Third Hour Exam Study Sheet

I. Big-O
   A. Definition
   B. Give Big-O, Big-Theta, Big-Omega of functions
   C. Using definition of Big-O to prove that the given Big-O is the correct one for the given function
   D. If f(x) is in O(g(x)), draw a picture of the relationship between f(x) and g(x)

II. Binary Relations
   A. Definition
   B. Properties (reflexive, irreflexive, non-reflexive, symmetric, antisymmetric, asymmetric, transitive)
   C. Matrix Representation
   D. Inverse Relation
   E. Composite Relations
   F. Closure Properties
   G. Special Types (equivalence relations, partially ordered sets)

III. Induction
   A. Prove closed form is true for a recursion, by induction
   B. Prove inequalities are true by induction
   C. Prove Big-O by induction
Third Hour Exam Practice

These problems are intended as extra practice for the third hour exam. They do not have to be turned in and will not be collected. However, they are recommended, since the same types of problems will be encountered on the test. A solution for each problem is attached.

Give a Big-O estimate for each of the following functions. (Give the function of smallest order.)

1. $n \log(n^2 + 1) + n^2 \log n$
2. $(n \log n + 1)^2 + (\log n + 1)(n^2 + 1)$
3. $n^2^n + n n^2$
4. $\frac{(n^2 + 8)}{(n + 1)}$
5. $\frac{(n \log n + n^2)}{(n^3 + 2)}$
6. $\frac{(n! + 2^n)}{(n^3 + \log(n^2 + 1))}$

Using the definition of Big-O, prove the following:

7. Given that $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$. Prove that $f_1(n)f_2(n)$ is $O(g_1(n)g_2(n))$. 

Solutions

1. \( n \log(n^2 + 1) + n^2 \log n \)

\[ n \log n^2 + n^2 \log n = 2n \log n + n^2 \log n = n \log n + n^2 \log n \Rightarrow O(n^2 \log n) \]

2. \( (n \log n + 1)^2 + (\log n + 1)(n^2 + 1) \)

\[
[(n \log n)^2 + 2n \log n + 1] + [n^2 \log n + \log n + n^2 + 1] =
(n \log n)^2 + n \log n + n^2 \log n + \log n + n^2 \Rightarrow O((n \log n)^2)
\]

3. \( n^{2n} + n^n \)

\( \Rightarrow O(n^{2n}) \)

4. \( \frac{(n^2 + 8)}{(n + 1)} \)

\[ \frac{n^2}{n} = n \Rightarrow O(n) \]

5. \( \frac{(n \log n + n^2)}{(n^3 + 2)} \)

\[ \frac{(n \log n + n^2)}{n^3} = \frac{n^2}{n^3} = \frac{1}{n} = O\left(\frac{1}{n}\right) \Rightarrow O(n^{-1}) \]

6. \( \frac{(n! + 2^n)}{(n^3 + \log(n^2 + 1))} \)

\[ \frac{(n! + 2^n)}{(n^3 + \log n^2)} = \frac{(n! + 2^n)}{(n^3 + 2 \log n)} = \frac{(n! + 2^n)}{(n^3 + \log n)} \Rightarrow O\left(\frac{n!}{n^3}\right) \]
7. **Given that** $f_1(n) = O(g_1(n))$ and $f_2(n) = O(g_2(n))$.

**Prove that** $f_1(n) f_2(n)$ is $O(g_1(n) g_2(n))$.

From the given, we know the following:

$f_1(n) \leq C_1 g_1(n)$ for $n \geq n_1$ and

$f_2(n) \leq C_2 g_2(n)$ for $n \geq n_2$

Now we can multiply the two:

$f_1(n) f_2(n) \leq C_1 C_2 g_1(n) g_2(n)$ for $n \geq \max(n_1, n_2)$

With $C = C_1 C_2$ and $n_0 = \max(n_1, n_2)$, the previous line meets the definition of Big-O.

All of this proves that $f_1(n) f_2(n)$ is $O(g_1(n) g_2(n))$. 