Closed Form, Recursion, and Mindreading; Defining Sequences by Various Means

by Harold Reiter

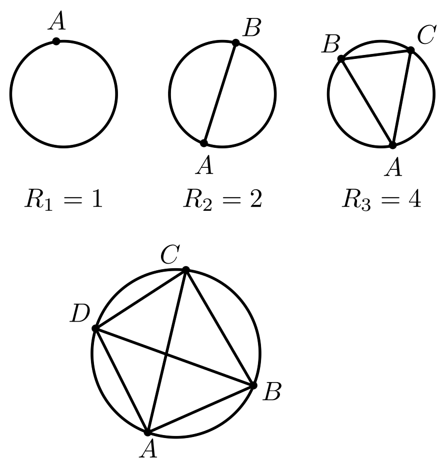
My friend said to me, 'I'm thinking of a sequence of positive integers the first four terms of which are $1, 2, 4, 8$'. Can you guess the next one. I said, 'How about 16?' and he replied, 'right, and the next one?'. I replied '32?' 'Wrong,' he said, 'its 31. And the next term after that is 57.' See below for more on this sequence.

Recently a preliminary version of an important mathematics contest had the following problem.

> Consider the sequence $2, 4, 8, 14, 22, \ldots$. What is the sum of the next two members of the sequence?

The editor pointed out to the author that including a definition of the sequence might improve the question and make it more acceptable to the reviewers. The author claimed that all good students would see that the differences of successive terms is an arithmetic sequence, so the next two terms are obviously $22 + 10 = 32$ and $32 + 12 = 44$. Is there a need to **define** the sequence, or is it sufficient to give just the first five terms.

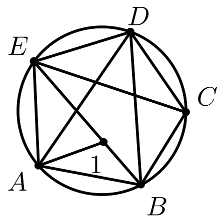
Let's get back to the first sequence, $1, 2, 4, 8,$
$16, 31, 57, \ldots$. The sequence is the (maximum) number of regions into which a circle can be cut using the chords joining n points on the circumference. Let R_n denote the number of such regions. The $R_1 = 1$, $R_2 = 2$, $R_3 = 4$, and in general $R_n = 1 + \binom{n}{2} + \binom{n}{4}$, as we will see below.



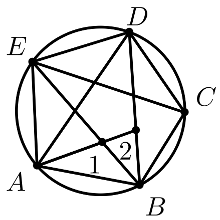
$R_1 = 1$ $R_2 = 2$ $R_3 = 4$

$R_4 = 8$

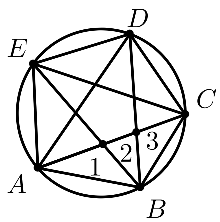
Let's consider the new regions that are formed when the line from A to C is drawn. The region marked with a 1 gets added when we reach the point of intersection of $\overline{AC}$ and $\overline{BE}$. This is one of the 5 points counted by $\binom{5}{4}$ in the table.



Extending this segment to the point where $\overline{AC}$ meets $\overline{BD}$ creates the new region marked 2.



Finally, extending to the point C creates the region marked 3. This region can be thought of as being created by the pair A, C. It is one of the 10 pairs counted by $\binom{5}{2}$.



$R_5 = 16$

The following table shows how to find each number is the sequence $R_1, R_2, \ldots$.

n	$\binom{n}{2}$	$\binom{n}{4}$	$1+\binom{n}{2}+\binom{n}{4}$
1	0	0	1
2	1	0	2
3	3	0	4
4	6	1	8
5	10	5	16
6	15	15	31
7	21	35	57

There are two standard ways to define sequences of real (or complex) numbers. The *closed form method* enables the reader to quickly compute the n^{th} term as a function of n. For example, the formula $a_n = 1/(2n-1)$ quickly enables the student to find the 10^{th} term of the sequence: $a_{10} = 1/(2 \cdot 10 - 1) = 1/19$. On the other hand a sequence like $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}$, $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$ is much more difficult to define in this way. Yet it is fairly easily defined *recursively.* What it takes to define a sequence recursively is (a) an anchor, and (b) a method for obtaining new members of the sequence in terms of those at hand. In the case above what is needed is the first term $a_1 = \sqrt{2}$ and the recognition that each term a_{n+1} can be obtained from the previous term a_n by adding 2 and then taking the square root of the sum: $a_{n+1} = \sqrt{2+a_n}$. That this method is a legitimate method for defining a sequence of numbers is not obvious. It depends on the **Principle of Mathematical Induction**. This principle, states that if $P(n)$ is a statement about the integer n for which (a) $P(1)$ is true and (b) for every positive integer n, the truth of $P(n)$ implies the truth of $P(n+1)$, the it follows that $P(n)$ is true for all positive integers. It is (provably) equivalent to the **Well Ordering Principle** of the natural numbers, which states the rather obvious fact that among any non-empty set of natural numbers there is a smallest one. But, let's not get too far off the path. We are talking about defining sequences. One method we could use to overcome the difficulty that students might have the wrong one sequence in mind when we write only the first few terms is the idea of *simplicity*. What the author mentioned above had in mind was the sequence starting out $a_1 = 2, a_2 = 4, a_3 = 8, a_4 = 14$, and $a_5 = 22$ that she felt was simplest. But what does simplest mean? Of course, arithmetic sequences are quite simple, and so are geometric sequences. What about reciprocals of such sequences. Or sums of such sequences.

There is a great website sequence puzzlers can use to see if their sequence is definable in some reasonable way. Its

`http://www.research.att.com/~njas/sequences/`

Thanks to Richard Askey for suggesting this addition. You can use the website to find the next terms of the sequence in the next paragraph. Of course that would defeat the purpose of the puzzle.

In conclusion, I think sequences *defined*? by listing the first half dozen terms have a rightful place in the mathematics classroom, but not on math contests. One of my most enjoyable classes includes the problem of finding the next few terms of the sequence $0, 1, 10, 2, 100, 11, 1000, 3, 20, 101, \ldots$. Picking out patterns and making conjectures are two highly enjoyable parts of the exercise. I invite you to try this one and the following exercises from my course in discrete math at UNC Charlotte. You can access the entire course at

`http://www.math.uncc.edu/~hbreiter/m1165/index.htm`

Exercises: Find both a closed form and a recursive definition of each of the following sequences.

1. $1, 4, 9, 16, 25, \ldots$.

2. $0, 0.1, 0.11, 0.111, 0.1111, \ldots.$

3. $0, 1, 3, 6, 10, 15, 21, \ldots.$

4. $0, 4, 16, 36, 64, 100 \ldots.$

5. $1, 3/2, 11/6, 50/24, 274/120 \ldots.$

6. $a_1 = \sqrt{2},\ a_2 = \sqrt{2+\sqrt{2}},$
   $a_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots.$

7. $1, 2, 3, 5, 8, 13, 21, \ldots$

8. $1, 3, 4, 9, 10, 12, 13, 27, 28, 30, 31, \ldots.$

9. $0, 1, 10, 2, 100, 11, 1000, 3, 20, 101, \ldots.$