UNIVERSITY OF NORTH CAROLINA CHARLOTTE
1995 HIGH SCHOOL MATHEMATICS CONTEST
March 13, 1995

1. $\dfrac{10^{12}-10^{11}}{9} =$

(A) $\frac{1}{9}$ **(B)** $\frac{10}{9}$ **(C)** 10^3 **(D)** $\frac{10^{11}}{9}$ **(E)** 10^{11}

(E)

$$\frac{10^{12}-10^{11}}{9} = \frac{10^{11}(10-1)}{9} = 10^{11}$$

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2. If $z = -x$, what are all the values of y for which

$$(x+y)^2 + (y+z)^2 = 2x^2?$$

(A) 0 **(B)** $0, 1$ **(C)** $-1, 0, 1$ **(D)** All positive numbers

(E) There are no values of y for which the equation is true

(A) Replace z with $-x$ in $(x+y)^2 + (y+z)^2 = 2x^2$ to get

$$(x+y)^2 + (y-x)^2 = 2x^2$$

or

$$2x^2 + 2y^2 = 2x^2$$

which implies that $2y^2 = 0$. Hence $y = 0$.

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3. It is known that $\log_{10} 3 = .4771$, correct to four places. How many digits are there in the decimal representation of 3^{100}?

(A) 46 **(B)** 47 **(C)** 48 **(D)** 49 **(E)** 50

(C) The number of decimal digits of an integer N is $\lfloor \log_{10} N \rfloor + 1$. Applying this to 3^{100}, we get $\lfloor \log_{10} 3^{100} \rfloor + 1 = \lfloor 100 \log_{10} 3 \rfloor + 1 = \lfloor 47.71 \rfloor + 1 = 48$.

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4. Given that $\frac{3}{2} < x < \frac{5}{2}$, find the value of

$$\sqrt{x^2 - 2x + 1} + \sqrt{x^2 - 6x + 9}$$

(A) 1 **(B)** 2 **(C)** $2x - 4$ **(D)** $4 - 2x$

(E) none of **A**, **B**, **C** or **D**

(B) Under the condition we have $\sqrt{x^2 - 2x + 1} + \sqrt{x^2 - 6x + 9} = x - 1 + 3 - x = 2$. Note that $\sqrt{x^2}$ is not always equal to x.

5. An archer misses the target on his first shot and hits the target on the next three shots. What is the least number of consecutive hits he must achieve following the first four shots in order to hit the target on more than nine tenths of his shots?

(A) 6 **(B)** 7 **(C)** 9 **(D)** 10 **(E)** 11

(B) If the archer hits the next seven consecutive shots then he will have hit 10 out of 11 shots and $\frac{10}{11} > \frac{9}{10}$. But six consecutive hits will not be enough.

6. For all integers n, $(-1)^{n^4+n+1}$ is equal to

(A) -1 **(B)** $(-1)^{n+1}$ **(C)** $(-1)^n$ **(D)** $(-1)^{n^2}$ **(E)** $+1$

(A) Since n and n^4 are both even or both odd, their sum is even. Hence $(-1)^{n^4+n+1} = (-1)^{n^4+n}(-1)^1 = (1)(-1) = -1$.

7. The graph of $|x| + |y| = 4$ encloses a region in the plane. What is the area of the region?

(A) 4 **(B)** 8 **(C)** 16 **(D)** 32 **(E)** 64

(D) The enclosed region is a square with vertices at $(4, 0), (0, 4), (-4, 0),$ and $(0, -4)$, which has area $(4\sqrt{2})^2 = 32$.

8. Find the minimum value of

$$1 \circ 2 \circ 3 \circ 4 \circ 5 \circ 6 \circ 7 \circ 8 \circ 9$$

where each "$\circ$" is either a "$+$" or a "$\times$".

(A) 36 **(B)** 40 **(C)** 44 **(D)** 45 **(E)** 84

(C) For integers larger than 1, addition produces smaller values than multiplication. Make the first operator $\times$ and the others $+$'s to get the minimum value $1 \times 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 44$.

9. Given $3 = \sqrt{a} + \dfrac{1}{\sqrt{a}}$ where $a \neq 0$, find $a - \dfrac{1}{a}$.

(A) 5 **(B)** 6 **(C)** $3\sqrt{5}$ **(D)** 7 **(E)** $5\sqrt{2}$

(C) Using the identity $(a-b)^2 = (a+b)^2 - 4ab$, we get

$$\left(\sqrt{a} - \frac{1}{\sqrt{a}}\right)^2 = \left(\sqrt{a} + \frac{1}{\sqrt{a}}\right)^2 - 4 = 3^2 - 4 = 5,$$

hence $\sqrt{a} - \dfrac{1}{\sqrt{a}} = \sqrt{5}$. Thus $a - \dfrac{1}{a} = \left(\sqrt{a} + \dfrac{1}{\sqrt{a}}\right)\left(\sqrt{a} - \dfrac{1}{\sqrt{a}}\right) = 3\sqrt{5}$.

10. The number $\log_{\frac{1}{4}} \sqrt[3]{1024}$ is equal to

(A) 5 **(B)** $\dfrac{20}{3}$ **(C)** $\dfrac{-5}{3}$ **(D)** $\dfrac{5}{3}$ **(E)** $\dfrac{-20}{3}$

(C) The log in question is the solution of $\sqrt[3]{1024} = (\frac{1}{4})^x$ or $\sqrt[3]{2^{10}} = \left[(\frac{1}{2})^2\right]^x$ so $2^{\frac{10}{3}} = (\frac{1}{2})^{2x}$, from which it follows that $x = -\frac{5}{3}$.

11. Given that

$$x(y-a)=0, \qquad z(y-b)=0, \quad \text{and} \quad a<b,$$

which of the following must be true?

(A) $xz<0$ **(B)** $xz>0$ **(C)** $x=0$

(D) $z=0$ **(E)** $xz=0$

(E) From the first condition, either $x=0$ or $y=a$. Case 1: If $x=0$, then $xz=0$. Case 2: If $x \neq 0$, then $y=a$ and since $a<b$, $y-b \neq 0$. Thus, from the second equation, $z=0$ and so $xz=0$. In either case $xz=0$.

12. If $2^{100} = 5m+k$, where k and m are integers and $0 \le k \le 4$, then k is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

(B) The sequence of powers of 2 has the property that every term of the form 2^{4k} is one larger than a multiple of 5, that is, has the form $5m+1$ for some integer m. To see this note that 2^4 is such a number and the product $(5k+1)(5j+1) = 25kj+5k+5j+1$ of two numbers of this form also has this form. Hence $2^{100} = 5m+1$ for some integer m.

13. If $\mathtt{a}$, $\mathtt{b}$, $\mathtt{c}$ and $\mathtt{d}$ represent distinct nonzero base ten digits for which $\mathtt{aaaa}_{\mathtt{ten}} + \mathtt{bbb}_{\mathtt{ten}} + \mathtt{cc}_{\mathtt{ten}} + \mathtt{d}_{\mathtt{ten}} = \mathtt{1995}$, then $\mathtt{a} \cdot \mathtt{b} \cdot \mathtt{c} \cdot \mathtt{d} =$

(A) 336 **(B)** 432 **(C)** 486 **(D)** 504 **(E)** 567

(D) Since $\mathtt{aaaa}_{\mathtt{ten}} + \mathtt{bbb}_{\mathtt{ten}} + \mathtt{cc}_{\mathtt{ten}} + \mathtt{d}_{\mathtt{ten}} = \mathtt{1995}$, it follows that $\mathtt{a} = 1$ and $\mathtt{bbb}_{\mathtt{ten}} + \mathtt{cc}_{\mathtt{ten}} + \mathtt{d}_{\mathtt{ten}} = 884$. Since $13 = 11+2 \le \mathtt{cc}_{\mathtt{ten}} + \mathtt{d}_{\mathtt{ten}} \le 99+8 = 107$, we have $\mathtt{b} = 7$, and then it follows that $\mathtt{c} = 9$ and $\mathtt{d} = 8$. Thus, $\mathtt{a} \cdot \mathtt{b} \cdot \mathtt{c} \cdot \mathtt{d} = 504$.

14. The number of rational solutions to $x^4 - 3x^3 - 20x^2 + 30x + 100 = 0$ is

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

(C) All rational solutions must be factors of 100. Neither ± 1 nor 2 are solutions, but $x = -2$ is a solution, and we find

$$x^4 - 3x^3 - 20x^2 + 30x + 100 = (x+2)(x^3 - 5x^2 - 10x + 50).$$

It is easy to see that $x = 5$ is a solution to $x^3 - 5x^2 - 10x + 50 = 0$, so by dividing, we find that

$$x^4 - 3x^3 - 20x^2 + 30x + 100 = (x+2)(x-5)(x^2 - 10).$$

The solutions to $x^2 - 10 = 0$ are $\pm\sqrt{10}$, and these are not rational, so there are just two rational solutions, $x = -2$ and $x = 5$.

15. The nine numbers $N, N+3, N+6, \cdots, N+24$, where N is a positive integer, can be used to complete a three by three magic square. What is the sum of the entries of a row of such a magic square? A magic square is a square array of numbers such that the sum of the numbers in each row, each column, and the two diagonals is the same.

(A) $3N$ **(B)** $3N+6$ **(C)** $3N+12$ **(D)** $3N+24$ **(E)** $3N+36$

(E) If K is the sum of the entries of each row, then $3K = N + (N+3) + (N+6) + \ldots + (N+24) = 9N + 3 + 6 + \ldots + 24 = 9N + 8(\frac{3+24}{2}) = 9N + 4 \cdot 27$, since the sum of the entries of an arithmetic sequence is the number of terms times the average of the first and last term. Hence, $K = \frac{9N+108}{3} = 3N + 36$.

16. Consider the function $F : N \to N$ defined by

$$F(n) = \begin{cases} n/3 & \text{if } n \text{ is a multiple of } 3 \\ 2n+1 & \text{if otherwise} \end{cases}$$

For how many positive integers k is it true that $F(F(k))) = k$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

(C) The two-fold composition $F \circ F(k)$ of F with itself evaluated at k results in one of the four values $\frac{k}{9}, 2(\frac{k}{3}) + 1, \frac{2k+1}{3}$ or $2(2k+1)+1$. Only the second and third of these give values of k for which $F(F(k))) = k$. These values are 1 and 3.

17. Define the sequence $a_1, a_2, a_3, \dots$ by

$$a_i = \left\lfloor 10^i \times \frac{1}{13} \right\rfloor - 10 \times \left\lfloor 10^{i-1} \times \frac{1}{13} \right\rfloor, \text{ for } i = 1, 2, \dots.$$

The largest value of any a_i is

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Note: For real x, $\lfloor x \rfloor$ is the largest integer that does not exceed x.

(D) Since $\frac{1}{13} = \overline{.076923}$ and a_i is the i^{th} digit in the decimal expansion of $\frac{1}{13}$, the largest value of a_i must be 9.

18. Triangle T has vertices $(0, 30)$, $(4, 0)$, and $(30, 0)$. Circle C with radius r circumscribes T. Which of the following is the closest to r?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

(B) The lines $y = x$ and $x = 17$ intersect at the center of C, so the radius is $\sqrt{(13^2 + 17^2)} \approx 21.40$ which is closest to 21.

19. If a, b and c are three distinct numbers such that

$$a^2 - bc = 7, \qquad b^2 + ac = 7, \quad \text{and} \quad c^2 + ab = 7,$$

then $a^2 + b^2 + c^2 =$

(A) 8 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 17

(D) Subtract the second given equation from the third to get

$$c^2 - b^2 - a(c - b) = 0. \qquad (1)$$

Dividing equation (1) by $(c - b)$ we have

$$b + c - a = 0. \qquad (2)$$

Add the three given equations to obtain

$$a^2 + b^2 + c^2 - bc + a(c + b) = 21. \qquad (3)$$

Using equation (2) we can substitute a for $c + b$ in equation (3) to obtain

$$a^2 + b^2 + c^2 - bc + a^2 = 21,$$

and then use the fact that $a^2 - bc = 7$ to obtain $a^2 + b^2 + c^2 = 14$. For completeness, note that the system *has* a solution, $a = 3, b = 2,$ and $c = 1$.

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20. The midpoints of the sides of a triangle are $(1, 1)$, $(4, 3)$, and $(3, 5)$. Find the area of the triangle.

(A) 14 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 22

(B) The triangle in question has area four times the one obtained by connecting the midpoints with line segments. (Draw the picture.) That triangle has area 4. You can find the area of any triangle with integer vertices by augmenting triangular regions to build a rectangle, and then subtracting areas. The rectangle here has vertices $(1, 1), (4, 1), (4, 5)$, and $(1, 5)$, so the area of the midpoint triangle is $12 - 3 - 4 - 1 = 4$. Thus the large triangle has area 16.

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21. Let x and y be two positive real numbers satisfying

$$x + y + xy = 10 \qquad \text{and} \qquad x^2 + y^2 = 40.$$

What integer is nearest $x + y$?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

(D) Square $x + y$ to get

$$(x + y)^2 = x^2 + y^2 + 2xy = 40 + 2(10 - x - y) = 60 - 2(x + y).$$

Let $t = x + y$, and note that $t^2 + 2t - 60 = 0$. Use the quadratic formula to find that $t = -1 + \sqrt{61} \approx 6.81$ or $t = -1 - \sqrt{61} \approx -8.81$.

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22. Let $f(n)$ be the integer closest to $\sqrt[4]{n}$. Then $\sum_{i=1}^{1995} \frac{1}{f(i)} =$

(A) 375 **(B)** 400 **(C)** 425 **(D)** 450 **(E)** 500

(B) The largest integer with fourth root closest to k is $\lfloor (k+\frac{1}{2})^4 \rfloor$. Why? Then

$$\begin{aligned}
\lfloor (k + \frac{1}{2})^4 \rfloor &= \lfloor k^4 + 2k^3 + \frac{3}{2}k^2 + \frac{1}{2}k + \frac{1}{16} \rfloor \\
&= \lfloor k^4 + 2k^3 + \frac{1}{2}(3k^2 + k) + \frac{1}{16} \rfloor \\
&= k^4 + 2k^3 + \frac{1}{2}(3k^2 + k) \text{ since } 3k^2 + k \text{ is even.}
\end{aligned}$$

Therefore, the number of integers with fourth root closest to k is

$$\begin{aligned}
\lfloor (k + \frac{1}{2})^4 \rfloor - \lfloor ((k-1) + \frac{1}{2})^4 \rfloor &= \\
\lfloor (k + \frac{1}{2})^4 \rfloor - \lfloor (k - \frac{1}{2})^4 \rfloor &= \\
\{k^4 + 2k^3 + \frac{1}{2}(3k^2 + k)\} - \{k^4 - 2k^3 + \frac{1}{2}(3k^2 - k)\} &= \\
4k^3 + k &
\end{aligned}$$

That is, $f(n) = k$ for $4k^3 + k$ (consecutive) values of n. Since $f(1995) = 7, \sum_{k=1}^{6}(4k^3 + k) = 1785$ and $f(1786) = f(1787) = \ldots = f(1995) = 7$, it follows that

$$\sum_{i=1}^{1995} \frac{1}{f(i)} = \sum_{i=1}^{1785} \frac{1}{f(i)} + \frac{210}{7} = \sum_{i=1}^{6} \frac{4k^3 + k}{k} + 30 = \sum_{i=1}^{6} (4k^2 + 1) + 30 = 400.$$
