PROBLEM DEPARTMENT

ASHLEY AHLIN   AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`. Electronic submissions using LATEX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2011. Solutions identified as by students are given preference.

Problems for Solution.

1235. *Proposed by Parviz Khalili, Christopher Newport University, Newport News, VA.*

Let x, y, and z be positive real numbers such that $x + y + z = 1$. Show that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq 3 + 2\sqrt{\frac{x^x\, y^y\, z^z}{x\, y\, z}}.$$

1236. *Proposed by Mohsen Soltanifar, University of Saskatchewan, Saskatoon, Canada.*

Prove or give a counterexample:Let $U_i (1 \leq i \leq n)$, be finite dimensional subspaces of a vector space V. Then, the dimension of $\sum_{i=1}^{n} U_i$ is given by :
$\dim(\sum_{i=1}^{n} U_i) = \sum_{r=1}^{n}(-1)^{r+1} \sum_{i_1 < i_2 < ... < i_r : n} \dim(U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_r})$,
where the summation $\sum_{i_1 < i_2 < ... < i_r : n} \dim(U_{i_1} \cap U_{i_2} \cap ... \cap U_{i_r})$ is taken over all of the $\binom{n}{r}$ possible subsets of the set $1, 2, ..., n$.

1237. *Proposed by Thomas Dence, Ashland University, Ashland, OH and Joseph Dence, St. Louis, MO.*

For each integer $n \geq 2$, determine the values of the integrals

$$I_{n,3} = \int_0^\pi \sin^3 x \sin(nx)\ dx \text{ and } I_{n,5} = \int_0^\pi \sin^5 x \sin(nx)\ dx.$$

1238. *Proposed by Tuan Le, student, Fairmont High School, Fairmont, CA.*

Given $a, b, c, d \in [0, 1]$ such that no two of them are equal to 0. Prove that:

$$\frac{1}{a^2 + b^2} + \frac{1}{b^2 + c^2} + \frac{1}{c^2 + d^2} + \frac{1}{d^2 + a^2} \geq \frac{8}{3 + abcd}$$

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1239. *Proposed by Matthew McMullen, Otterbein College, Westerville, OH.*

For $i \in \{1, 2, \ldots, 9\}$, define D_i to be the set of all positive integers that begin with i. For all positive integers n, define

$$a_{n,i} = \frac{1}{n} \cdot |D_i \cap \{1, 2, \ldots, n\}|.$$

Find $\limsup_{n \to \infty} a_{n,i}$ and $\liminf_{n \to \infty} a_{n,i}$.

1240. *Proposed by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, via della ricerca scientifica, Rome, Italy.*

Let x, y be positive real numbers. Prove that

$$\frac{2xy}{x+y} + \sqrt{\frac{x^2 + y^2}{2}} \leq \sqrt{xy} + \frac{x+y}{2} + \frac{(L(x,y) - \sqrt{xy})^2}{\frac{2xy}{x+y}}$$

where $L(x, y) = (x - y)/(\ln(x) - \ln(y))$ if $x \neq y$ and $L(x, x) = x$.

1241. *Proposed by Arthur Holshouser, Charlotte, NC. and Johannas Winterink, Albuquerque, NM.*

For each $i = 0, 1, 2, 3, \ldots, 8$, does a set $\{A, B, C\}$ of three circles in the plane exist such that there are exactly i circles in the plane that are tangent to each of A, B, and C?

1242. *Proposed by Stas Molchanov, University of North Carolina Charlotte.*

This problem has an interesting history. It is mentioned in the book **Mathematicians Also Like Jokes**, by A. Fonin, Moskow, 'Nauka' 2010. The problem appeared on the admissions test for Moskow State University. The exams were very hard, given that there was one place for every 700 applicants. The applicant took the problem to physicist E. Lefshitz, who himself was unable to solve. Lefshitz asked his friend Lev Landau, nobel prize winning physicist about the problem. Landau rightly considered himself an expert in elementary mathematics and the same evening called Lefshitz back to say he'd solved it in an hour, and that nobody, 'except possibly Yakov Zelodovich could solve it faster'. Landau sent the problem to Zeldovich, and indeed he solved the problem in 45 minutes.

Given pyramid $ABCD$ with bottom face triangle ABC with $BC = a, AC = b, AB = c$. Let the lateral faces BCD, ACD, ABD form with the bottom angles α, β, γ, in radians, all acute angles. Find the radius r of the sphere inscribed in the pyramid.