UNIVERSITY OF NORTH CAROLINA CHARLOTTE
1999 HIGH SCHOOL MATHEMATICS CONTEST
March 8, 1999

1. The sides of a triangle are in the ratio $3 : 5 : 9$. Which of the following words best describes the triangle?

   (A) obtuse **(B)** scalene **(C)** right **(D)** isosceles **(E)** impossible

   (E) The triangle inequality, which states that the sum of any two sides of a triangle is at least as large as the third side, can be invoked. There are no such triangles.

2. The product of a positive number, its reciprocal, and its square is 7. Which of the following is closest to the sum of the number and its reciprocal?

   (A) 2.64 **(B)** 2.86 **(C)** 3.02 **(D)** 3.33 **(E)** 3.51

   (C) Let x denote the number. Then it follows from the given information that $x \cdot 1/x \cdot x^2 = 7$. This means that $x = \pm\sqrt{7}$. Since it is given that x is positive, $x + \frac{1}{x} \approx 2.645 + 1/2.645 \approx 3.02$.

3. Given that $a = 1/x, b = 9a, c = 1/b, d = 9c, e = 1/d$, and a, b, c, and d are all distinct non-zero numbers, then x must be the same as

   (A) a **(B)** b **(C)** c **(D)** d **(E)** e

   (D) Note that $b = 9(1/x) = 9/x$. Thus, $c = x/9$ and $x = 9c = d$.

4. Let $f(x) = \sqrt{(x-2)^2}$. Compute $\sum_{x=-2}^{x=2} f(2x)$.

   (A) -7 **(B)** 0 **(C)** 7 **(D)** 14 **(E)** 16

   (D) Another way to write f is $f(x) = |x-2|$, so $f(2x) = |2x-2|$, and the sum in question is

   $$|2(-2)-2|+|2(-1)-2|+|2(0)-2|+|2(1)-2|+|2(2)-2| = 6+4+2+0+2 = 14.$$

5. What is the product of the roots of

$$(x-1)(x-3)+(x-4)(x+5)+(x-3)(x-7)=0?$$

(A) -1260 **(B)** -420 **(C)** $4/3$ **(D)** 10 **(E)** 36

(C) Group the terms together and factor $(x-3)$ from the first two to get $(x-1)(x-3)+(x-4)(x+5)+(x-3)(x-7) = (x-3)[(x-1)+(x-7)]+(x-4)(x+5) = (x-3)[(2x-8)]+(x-4)(x+5) = 2(x-3)[(x-4)]+(x-4)(x+5) = (x-4)[2(x-3)+(x+5) = (x-4)(3x-1) = 0$, so the product of the two roots is $4 \cdot 1/3 = 4/3$.

OR

Combine the terms to get $x^2 - 4x + 3 + x^2 + x - 20 + x^2 - 10x + 21 = 3x^2 - 13x + 4 = 0$ which is equivalent to

$$x^2 - 13x/3 + 4/3 = 0,$$

the product of whose roots is the constant term, $4/3$.

6. Let $ABCD$ be a convex quadrilateral with the area s and let P, Q, R, and S be the midpoints of sides AB, BC, CD, and DA respectively. The sum of the areas of the triangles PBQ and RDS equals

(A) $3s/4$ **(B)** $2s/3$ **(C)** $s/2$ **(D)** $s/4$

(E) the ratio in question cannot be determined

(D) The area of PBQ is a quarter of the area of ABC and the area of RDS is a quarter of ACD.

7. If f is a function such that $f(3) = 2, f(4) = 2$ and $f(n+4) = f(n+3) \cdot f(n+2)$ for all the integers $n \geq 0$, what is the value of $f(6)$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 8

(E) it cannot be determined from the information given.

(D) $f(6) = f(2+4) = f(5) \cdot f(4) = (f(4) \cdot f(3)) \cdot f(4) = 2^2 \cdot 2 = 8$.

8. Which one of the following five numbers can be expressed as the sum of the squares of six odd integers (repetitions allowed).

   (A) 1996 **(B)** 1997 **(C)** 1998 **(D)** 1999 **(E)** 2000

(C) The square of an odd number is one bigger than a multiple of 4: $(2k+1)^2 = 4k^2+4k+1$. Therefore the sum of squares of six odd integers is necessarily 2 bigger than a multiple of 4. Only 1998 satisfies this requirement. On the other hand, $1998 = 43^2+9^2+7^2+3^2+3^2+1^2$.

9. There exist positive integers x, y, and z satisfying

$$28x + 30y + 31z = 365.$$

   Compute the value of $z - 2x$ for some such triplet.

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

(A) There are two solutions, $x = 1, y = 4$, and $z = 7$; and $x = 2, y = 1$, and $z = 9$. In both cases, $z - 2x = 5$. Do you notice anything special about the numbers $28, 30, 31$, and 365?

OR

Using the notation of modular congruences,

$$\begin{aligned} 28x + 30y + 31z &\equiv 30x + 30y + 30z - 2x + z \\ &\equiv 30(x + y + z) + z - 2x \\ &\equiv z - 2x \\ &\equiv 365 \\ &\equiv 5 \pmod{10}. \end{aligned}$$

10. If x and y are positive integers for which

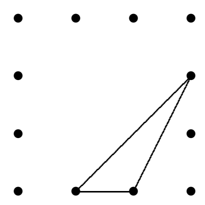
$$2(x-y)^2 + 4y^2 = 54,$$

then x could be

(A) 2 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** 10

(C) Divide every term by 2 to get $(x-y)^2 + 2y^2 = 27$. Note that $27 - 2y^2$ is an odd perfect square which could by 9 (for $y = 3$) or 25 (for $y = 1$). Both these lead to values of 6 for x. Other possible values of x are $\pm 4, 0$, and -6, but $x = 6$ is the only one for which both x and y are positive.

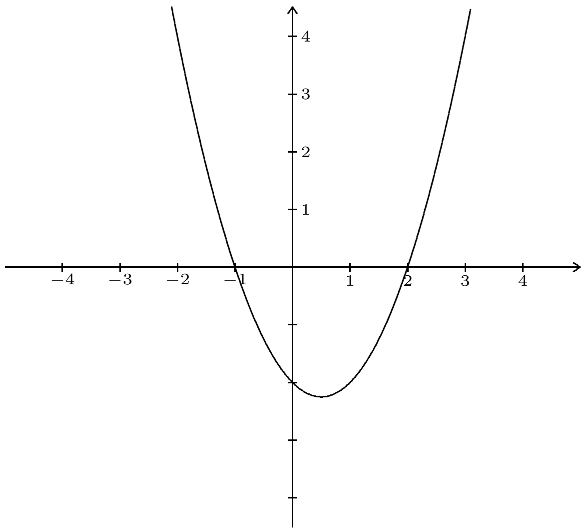
11. Twelve lattice points are arranged along the edges of a 3×3 square as shown. How many triangles have all three of their vertices among these points? One such triangle is shown.

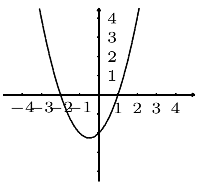


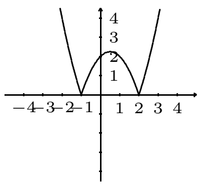
(A) 48 **(B)** 64 **(C)** 204 **(D)** 220 **(E)** 256

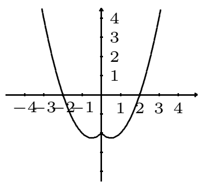
(C) Every set of three points except those which are collinear can be the vertices of a triangle. There are $\binom{12}{3} = 220$ three element subsets, and $4 \cdot \binom{4}{3} = 16$ which are collinear, so there are $220 - 16 = 204$ triangles.

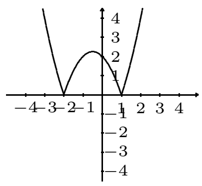
12. Let f be the function whose graph is shown. Which of the following represents the graph of $f(|x|)$?

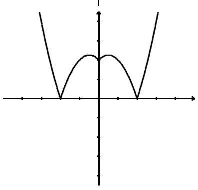


(A) 

(B) 

(C) 

(D) 

(E) 

(C) The functions shown are A, $f(-x)$; B, $|f(x)|$; C, the answer key; D, $|f(-x)|$; and E, $|f(|x|)|$.

13. Statistics have shown that in a certain college course, 65% of the students pass the first time they take it. Among those who have to repeat it, 70% pass on the second attempt, and among those who have to take it three times, 50% pass on the third attempt. What percentage of students have to take the course more than three times?

(A) 50% **(B)** 35% **(C)** 22.75% **(D)** 5.25% **(E)** 1%

(D) The percent of students who have to take the course twice is 35%. The percent of students who have to take the course 3 times is $.3 \times .35 = .105$. The percent of students who have to take the course more than 3 times is $.3 \times .35 \times .5 = 0.0525$.

14. Three integers a, b, and c have a product of 27,846 and the property that the same number N results from each of the following operations:

- a is divided by 6.
- 4 is added to b.
- 4 is subtracted from c.

What is $a + b + c$?

(A) 102 **(B)** 136 **(C)** 152 **(D)** 160 **(E)** 177

(B) Solve for a and c in terms of b to get $(6b+24) \cdot b \cdot (b+8) = 27{,}846$. Thus $b \cdot (b+4) \cdot (b+8) = 4641$. Factor the right side to get $4641 = 3 \cdot 7 \cdot 13 \cdot 17$, which can be written in the form $b \cdot (b+4) \cdot (b+8)$ only when $b = 13$. Thus $a + b + c = 102 + 13 + 21 = 136$.

15. A non-constant polynomial function $f(x)$ satisfies

$$f(-4) = f(-2) = f(1) = f(3) = 2.$$

What is the smallest possible degree of f?

(A) 1 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

(C) The polynomial $h(x) = f(x) - 2$ has four distinct zeros, and is not constant. Therefore its degree is at least 4, and so is f's degree.

16. Let

$$g(x) = \begin{cases} |x| - 2 & \text{if } x \le 0 \\ x - 3 & \text{if } 0 < x < 4 \\ 3 - x & \text{if } 4 \le x \end{cases}$$

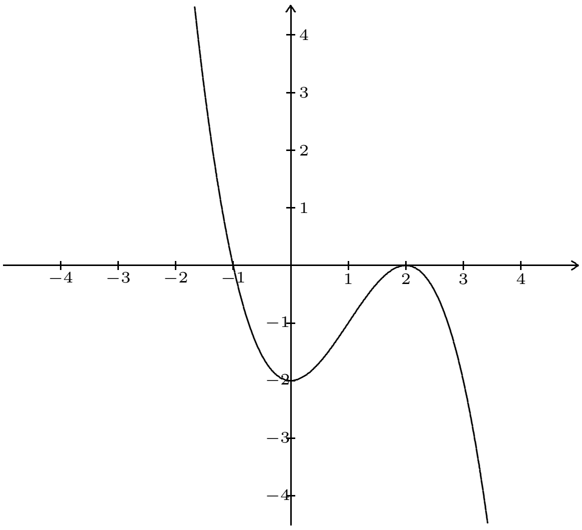
Find a number x such that $g(x) = -4$.

(A) -2 **(B)** -1 **(C)** 3 **(D)** 4 **(E)** 7

(E) Set each piece of g equal to $-4 : |x| - 2 = -4$ has no solutions; $x - 3 = -4$ only for $x = -1$ which is not in the domain of that piece; and
$3 - x = -4$ only when $x = 7$, and 7 does belong to the domain of that piece. Alternatively, you can plug into g the five options.

17. A cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ has a graph which is tangent to the x-axis at 2, has another x-intercept at -1, and has y-intercept at -2 as shown. Find the constants a, b, c, and d. Then, $a + b + c + d =$

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2



(B) The polynomial must be of the form $f(x) = a(x-2)^2(x+1)$ because it has a double zero at 2 and a zero at -1. To solve for a, note that $f(0) = a(0-2)^2(0+1) = -2$. It follows that $a = -1/2$ and that $a = -1/2, b = 3/2, c = 0$, and $d = -2$, so their sum is -1.

18. Find the sum of all values of x that satisfy

$$|x+1|+3|x-2|+5|x-4|=20.$$

(A) 2 **(B)** 5 **(C)** 6 **(D)** 9 **(E)** 11

(C) Consider the four cases, $x<-1, -1<x<2, 2<x<4,$ and $4<x$. Each of these gives rise to a linear equation in x. Just two of these have solutions in the appropriate intervals, $x=1$ and $x=5$. Their sum is 6.

19. You have 10 coins, all of different weights and you can weigh them only in pairs in a two-pan balance. What is the minimal numbers of weighings needed to find the heaviest coin?

(A) 5 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 45

(B) Each time you compare the new coin with the most heavy from the previous pair. You cannot do any better because the last coin might be the heaviest.

20. The area of a circle circumscribed about a regular hexagon is 200π. What is the area of the hexagon?

(A) $60\sqrt{3}$ **(B)** 600 **(C)** 1200 **(D)** $300\sqrt{3}$ **(E)** $600\sqrt{3}$

(D) A circle with an area of $200\pi=\pi r^2$ has a radius of $10\sqrt{2}$. A regular hexagon is made up of six equilateral triangles each with side equal to the radius of the circle. Since a hexagon has six sides the sum of the angles is $(4-2)180=720$, so that each interior angle has measure $120°$. The side of the triangle bisects this angle resulting in a $30-60-90$ triangle. Using this information, the area of each triangle is computed to be $50\sqrt{3}$ so the area of the hexagon is $6\cdot 50\sqrt{3}=300\sqrt{3}$.

21. From a group of three female students and two male students, a three student committee is selected. If the selection is random, what is the probability that exactly 2 females and 1 male are selected?

(A) 0.3 **(B)** 0.4 **(C)** 0.5 **(D)** 0.6 **(E)** 0.7

(D) There are just $\binom{5}{3} = 10$ ways to the committee. One of these has three females and three have two males. The other all have two females and one males. Thus the probability is 0.6.

22. What is the units digit of integer 3^{1999}?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 7 **(E)** 9

(D) The sequence of units digits of 3^n is periodic with period four: $3, 9, 7, 1, \ldots$. Therefore the units digit of 3^{1999} is the digit in the 1999^{th} term, which is 7.

23. The set of all x such that

$$(|x| - 2)(1 + x) > 0$$

is exactly

(A) $x > 2$ **(B)** $|x| > 2$ **(C)** $-2 < x < -1$ or $x > 2$ **(D)** $-1 < x < 2$
(E) $x < -2$ or $x > 2$

(C) The inequality $(|x| - 2)(1 + x) > 0$ is satisfied if both factors are positive or if both are negative. Both are positive if $x > 2$ and both are negative if $-2 < x < -1$.

24. The product of four distinct positive integers, a, b, c, and d is $8!$. The numbers also satisfy

$$\begin{aligned} ab + a + b + 1 &= 323 \quad (1) \\ bc + b + c + 1 &= 399. \quad (2) \end{aligned}$$

What is d?

(A) 7 **(B)** 14 **(C)** 21 **(D)** 28 **(E)** 35

(A) From (1) it follows that $ab+a+b+1 = (a+1)(b+1) = 323 = 17 \cdot 19$ and from (2) it follows that $bc+b+c+1 = (b+1)(c+1) = 399 = 19 \cdot 21$. Thus, $b = 18, a = 16$, and $c = 20$. Then $d = 8! \div (a \cdot b \cdot c) = 7$.

25. Which of the equations below has roots that are the reciprocals of the roots of the equation

$$x^2 - 3x - 2 = 0?$$

(A) $2x^2 + 3x - 1 = 0$ **(B)** $2x^2 - 3x - 1 = 0$ **(C)** $2x^2 + 3x + 1 = 0$
(D) $2x^2 - 3x + 1 = 0$ **(E)** none of **A**, **B**, **C** or **D**

(A) Let x_1, x_2 be the roots. Then $x_1 + x_2 = 3$ and $x_1 \cdot x_2 = -2$. So $\frac{1}{x_1} \cdot \frac{1}{x_2} = -\frac{1}{2}$ and $\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1+x_2}{x_1 \cdot x_2} = \frac{3}{-2}$. Therefore the quadratic equation with roots x_1 and x_2 must be

$$x^2 + \frac{3}{2}x - \frac{1}{2},$$

or equivalently, $2x^2 + 3x - 1 = 0$.

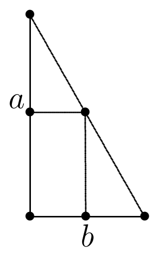
26. How many two-digit integers are there where the tens digit is greater than the units digit?

(A) 35 **(B)** 36 **(C)** 45 **(D)** 55 **(E)** 85

(C) If the units digits is 0 then there are 9 possible tens digit which is greater than 0. Now discuss the cases when the units digit is 1 through 9. We have total $9 + 8 + \cdots + 2 + 1 = 45$.

27. What is the area of the largest rectangular region that can be inscribed in a right triangle with legs of length 3 and 4?

(A) 2 **(B)** 2.5 **(C)** 3 **(D)** 3.5 **(E)** 4



(C) Let the length and the width of the rectangle be a and b respectively and let x and y be as shown in the diagram. Then $b = 4 - y$, $\frac{y}{a} = \frac{4}{3}$, and $a = \frac{4}{3}y$. This implies that the area $A = ab = \frac{3}{4}y(4-y) = 3y - \frac{3}{4}y^2$ is a quadratic function of y. Its maximum value occurs at its vertex, which occurs at $y = 2$, and corresponds to $A(2) = 3 \cdot 2 - \frac{3}{4}2^2 = 3$.

28. How many digits are there in the (decimal representation of the) integer 19^{9^9}? Recall that $2^{3^4} = 2^{(3^4)}$.

Note that $19^{9^9} = 10^{(9^9 \log 19)} = 10^{9^9 \cdot 1.27875} \approx 10^{495,415,345.3}$, so the number requires 495,415,346 digits.