UNIVERSITY OF NORTH CAROLINA CHARLOTTE
2000 HIGH SCHOOL MATHEMATICS CONTEST
March 6, 2000

1. How many six-digit multiples of 5 can be formed from the digits $1, 2, 3, 4, 5$, and 6 using each of the digits exactly once?

   (A) 21 **(B)** 32 **(C)** 36 **(D)** 64 **(E)** 120

   (E) Only the numbers ending in 5 are divisible by 5. There are $5! = 120$ ways to permute the other five digits.

2. Laura jogs seven blocks the first day of her training program. She increases her distance by two blocks each day. On the last day, she jogs 25 blocks. How many days was she in training?

   (A) 5 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 15

   (D) By making a table, the pattern can easily be seen:

Day	Number of Blocks
1	7
2	9
3	11
4	13
5	15
6	17
7	19
8	21
9	23
10	25

   OR

   Note that $d_1 = 7, d_2 = 7+1\cdot 2, d_3 = 7+2\cdot 2, \ldots, d_n = 7+(n-1)\cdot 2$. Hence, $d_n = 25 = 7 + (n-1)\cdot 2 \Longrightarrow n = 10$.

3. The line that passes through the points $(2, 5)$ and $(7, -2)$ also passes through the point $(17, y)$ for some y. What is y?

   (A) -16 **(B)** -15 **(C)** -14 **(D)** -5 **(E)** 5

   (A) The slope of the line is $-7/5$ and an equation for it is $y-5 = -\frac{7}{5}(x-2)$. Let $x = 17$ to find that $y = -16$.

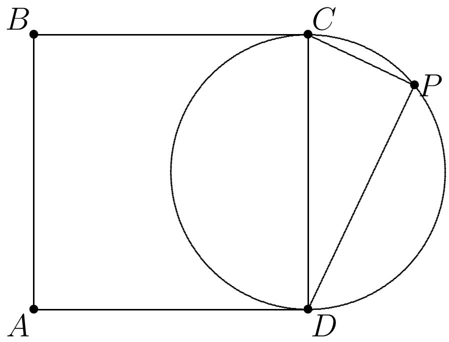
4. The number $((N-2)(N-4)(N-6)(N-8)-1)/2$ is an integer if N is

(A) 1 only **(B)** 2 only **(C)** 9 only

(D) any odd integer **(E)** any even integer

(D) If N is any odd integer, then all of $(N-2), (N-4), (N-6),$ and $(N-8)$ are odd, so their product is odd. Thus $(N-2)(N-4)(N-6)(N-8)-1$ is even.

5. In the diagram, $ABCD$ is a square and P is a point on the circle with diameter CD, $CP = 7$, and $PD = 11$. What is the area of the square?



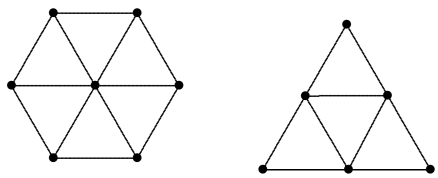
(A) 144 **(B)** 169 **(C)** 170 **(D)** 180 **(E)** 225

(C) Angle CPD is a right angle. Therefore $CD = \sqrt{7^2 + 11^2} = \sqrt{49 + 121} = \sqrt{170}$, and $CD^2 = 170$.

6. An equilateral triangle and a regular hexagon have the same perimeter. What is the ratio of the area of the triangle to the area of the hexagon?

(A) 1/2 **(B)** 2/3 **(C)** 3/4 **(D)** $\sqrt{2}/2$ **(E)** $\sqrt{3}/3$

(B) Each of the small triangles has the same area.



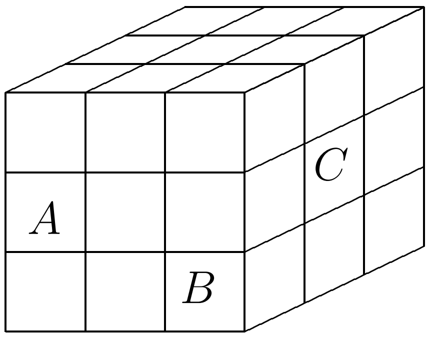
7. A school has b boys and g girls, where $g < b$. How many girls must be enrolled so that 60% of the student body is female.

(A) $0.6g$ **(B)** $0.6g - 0.4b$ **(C)** $0.6b - 0.4g$ **(D)** $1.5b - g$ **(E)** $2b - g$

(D) Let x denote the required number of girls. Solve the equation $\frac{g+x}{b+g+x} = .60$ for x to get $x = 1.5b - g$.

8. What is the surface area of the figure obtained by removing the three labeled unit cubes from the large $3 \times 3 \times 3$ cube shown?

(A) 50 **(B)** 54 **(C)** 58 **(D)** 60 **(E)** 64



(D) The removal of the cube B does not change the surface area, the removal of A adds 2 and the removal of C adds 4, so the total is $9 \cdot 6 + 2 + 4 = 60$.

9. How many ordered triples (x, y, z) satisfy the equation

$$(x^2 - 1)^2 + (y^2 - 4)^2 + (z^2 - 9)^2 = 0?$$

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

(E) Each term must be zero. Thus $x = \pm 1, y = \pm 2$, and $z = \pm 3$. There are 8 ways to put these values into a triplet.

10. Let $f(x) = (x-b)/(x-a)$ for constants a and b. If $f(2) = 0$ and $f(1)$ is undefined, what is $f(1/2)$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

> **(D)** Note that $f(1) = \frac{1-b}{1-a}$, so $f(1)$ is undefined for $a = 1$. Because $f(2) = 0$, it follows that $2-b = 0$, and $b = 2$. Finally, $f(1/2) = (1/2-2) \div (1/2-1) = 3$.

11. The slope of the line tangent to the graph of $y = x^2$ at the point $(2, 4)$ is 4. What is the y-intercept of the line?

(A) -12 **(B)** -4 **(C)** 0 **(D)** 4 **(E)** 12

> **(B)** The equation is given by $y - 4 = 4(x - 2)$ which is equivalent to $y = 4x - 4$, so the y-intercept is -4.

12. What is the coefficient of x^7 in the polynomial $(x + 3)^{10}$?

(A) 120 **(B)** 2187 **(C)** 3240 **(D)** 3402 **(E)** 5670

> **(C)** Using the Binomial Expansion Theorem or by contructing the 10th row of Pascal's triangle, the expansion of $(x + 3)^{10}$ can be found to be $x^{10} + 10(3)x^9 + 45(3^2)x^8 + 120(3^3)x^7 + 210(3^4)x^6 + 252(3^5)x^5 + 210(3^6)x^4 + 120(3^7)x^3 + 45(3^8)x^2 + 10(3^9)x^1 + 1(3^{10})$. So the coefficient of x^7 is $120 \cdot 3^3 = 120 \cdot 27 = 3240$.

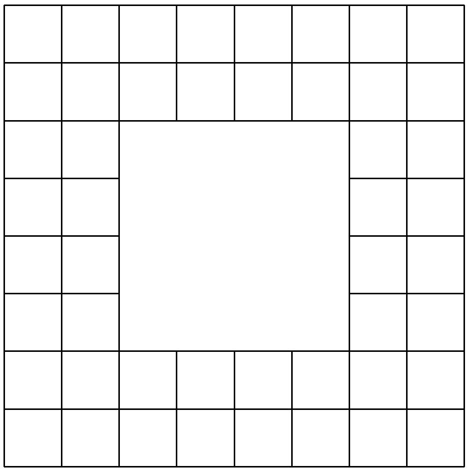
13. Niki just completed a 10 mile bike trip. If she had been able to ride 2 miles per hour faster, she would have completed her trip in 20 fewer minutes. Find her speed to the nearest tenth of a mile per hour.

(A) 6.2 **(B)** 6.3 **(C)** 6.5 **(D)** 6.7 **(E)** 6.8

> **(E)** Let r and t denote the rate in mph and time in hours, respectively. Then $rt = 10$ and $(r + 2)(t - 1/3) = 10$. Solve these simutaneously to get the quadratic $r^2 + 2r - 60 = 0$. Solve this using the quadratic formula to get the positive root $r = -1 + \sqrt{61} \approx 6.81$.

14. How many squares of all sizes have sides determined by the grid lines below?

(A) 48 **(B)** 72 **(C)** 75 **(D)** 89 **(E)** 91



(E) Count them by size. There are 48 unit squares, 24 with area 4, one with area 16, four with area 25, nine with area 36, four with area 49, and one with area 64, for a total of $48 + 24 + 1 + 4 + 9 + 4 + 1 = 91$.

15. An ant located at a corner of a 2in. $\times$ 3in. $\times$ 5in. rectangular block of wood wants to crawl along the surface to the opposite corner of the block. What is the length of the shortest such path?

(A) $\sqrt{50}$ **(B)** $\sqrt{58}$ **(C)** 8 **(D)** $\sqrt{68}$ **(E)** 10

(A) Each of the six routes takes the ant across two rectangular faces. The block can be cut and unfolded so the two faces are adjacent as shown below. The three possible routes across two faces of the block have lengths $\sqrt{2^2 + (3+5)^2} = \sqrt{68}$, $\sqrt{3^2 + (2+5)^2} = \sqrt{58}$, and $\sqrt{(2+3)^2 + 5^2} = \sqrt{50}$.

16. The students in Professor Einstein's class decided to reward the fine teacher with a CD player at the end of the course. A total of \$529 was collected from the students, with each student contributing the same amount, which was equal to the total number of students in the class. Only ordinary US bills were used and none of these were \$2 dollar bills. In addition, each student paid using the same five bills. How many ten-dollar bills were collected?

(A) 10 **(B)** 12 **(C)** 15 **(D)** 23 **(E)** 46

(E) Let x equal the amount each student contributed. If each student gave an amount equal to the total number of students, then x^2 represents the total value of the collection, \$529. Hence, $x = \$23$. Since each student used the same five bills, the only combination of bills that each student paid must be 2 ten-dollar bills and 3 one-dollar bills. Therefore there are $23 \cdot 2 = 46$ ten dollar bills collected.

17. If the operation $\oplus$ is defined for all positive x and y by $x \oplus y = (xy)/(x+y)$, which of the following must be true for positive x, y, and z?

    i. $x \oplus x = x/2$

    ii. $x \oplus y = y \oplus x$

    iii. $x \oplus (y \oplus z) = (x \oplus y) \oplus z$

    (A) i. only **(B)** i. and ii. only **(C)** i. and iii. only

    (D) ii. and iii. only **(E)** all three

(E) The operation can be written $x \oplus y = \left(\frac{1}{x} + \frac{1}{y}\right)^{-1}$. It is not hard to see that it is commutative. A straightforward calculation shows that it is associative as well. Also, $x \oplus x = \left(\frac{1}{x} + \frac{1}{x}\right)^{-1} = x/2$.

18. Suppose that $f(n+1) = f(n) + f(n-1)$ for $n = 2, 3, \ldots$. Given that $f(6) = 23$ and $f(4) = 8$, what is $f(1) + f(3)$?

    (A) 6 **(B)** 7 **(C)** 8 **(D)** 12 **(E)** 13

(E) Substituting $n = 5$ into the given equation, we find that $f(5) = 23 - 8 = 15$. Then substituting $n = 4$, we obtain $f(3) = 15 - 8 = 7$. Then $n = 3$ yields $f(2) = 8 - 7 = 1$. Finally, $n = 2$ gives $f(1) = 7 - 1 = 6$. Thus, $f(1) + f(3) = 13$.

19. The point $A = (2, 3)$ is reflected about the x-axis to a point B. Then B is reflected about the line $y = x$ to a point C. What is the area of the triangle ABC?

    (A) 12 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** 24

(C) The coordinates of B are $(2, -3)$, and those of C are $(-3, 2)$, so the triangle can be viewed as one with base $AB = 3 - (-3) = 6$ and altitude of $2 - (-3) = 5$, so the area is $\frac{1}{2}(6 \cdot 5) = 15$.

20. The sides a, b, and c of a triangle satisfy $\sqrt{a}+\sqrt{b}=\sqrt{c}$. Which of the following best describes the triangle?

(A) acute **(B)** scalene **(C)** isosceles

(D) non-existent **(E)** equilateral

(D) Square both sides of $\sqrt{a}+\sqrt{b}=\sqrt{c}$ to obtain $c=a+b+2\sqrt{ab}$, which means that c is larger than the sum of a and b.

21. In triangle ABC, $AB=9, BC=10$, and $AC=11$. Among the following, which number is closest to $\cos(\angle ABC)$?

(A) 2/9 **(B)** 1/4 **(C)** 2/7 **(D)** 3/10 **(E)** 1/3

(E) By the Law of Cosines, $11^2=9^2+10^2-2\cdot 9\cdot 10\cos(\angle ABC)$. Therefore, $\cos(\angle ABC)=(181-121)\div 180=1/3$.

22. If $a>0$ and $ab=3, bc=5$, and $ac=7$, what is c.?

(A) 3 **(B)** $\sqrt{3}$ **(C)** $\sqrt{\dfrac{35}{3}}$ **(D)** 2 **(E)** 1

(C) We have $a^2b^2c^2=3\cdot 5\cdot 7=105$. Since $a>0$, so are b and c. Hence $abc=\sqrt{105}$. Thus $c=\frac{abc}{ab}=\frac{\sqrt{105}}{3}=\sqrt{\frac{35}{3}}$.

23. Let N be the smallest four digit number such that the three digit number obtained by removing the leftmost digit is one ninth of the original number. What is the sum of the digits of N?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

(D) The number N can be written in the form $N = 1000a + b$, where a is a digit and b is a three digit number. The condition implies that $9b = 1000a + b$, or equivalently, $8b = 1000a$. The smallest nonzero digit a for which $1000a$ is a multiple of 8 is $a = 1$, which leads to $b = 125$. Hence $N = 1125$.

24. There are some elevens in a collection of numbers and the rest of the numbers are twelves. There are three more elevens than twelves. Which of the following could be the sum of the numbers in the collection?

(A) 232 **(B)** 234 **(C)** 235 **(D)** 240 **(E)** 256

(D) Suppose there are k twelves. Then the sum is $11(k+3) + 12k = 23k + 33 = 23(k+1) + 10$ so the sum must be 10 bigger than some multiple of 23. Only 240 of the options is such a number.

25. A treasure is located at a point along a straight road with towns $A, B, C,$ and D in that order. A map gives the following instructions for locating the treasure:

(a) Start at town A and go 1/2 of the way to C.

(b) Then go 1/3 of the way towards D.

(c) Then go 1/4 of the way towards B, and dig for the treasure.

If $AB = 6$ miles, $BC = 8$ miles, and the treasure is buried midway between A and D, find the distance from C to D.

(A) 4 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** None of the above

(B) Let the distance from C to D be x. Using the information given, we can represent the location of the treasure in two ways, which gives the following equation:

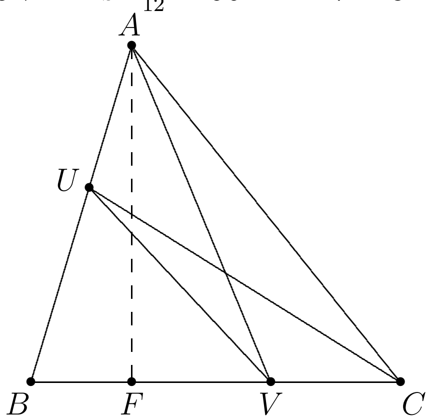
$$\frac{x+14}{2} = 7 + (1/3)(x+7) - \frac{1}{4}\left[\frac{1}{3}(x+7) + 1\right].$$

Solve this to get $x = 6$.

26. The area of $\triangle ABC$ is 144 square units. Point U is on $\overline{AB}$ such that the ratio AU to UB is 5 to 7. Point V is on $\overline{BC}$ such that the ratio $BV$to VC is 2 to 1. What is the area of $\triangle UVB$?

(A) 50 **(B)** 54 **(C)** 55 **(D)** 56 **(E)** 60

(D) The area of $\triangle ABV$ is $\frac{2}{3} \cdot 144 = 96$ (because the altitude AF to the base $\overline{BC}$ is the same for both triangles ABC and ABV), so the area of $\triangle UVB$ is $\frac{7}{12} \cdot 96 = 7 \cdot 8 = 56$.



27. When a missile is fired from a ship, the probability it is intercepted is 1/3. The probability that the missile hits the target, given that it is not intercepted, is 3/4. If three missiles are fired independently from the ship, what is the probability that all three hit their targets?

(A) 1/12 **(B)** 1/8 **(C)** 9/64 **(D)** 3/8 **(E)** 3/4

(B) The probability that a missile is **not** intercepted is 2/3. The probability that any one missile hits its target is $P(hits) = (3/4) \cdot (2/3) = 1/2$. The three missiles are fired independently, so the desired probability is $1/2 \cdot 1/2 \cdot 1/2 = 1/8$.