 form a *channel element* whose width limits the feasible configurations for the objects. We refer to this channel element as a *limiting channel element of CF-compliance*, or a LIMITEL, to be formally defined in the next Section.

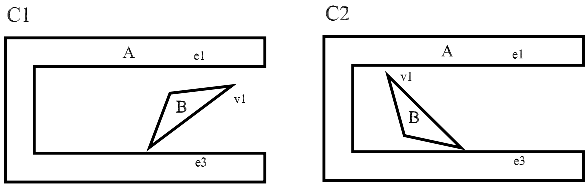


Figure 1: Non-CF-connected configurations with the same CF.

3 2-D LIMITELs

In this section we formalize the idea of *channel elements* and use them to further develop the concept of LIMITELs for 2-D polygons. We then examine under what conditions LIMITELs may exist and give an algorithm for determining the possible LIMITELs in an arbitrary polygon A given the dimensions of a second, contacting polygon B.

3.1 Channel Elements

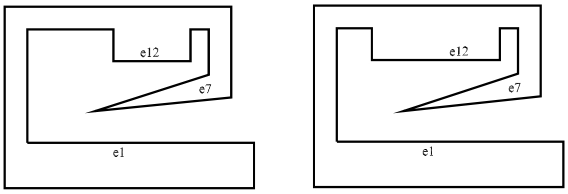
Let e_1 and e_2 be two edges of a polygon A. Let $\mathbf{N}_{e_i}$ be the outward unit normal of e_i, $i = 1, 2$. Let $d(e_1, e_2)$ be the shortest distance between e_1 and e_2, and let $Proj(e_1, e_2)$ be the segment of e_2 corresponding to the projection of e_1 onto e_2.

Definition 1 *e_1 and e_2 form a* channel element *if*

1. *$\mathbf{N}_{e_1} \cdot \mathbf{N}_{e_2} < 0$,* ***AND***
2. *$Proj(e_2, e_1) \neq \emptyset$* ***AND***
3. *$Proj(e_2, e_1) \not\subseteq Proj(g_1, e_1) \cup Proj(g_2, e_1) \cup ...$ where $\{g_1, g_2, ...\}$ is the set of features other than e_2 that fulfill 1 and 2 above,* ***OR*** *$d(e_2, e_1) < min(d(g_1, e_1), d(g_2, e_1), ...)$.*

Note that for adjacent edges, this definition of channel elements leads to the conclusion that only those making an acute interior angle can form a channel element[1].

[1]From the theorems in Section 3.4, indeed, it can be seen that if the angle between two edges is obtuse or right, no CF involving them can have disjoint regions of configurations.



a. (e_1- e_7) is a channel element (e_1- e_{12}) is not

b. Both (e_1- e_7) and (e_1- e_{12}) form channel elements

Figure 2: Some edges and their relations

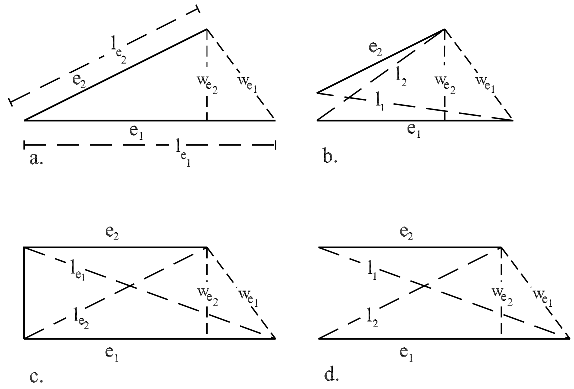


Figure 3: Types and dimensions of channel elements

Figure 2 illustrates some examples of channel elements based on Definition 1.

3.2 Types and Dimensions of Channel Element

We can classify channel elements as either of the $\angle$-*channel* type (Figure 3a. and b.) or of the $\|$-*channel* type (Figure 3c. and d.). We can further classify channel elements as either *closed* (Figure 3a. and c.) or *open* (Figure 3b. and d.)[2].

Whether a channel element H of polygon A can cause disjoint regions of a CF between A and another polygon B depends on its dimensions relative to the dimensions of B. Hence, it is necessary to define related dimensions of H.

Definition 2 *A channel element H with edges e_1 and e_2 has the following dimensions:*

- *width w_{e_i} with respect to edge $e_i, i = 1, 2$:*

$$w_{e_i} = max(d(v_1^{e_i}, e_j), d(v_2^{e_i}, e_j))$$

where e_j is the other edge, $d(v_k^{e_i}, e_j)$ is the minimum distance between the $k-th$ vertex ($k = 1, 2$) of e_i and edge e_j;

- *length l_{e_i} with respect to $e_i, i = 1, 2$, for closed H: l_{e_i} is the diagonal distance from the vertex of e_i at the open end of H to the vertex of the other edge at the closed end;*

- *lengths l_1 and l_2 for open H: the lengths of the two diagonals respectively.*

Figure 3 illustrates the dimensions.

3.3 Dimensions of a Polygon

In this research we consider the dimensions of a polygon to be the distances between pairs of vertices of the convex hull of the polygon. We consider the vertices that determine a dimension to be the vertices of that dimension, and the edges that contain those vertices to be the adjacent edges to the dimension.

As mentioned above, whether a channel element H of polygon A will cause disjoint regions in a contact formation between A and another polygon B that involves H depends on how the dimensions of H relate to the relevant dimensions of B. In the next subsection, we analyze such relations systematically.

3.4 LIMITELs

With the above definitions of channel elements and object dimensions, we can characterize under what conditions a channel element could become a LIMITEL, as formally defined below.

Definition 3 *A channel element is a LIMITEL for a given contact formation (CF) if it causes disjoint regions of contact configurations for that CF.*

We begin by describing cases where a channel element is *not* a LIMITEL.

Theorem 1 *Let H be a channel element of polygon A with dimensions defined in Definition 2. Let CF be a contact formation between A and polygon B where B is in contact with only one edge e_i of H at its vertex v^B (besides possibly contacting features of A other than the considered H). Let d_{min} and d_{max} be the*

[2] Note that we do not consider the cases where both ends of a channel element are closed, because nothing can enter the channel element in the planar case.

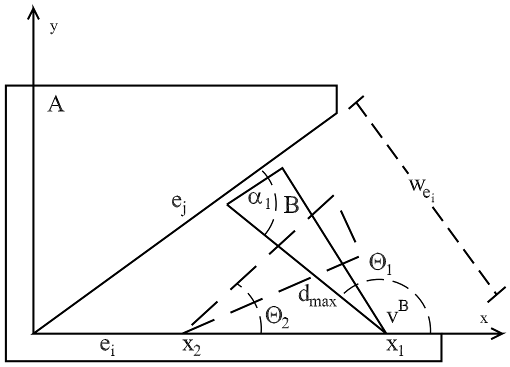


Figure 4: A contact formation between a polygon, B, and a channel element consisting of edges e_i and e_j, where d_{max} of B is smaller than w_{e_i}. This CF satisfies the constraints of the first case of Proof 1.

minimum and maximum dimensions of v^B. H is not a LIMITEL for CF if one of the following conditions holds:

- $d_{max} < w_{e_i}$,
- $d_{min} > l_{e_i}$ *if H is closed,*

Proof 1 For $d_{max} < w_{e_i}$: Define an x axis along e_i and pointing towards the wider (open) end of H and θ as the counterclockwise angle between x and d_{max} (see Figure 4). Let (x_1, θ_1) and (x_2, θ_2) be two contact configurations that share the same contact formation. Furthermore, let α_1 be the counterclockwise angle from d_{max} to e_j (the other edge) of H at configuration (x_1, θ_1). Then for each possible relationship between (x_1, θ_1) and (x_2, θ_2), there exists a valid CF-compliant motion of simultaneous translation of B and rotation of B about v^B respectively, as described below:

1. $x_1 > x_2$ **and** $\theta_1 \geq \theta_2$**:** If $\theta_1 = \theta_2$, a translation along -x suffices. Otherwise, if $\alpha_1 < 90°$, the motion can consist of two parts: (1) simultaneous translation along +x and clockwise rotation until $\alpha_1 = 90°$, and then (2) simultaneous translation along -x and clockwise rotation. Note that motion (1) will not result in v^B moving out of e_i because $d_{max} < w_{e_i}$. Finally, if $\alpha_1 \geq 90°$, just motion (2) will suffice.
2. $x_1 \geq x_2$ **and** $\theta_1 < \theta_2$**:** counterclockwise rotation and translation along -x (if needed) simultaneously.
3. $x_1 < x_2$ **and** $\theta_1 \geq \theta_2$**:** clockwise rotation (if needed) and translation along +x simultaneously.
4. $x_1 \leq x_2$ **and** $\theta_1 < \theta_2$**:** counterclockwise rotation and translation along +x (if needed) simultaneously; the rotation is always possible without B contacting e_j because $d_{max} < w_{e_i}$.

For $d_{min} > l_{e_i}$ when H is closed: Define an x axis in the same way as above. Let θ be the counterclockwise angle from x to d_{min}. Let (x_1, θ_1) and (x_2, θ_2) be any two configurations satisfying the CF. Further define α_1 as the counterclockwise angle between d_{min} and e_j. If $d_{min} > l_{e_i}$, then $\theta < 90°$ for both θ_1 and θ_2 always, i.e., object B can only be tilted toward the open end of H, and $\alpha_1 \geq 90°$ all the time. For each possible relation between (x_1, θ_1) and (x_2, θ_2), the motion described in the corresponding case (of cases 1-4) above is always feasible.

Hence, H is not a LIMITEL for CF if either of the constraints holds.

Theorem 2 *Let H be a channel element of A with dimensions defined in Definition 2. Let CF be a contact formation between A and polygon B where B is in contact with both edges of H (besides possibly contacting features of A other than the considered H), with vertex v_1^B contacting edge e_1 of H and vertex v_2^B contacting edge e_2 of H. Let d be the dimension between v_1^B and v_2^B. H is not a LIMITEL for CF if one of the following conditions holds:*

- $d < min(w_{e_1}, w_{e_2})$,
- $d > min(l_{e_1}, l_{e_2})$ *if H is closed,*
- $d > min(l_1, l_2)$ *if H is open.*

Proof 2 For $d < min(w_{e_1}, w_{e_2})$ (Note: H can only be an $\angle$-channel): The proof here is similar to the first part of Proof 1 by changing d_{max}, i, j, v^B to $d, 1, 2, v_1^B$ respectively, except that only cases 1 and 4 are possible with v_2^B always contacting e_2.

For $d > min(l_{e_1}, l_{e_2})$ if H is closed and $d > min(l_1, l_2)$ if H is open: If H is a $\|$-channel, then $\theta_1 = \theta_2$, and there is always a CF-compliant translation from (x_1, θ_1) to (x_2, θ_2). If H is an $\angle$-channel, the proof is similar to that when $d < min(w_{e_1}, w_{e_2})$, with additional observation similar to that in Proof 1 for $d_{min} > l_{e_i}$.

Finally, if a CF contains an edge-edge PC or equivalent, the only possible kind of CF-compliant motion is translation. For such a CF, the following theorem describes the sufficient conditions for a channel element not to be a LIMITEL.

Theorem 3 *Let H be a channel element of polygon A with dimensions as defined in Definition 2. Let CF be a contact formation between A and another polygon B which involves an edge-edge PC or equivalent[3]. H is not a LIMITEL for CF if it is closed. If H is open, it is not a LIMITEL if no distance between the edge e^B of B (or its convex hull) in the edge-edge PC and another feature of B equals to the minimum distance, w_{min}, between e_1 and e_2 of H on A, i.e., there is no degeneracy.*

Proof 3 If H is closed, it is evident that no disjoint contact configuration regions for CF are possible. For H open (see Figure 7), let d be the maximum distance between the edge e^B of B (or its convex hull) in the edge-edge PC with, say, edge e_1, and another feature of B. For $d < w_{min}$, B can pass through H without gaining a contact with e_2. For $d > w_{min}$, B can not pass into H, and so can not gain a contact with e_2. Therefore, no change in contact formation is possible due to H, and H is not a LIMITEL.

With the above theorems, we can now look at situations where the conditions in those theorems are not satisfied and a channel element is a LIMITEL.

Figures 5, 6, and 7 show examples of LIMITELs and the corresponding CFs, where the conditions of Theorem 1, 2, and 3 respectively are violated. In each figure, B and B* are in disjoint contact configuration regions.

3.5 An Algorithm for Discovering LIMITELs

Here we propose a simple algorithm for finding the channel elements for a polygon A with N edges, and determining for which (if any) contact formations with another polygon, B with M vertices, each channel element might be a LIMITEL.

1. **Determine the convex hull for polygon B**. Any convex hull algorithm, such as the Graham scan[5], will work here, with time complexity $O(MlgM)$.

2. **Determine the dimensions of the hull for polygon B**. This requires calculating the distances between every pair of vertices of the hull, with time complexity $O(M^2)$.

[3] The equivalent to an edge-edge PC is a PC where the convex hull of B has an "edge-edge" contact with A.

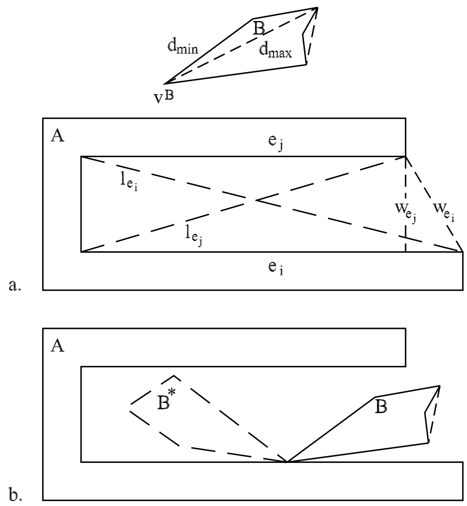


Figure 5: (a.)The dimensions and a vertex of polygon B and the dimensions and edges of a closed channel element, H, of polygon A with edges e_i and e_j. Note that $w_{e_i} < d_{max}$ and $l_{e_i} > d_{max}$, which violates the constraints of Theorem 1. (b.)A vertex-edge contact formation between v^B and e_i for which H is a LIMITEL.

3. **Determine the channel elements of polygon A**. This requires applying the channel element tests (Section 3.1) to every pair of edges in A, which gives a time complexity $O(N^2)$.

4. **For each channel element of A, determine for which, if any, contact formations with B that channel element might be a LIMITEL**. Pseudocode for this step follows:

   FOR (every channel element H of A) DO

   FOR (every edge or vertex, f, of the hull of B) DO

   - Check the non-LIMITEL constraints of Section 3.4 for H using the dimensions of B associated with f. If any constraint is violated, add H to a list of possible LIMITELs.

   Assuming the number of channel elements is of the same order as N, then, Step 4 has time complexity $O(NM^2)$.

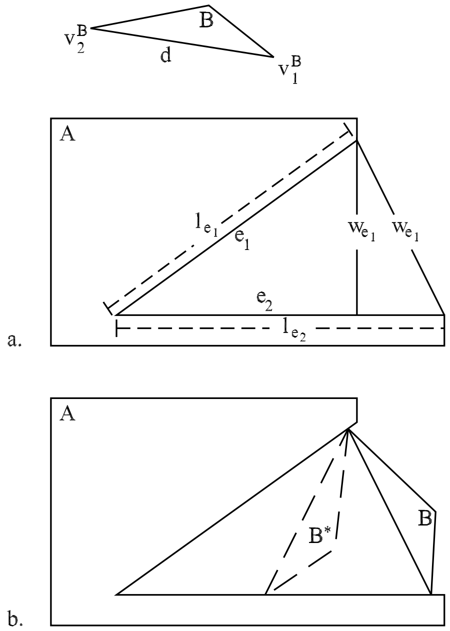


Figure 6: (a.)The dimensions and two vertices of polygon B and the dimensions and edges of a closed channel element, H, of polygon A with edges e_1 and e_2. Note that $min(w_{e_1}, w_{e_2}) < d < max(w_{e_1}, w_{e_2})$ and $d < min(l_{e_1}, l_{e_2})$, which violates the constraints of Theorem 2. (b.)A two vertex-edge contact formation between v_1^B and e_2, and v_2^B and e_1 for which H is a LIMITEL.

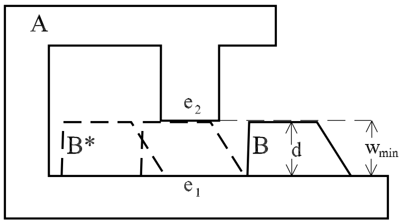


Figure 7: An edge-edge contact between a polygon, B, and a channel element, H, of another polygon, A, where the dimension, d of B equals the minimum width, w_{min} of H (thus violating the constraints of Theorem 3). The unlabdled, intermediate dashed polygon is in contact with both edges of H.

4 Areas for Future Research and Conclusion

This paper has presented a method for systematically categorizing the geometric features of a 2-D polygon A that, when in certain contacts with another polygon B, can lead to disjoint regions of contact configurations with the same contact formation. We have identified such features of A as limiting channel elements or LIMITELs with respect to certain contact formations between A and B, analyzed the necessary conditions for their existence, and presented an algorithm for automatic detection of potential LIMITELs.

There are several areas for future research. So far we have considered how a single channel element of polygon A may cause disjoint regions of contact configurations for a contact formation between A and another polygon B. However, we have not considered the combined effect of multiple channel elements of A, the interacting effect of a channel element of A with that of a channel element of B, and the effect of multiple channel elements of both A and B on the disconnectedness of CFs between A and B. These issues need to be studied. Another subject of future investigation is how we may extend this work into 3-D objects.

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